ICIC Express Letters
Part B: Applications
Volume **1**, Number **1**, September **2010**

ICIC International ©2010 ISSN 2185-2766
pp. **15–20**

A SIMPLE APPROACH FOR COMPARTMENTAL SYSTEMS MODEL ORDER REDUCTION

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Received May 2010; accepted July 2010

Abstract. *Compartmental linear systems naturally arise when someone aims to develop dynamic models for populations, economic systems, chemical reactions, hydraulic systems, among others. General theories for this class of systems have been well developed in the last decades. This paper aims to present a simple methodology of model order reduction for asymptotically stable compartmental linear systems, by observing the fact that even for higher order systems an approximately first order-like between input and output is observed in any case that involves the driven-point function. Then, a step response error minimization is carried out with DC gain retention, since by solving a single nonlinear equation it is possible to obtain the parameters of first order approximated model. Numerical examples, both theoretical and real world models support the proposed methodology.*
Keywords: Linear systems, Compartmental systems, Model order reduction

1. **Introduction.** Linear compartmental systems comprehend a class of linear systems, which naturally occur in the study of dynamic models in which the state variables are quantities that are physically meaningful only if they are non-negative or strictly positive. Examples of this class are economic models, mass balance, population dynamics, among others [1]. Model order reduction for linear systems is an extremely valuable technique in order to reduce model complexity, culminating, for example, in the use of a simple controller in order to control a higher order system [2]. For example, in H_2/H_∞ control and filtering design, the order of obtained controller/filter may be even higher than plant order A widely used method for model order reduction is the method of balanced realization [3, 4], where the controllability and observability grammians of the system are equal and diagonal, and the Hankel singular values appear in the main diagonal. One possible disadvantage of balanced truncation model reduction occurs when extra zeros appear in the final model. Continuous-time compartmental linear system class is the main object of study in this paper, having its properties well defined and studied, which can be appealing when the goal is to reduce the system order. In the following, some relevant definitions and theorems taken from [5] are presented.

Definition 1.1. *Let be the time-invariant linear, single-input single-output system in the form*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

Such a system is said compartmental if $A = [a_{ij}]$, B and C obey:

$$a_{ij} \geq 0,\ i \neq j;\ a_{ii} + \sum_{j \neq i} a_{ij} \leq 0;\ B \in \mathbb{R}_+^{n \times 1};\ C \in \mathbb{R}_+^{1 \times n}$$

Matrix A in the previous definition is said compartmental, being a particular case of Metzler matrix. For the matrix A to be Metzler, only the first condition on its entries needs to be satisfied [6].

The following lemma gives an important property that will be explored in this work:

Lemma 1.1. *If A is compartmental, then there is only one real eigenvalue of A, $\lambda_k \leq 0$ such that $Re\,(\lambda_l) \leq \lambda_k, \forall l \neq k$.*

At this point, we consider only asymptotically stable systems, also known as systems without *traps* [7]. Together with this lemma, the following property is very useful:

Property 1.1. [4] *Whenever there is k such that $b_k = c_k = 1$, $b_j = c_j = 0$ and $j \neq k$, the transfer function*

$$G(s) = C\,(sI - A)^{-1}\,B \tag{2}$$

is called the driven-point function (DPF). As an example, for an RC cascade, the DPF is the cascade impedance between k-th compartment and the ground. Moreover, the impulse response $g(t)$ is such that (i) $g(0) \geq g(t) \geq 0$ and (ii) $\dot{g}(0) \leq 0$, which becomes strong if $ord\,[G(s)] \geq 2$.

Summarily, the concepts presented here demonstrated that the i-th state variable x_i in a compartmental system obeys to the mass balance, i.e., the i-th compartment gets incoming material from the input at rate bi, from another j-th compartment at rate $a_{ij}x_j$, and loses material to the exterior at rate $a_{0i}x_i$ and to another k-th compartment at rate $a_{ki}x_i$, such that:

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}x_j - \left(a_{0i} + \sum_{j \neq i} a_{ji}\right) x_i(t) + b_i u(t)$$

The theorems and property previously presented have a central role in this paper, because of two main facts: (i) states that the system has one dominant eigenvalue in the sense of origin proximity, also known by Frobenius eigenvalue, and (ii) $g(t)$ is positive and decreasing in the vicinity of $t = 0$; henceforth, we can conjecture that the system has a first-order-like impulse response.

