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COMPUTATION OF OPTIMAL CONTROL OF LINEAR SYSTEMS USING HAAR WAVELETS

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Abstract. *A new method is presented for computation of optimal control for linear systems using Haar wavelets. The method is based on a novel operational matrix derived from integration of Haar wavelets. The optimal control problem is converted to a two-point-boundary-value problem, which is then solved using the Haar wavelet transformation. The proposed method is then extended to the numerical solution for optimization of singular systems. Accuracy of the solution can be improved by increasing the resolution of wavelet expansion, i.e., by increasing the order of transformation. Compared with known methods in the literature, the proposed method does not require explicit computation of wavelet coefficients, which makes it computationally more efficient and requires less computer memory. Simulation results are presented to illustrate the method.*
Keywords: Optimal control, Two-point-boundary-value problem, Haar wavelet, Singular systems, Numerical solution

1. **Introduction.** It is well known that the optimum control that minimizes a cost function subject to a dynamic constraint can be obtained using the Pontryagin's minimum principle or Bellman's principle of optimality. Although these methods are conceptually elegant, computation of the optimal control law remains a very difficult problem, as it involves numerical solution of a two-point-boundary-value problem, the Riccati equation, or the Hamilton-Jacobi-Bellman equation. There are various methods available in the literature for computation of optimal control that can be found in standard textbooks. Numerical solution of the Hamilton-Jacobi-Bellman equation is difficult because of its computational overhead, in terms of memory and computation time, as it involves the solution of a partial differential equation. The gradient method is one of the best known methods of computation of optimal control problems; however, it is difficult to adopt when there are equality or inequality constraints. The control parametrization method [1] is more versatile as it can easily include various types of constraints. Nevertheless, computation of optimal control still remains an active research area. This paper presents a novel wavelet approach to a numerical solution of optimal control for linear systems, and its extension for singular systems.

The wavelet transform [2, 3] is a very powerful tool that has been applied in various engineering applications. It provides information in the time domain as well as the frequency domain, which is often not possible in conventional signal processing methods.

Due to this reason, wavelets have been successfully used in various signal processing applications, such as voice recognition, filtering, image processing, pattern recognition and medical diagnosis. Numerous studies have also been presented in the last two decades on wavelet applications in analysis and design of control systems, for example, in adaptive control [4, 5], system identification [6, 7], fault diagnosis [8, 9, 10], and optimal control [11, 12, 13, 14, 15, 16] of dynamic systems. Artificial neural networks are better known for their ability to map unknown nonlinear functions in which the neural activation functions play a significant role. Because of their ability to discriminate signals locally, wavelets are ideal for neural activation function as shown in adaptive wavelet neural networks [17] for robot control. There is an extensive body of literature on wavelets and their applications; references included in this paper represent only a small segment research articles available in the open literature, and by no means, are meant to be exhaustive.

Optimization of singular linear time invariant systems has been considered by various researchers, for example [18, 19, 20, 21, 22, 23, 24, 25]. There are several well-known numerical methods for solution of singular optimal control problems including the control parametrization method [1, 26], and genetic programming [32], and orthogonal transformation using the Chebyshev polynomial [27]. However, the presence of the singular matrix in the state equation and the adjoint equation causes various numerical difficulties.

This paper concerns the analysis of linear time invariant systems and their numerical optimization using wavelets. Fundamental to the optimization of dynamic systems is the solution of differential equations. Some of the earliest works on wavelet applications on system analysis were done by Chen and Hsiao in a series of papers, such as [11, 12, 28, 29], and by several other researchers including [15, 30, 31]. This paper presents a new operational matrix based on integration of Haar wavelets, which is then used for computation of optimal control of linear time invariant systems. The proposed method of solving linear systems has similarities with that of references [11, 28], but has the advantage of simplicity and reduced computational overhead. To the best of our knowledge, all known methods of wavelet analysis require explicit computation of wavelet coefficients, which is where the proposed method is fundamentally different. Using the new Haar operational matrix, the solution of the differential equation is computed without explicit computation of wavelet coefficients, which reduces computer memory utilization, and minimizes computation time. We also extend the proposed method to the solution of singular optimal control problems which was found to be numerically stable. Research in this direction can also be found in many references, including [14, 35, 36].