2. The Methodology.

The transfer function of a linear system of any order can be obtained from the classical relation given by (2). The spectrum of A can be partitioned with indices $1, 2, \ldots, i$, with respective multiplicities $m_1, m_2, ..., m_i$, such that the condition $ord\,[G(s)] = \sum m_i$ is satisfied. In this manner, the impulse response and the corresponding transfer function for the system are given respectively by:

$$\begin{aligned} g(t) &= \sum_{k=1}^{i} \sum_{n=0}^{m_k-1} \tfrac{a_n^{(k)}}{n!} t^n \exp(\lambda_k t) \\ G(s) &= \sum_{k=1}^{i} \sum_{n=0}^{m_k-1} \tfrac{a_j^{(k)}}{(s-\lambda_k)^{n+1}} \end{aligned} \tag{3}$$

For the approximated model, these functions are represented respectively by:

$$\begin{aligned} \hat{g}(t) &= \hat{b}\exp(\hat{\lambda}t) \\ \hat{G}(s) &= \tfrac{\hat{b}}{s-\hat{\lambda}} \end{aligned} \tag{4}$$

where $\hat{b} > 0$, $\hat{\lambda} < 0$. The first criterion used for a good approximation between the responses is the equality of DC gains $G(0) = \hat{G}(0)$ that results in:

$$\hat{b} = \hat{\lambda} \sum_{k=1}^{i} \sum_{n=0}^{m_k-1} \frac{(-1)^{n+2} n! a_n^{(k)}}{\lambda_k^{n+1}} \tag{5}$$

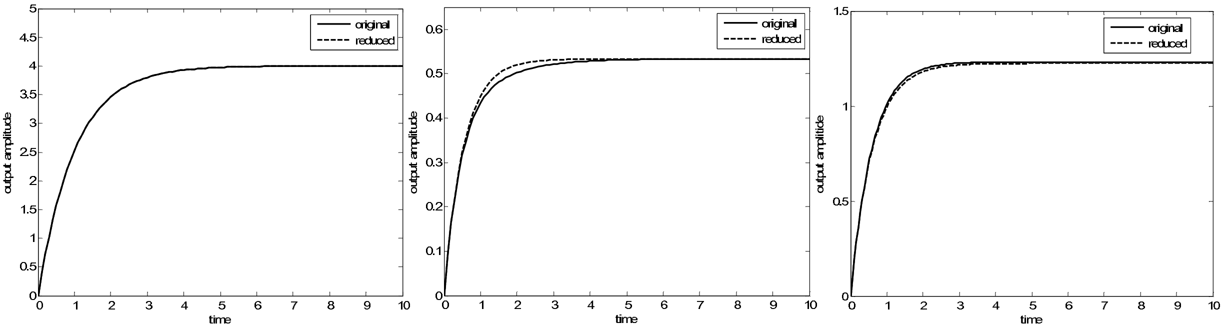

FIGURE 1. Step responses for the systems S_1, S_2, S_3 and their approximations

The second criterion is established by taking the minimization of the integral for square error between the impulse responses:

$$\varepsilon\left(\hat{\lambda}\right) = \int_0^{+\infty} \left[g(\tau) - \hat{g}(\tau)\right]^2 d\tau$$

by making $\frac{d\varepsilon}{d\hat{\lambda}} = 0$, one obtains a nonlinear algebraic equation, whose stable solution is the appropriate value for $\hat{\lambda}$:

$$\begin{gathered} \sum_{k=1}^{i} \sum_{n=0}^{m_k-1} \frac{(-1)^{n+1} a_n^k (\lambda_k - n\hat{\lambda})}{(\lambda_k + \hat{\lambda})^{n+2}} - \frac{\eta}{4} = 0 \\ \eta = \sum_{k=1}^{i} \sum_{n=0}^{m_k-1} \frac{(-1)^{n+2} a_n^k}{\lambda_k^{n+1}} \end{gathered} \tag{6}$$

3. **Numerical Examples.**

3.1. **Three compartmental systems.**

Consider three compartmental systems $S_1 : \{A_1, B_1, C_1\}$, $S_2 : \{A_2, B_2, C_2\}$ and $S_3 : \{A_3, B_3, C_3\}$ where:

$$A_1 = A_2 = \begin{bmatrix} -2 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; B_1 = C_1^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; B_2 = C_2^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$A_3 = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; C_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T \tag{8}$$

Note that in the system S_2 the transfer function is DPF, meanwhile the systems S_1 and S_3 are slightly different: S_2 is excited by all compartments as well as the output is for all compartments, and in S_3, the excitation is made by two compartments with the same type of output of S_1.

For these systems, the spectrum of A is:

$$S_1, S_2 : \lambda_1 = -1, \; m_1 = 1; \; \lambda_2 = -3, \; m_2 = 1, \; \lambda_3 = -2 + j, \; m_3 = 1,$$

$$\lambda_4 = -2 - j, \; m_4 = 1,$$

$$S_3 : \lambda_1 = -2, \; m_1 = 2; \; \lambda_2 = -3, \; m_2 = 3$$

By applying the proposed methodology, one obtains results that are a very good approximation of the original systems. The obtained parameters are respectively:

$$S_1 : \hat{b} = 4, \; \hat{\lambda} = -1, S_2 : \hat{b} = 0.986, \; \hat{\lambda} = -1.849, \; S_3 : \hat{b} = 2.038, \; \hat{\lambda} = -1.655$$

Figure 1 depicts the step response for the original and reduced systems. For the case of S_1, the obtained result certifies the merit of the methodology, because in this case the

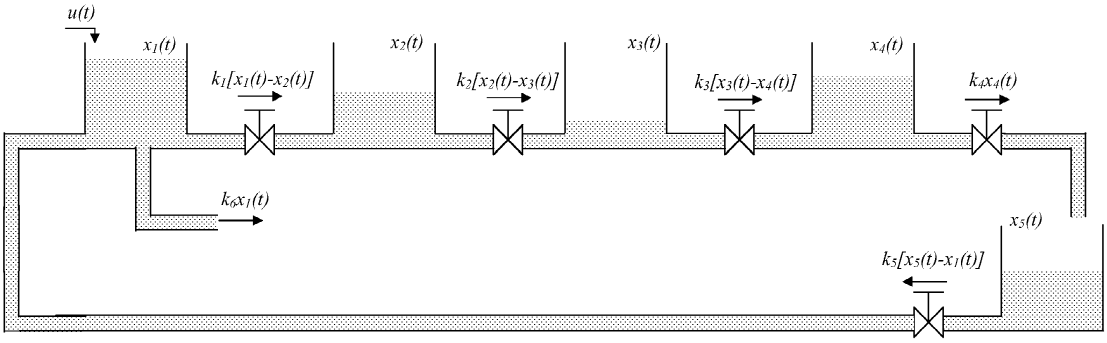

Figure 2. A tank system for study

system is an exact minimal order model. For the case of S_2, where even the complex eigenvalues have an appreciable modal index, a very good approximation is obtained. Also in the case of the system S_3 a good approximation is obtained.

Table 1 shows the integral of square error for the systems S_2 and S_3, providing a graphical comparison between the reduction by balanced truncation and by the proposed methodology.

Table 1. Comparison between the integral of square error for the proposed methodology and the balanced realization

	Balanced truncation	Proposed methodology
S_2	0.0412	0.0388
S_3	0.0468	0.0302

3.2. **Modeling the control of a tank system.** Consider the five tanks system with laminar flow shown in Figure 2. The state-space model for this system is:

$$\frac{dx}{dt} = \begin{bmatrix} -k_1-k_5-k_6 & k_2 & 0 & 0 & k_5 \\ k_1 & -k_1-k_2 & k_2 & 0 & 0 \\ 0 & k_2 & -k_2-k_3 & k_3 & 0 \\ 0 & 0 & k_3 & -k_3-k_4 & 0 \\ k_5 & 0 & 0 & k_4 & -k_5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \quad (9)$$

The state variables in this model are: the volume of material in each tank, and the parameters k_i, for $i = 1, ..., 6$, are all unitary. This is a compartmental model, and the spectrum of A is given by: $\lambda_1 = -4$; $\lambda_2 = -3.247$; $\lambda_3 = -0.198$; $\lambda_4 = -1.555$; $\lambda_5 = -1$. The output be the volume of material in the first tank, so the transfer function will be the DPF. Henceforth, one has $C = B^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. Parameters for reduced first-order model are: $\hat{b} = 0.5239$; $\hat{\lambda} = -0.5239$. Figure 3 shows the closed-loop system response to the reference:

$$r^*(t) = (1 - e^{-2t}) \left[3 + \left(0,5\text{sen}\frac{t}{2} - 0,2\cos\frac{3t}{2} \right)^2 \right]$$

by using a simple PI controller tuned for the following requirements: damping $\xi = \frac{1}{\sqrt{2}}$ and a zero $z = -4$, designed based on reduced model. The responses of nominal and approximated systems are quite similar. Figure 4 depicts the frequency response of the original and reduced model with feedback, in which one can see a good approximation up to the vicinity of system cutoff frequency. In the same scenario, a comparison between the proposed methodology for order reduction and the balanced truncation can be seen