The rest of the paper is organized as follows. Section 2 presents a brief introduction to wavelets, and derivation of the operational matrix based on Haar wavelet transformation. In Section 3, we present the basics of solution of two-point-boundary-value problems arising from optimal control problem of linear systems which is followed by optimization of linear singular systems in Section 4. The paper is concluded with some remarks in Section 5.

2. **Wavelet Fundamentals.**

A wavelet is a small wave defined on a compact support, and the simplest among all wavelets is the Haar wavelet. The function $h_0(t)$ in Figure 1 is called the *Scaling function*, and $h_1(t)$ is called the *Wavelet function*, or mother wavelet. Additional functions are generated by translation and dilation of the mother wavelet. In general, we have

$$h_n(t) = h_1(2^j t - k), \quad n = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j$$

Figure 1 shows the first eight Haar wavelets. In this paper, we shall use unnormalized Haar wavelets although one may use normalized wavelet functions if appropriate.

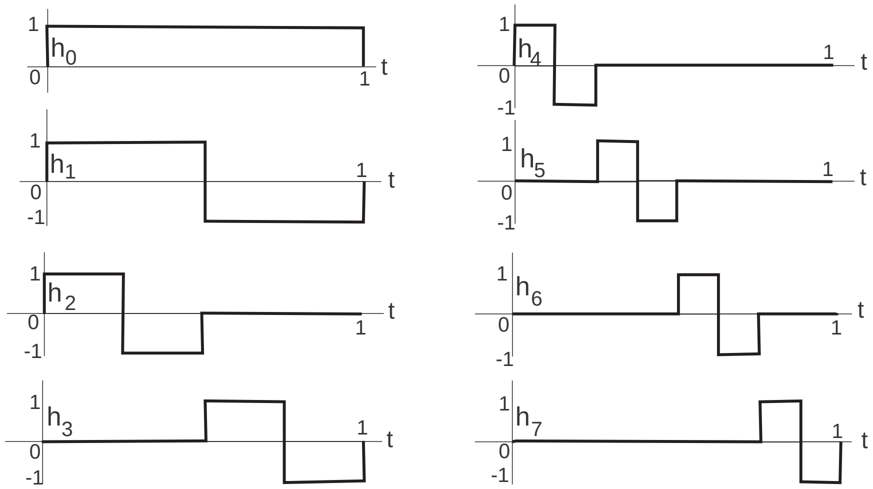

FIGURE 1. Haar wavelets

Haar wavelets form the simplest orthogonal sequence with compact support, and form a basis for $L_2(0,1)$. Without any loss of generality, the time domain is normalized to $[0,1]$. Therefore, given a square integrable function $\varphi \in L_2(0,1)$, we have the wavelet expansion

$$\varphi(t) = \sum_{i=0}^{\infty} c_i h_i(t) \tag{1}$$

where the coefficients are computed as

$$c_j = \int_0^1 \varphi(t) h_j(t) dt \Big/ \int_0^1 h_j^2(t) dt \tag{2}$$

For piecewise smooth functions, the infinite series (1) truncates to a finite series.

Notations:

In general, we shall use $y(t)$ to denote the time varying function and $\hat{y}$ to denote its discrete time representation with values at certain collocation points.