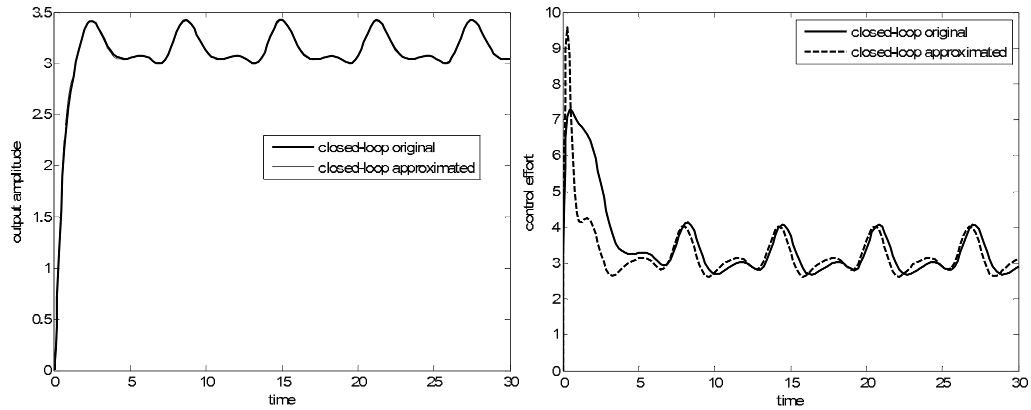

FIGURE 3. Output and control effort for the tank system example

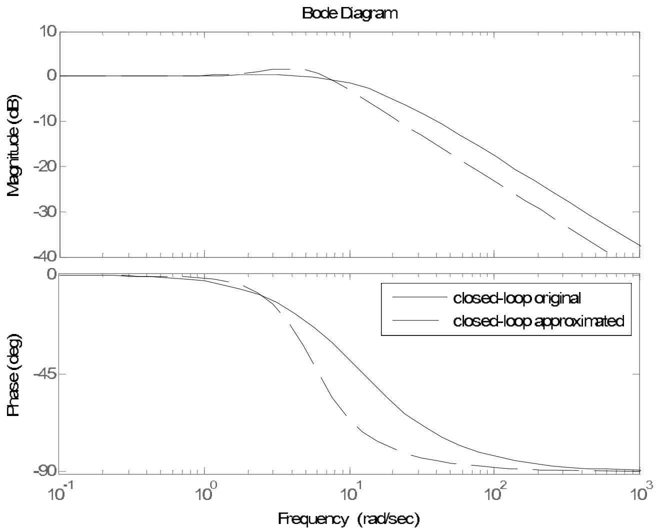

FIGURE 4. Closed-loop frequency response for the tank system example

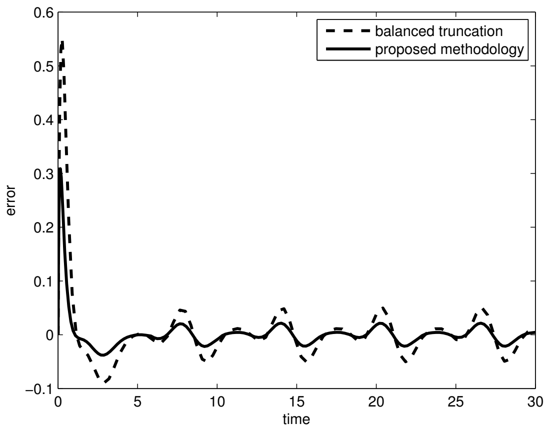

FIGURE 5. Tracking error comparison between balanced truncation and proposed approach, using PI controller

on Figure 5. The tracking error is smaller when the PI controller is designed based on the reduced order model obtained with the proposed approach.

4. **Conclusion.** Compartmental linear systems have a prominent characteristic: the Frobenius dominant eigenvalue. Based on this feature and another important properties, a methodology for order reduction from the original order to first order for this class of systems was proposed, by minimizing the integral of square error of impulse response and by enforcing the same DC gain for approximated and original systems. The obtained results show the effectiveness of methodology for theoretical and real word cases.

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