For the sake of simplicity, we consider wavelet expansion using only the first four Haar wavelets, however, numerical simulations will be carried out using higher order wavelets of arbitrary order m. Consider fourth-order wavelets $h_0(t)$, $h_1(t)$, $h_2(t)$ and $h_3(t)$ over the normalized time domain $t \in [0,1]$ as shown in Figure 1. We use the symbols $\hat{h}_0$, $\hat{h}_1$, $\hat{h}_2$ and $\hat{h}_3$ for their discrete time vector representation. A wavelet of order m has discrete time representation at m collocation points that are separated by $\Delta t = \frac{1}{m}$ with the first collocation point at $t = \frac{1}{2m}$. Thus for the fourth order Haar wavelets, the collocation points are: 0.125, 0.375, 0.625, 0.875. Using these notations, the first four Haar functions have the discrete time representation

$$\begin{aligned}
\hat{h}_0 &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\
\hat{h}_1 &= \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \\
\hat{h}_2 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \\
\hat{h}_3 &= \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}
\end{aligned} \tag{3}$$

2.1. **Haar operational matrix.** Since solution of differential equation involves integration of certain functions, which are represented using wavelets, it is necessary to integrate

Haar wavelet functions. For the fourth order wavelets as shown in Figure 1, define the Haar transformation

$$H_4(t) = \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} \tag{4}$$

and its discrete time representation

$$\widehat{H}_4 = \begin{bmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{5}$$

By direct integration of the Haar wavelet functions, we also construct the integral of the Haar transformation (4). For fourth order Haar transformation, we denote the integral of $H_4(t)$ as

$$N_4(t) = \int_0^t H_4(\tau) d\tau \tag{6}$$

where the integral on the right hand side can be easily computed. Then the corresponding discrete time representation of $N(t)$ at collocation points is obtained as

$$\widehat{N}_4 = \frac{1}{8} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{7}$$

Since the Haar wavelets form a basis for L_2, we can also express

$$\int_0^t h_i(\tau) d\tau = \sum_j p_{ij} h_j(t)$$

where p_{ij} are orthogonal expansion coefficients which can be computed using (2). It can be verified that

$$\widehat{N}_4 = P_4 \widehat{H}_4 \tag{8}$$

where the *Haar operational matrix* P_4 as derived in [28] is given by

$$P_4 = \frac{1}{8} \begin{bmatrix} 4 & -2 & -1 & -1 \\ 2 & 0 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix} \tag{9}$$

Higher order Haar operational matrices and their characteristics have been discussed in details in [28] which also presents a recursive method for computation of the operational matrix.

This paper develops a new *Haar Operational Matrix* Q, which provides numerical simplicity in solving linear differential equations and their optimization compared with that in [28]. For the fourth order Haar transformation matrix, we define

$$\widehat{N}_4 = \widehat{H}_4 Q_4 \tag{10}$$

where the new Haar operational matrix Q_4 is given by

$$Q_4 = \frac{1}{4}\begin{bmatrix} \frac{1}{2} & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{11}$$

While the matrix P_4 is generated [28] using a recursive algorithm, the operational matrix Q_4 has a very simple structure, and can be easily constructed for Haar transformation of any arbitrary order.

We also introduce integration of the Haar transformation from t to 1

$$M_4(t) = \int_t^1 H_4(\tau)d\tau = \int_0^1 H_4(\tau)d\tau - \int_0^t H_4(\tau)d\tau \tag{12}$$

Carrying out the integration of various Haar wavelets directly, and expressing the results in discrete form, we obtain the corresponding discrete time representation

$$\widehat{M}_4 = D_4 - P_4\widehat{H}_4 = \widehat{H}_4 Q_4' \tag{13}$$

with

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

The operational matrix Q forms the basis of analysis in the rest of the paper. The advantages of this operational matrix over that proposed in [28] are its simplicity and ease of numerical implementation. In general, for a Haar transformation of order m, the corresponding operational matrix is of dimension $m \times m$, and has the simple form

$$Q_m = \frac{1}{m}\begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 \\ 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 \\ 0 & 0 & \frac{1}{2} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix} \tag{15}$$

This operational matrix Q_m will be used in the sequel in the next section for solving linear systems and for their optimization. For notational simplicity, in the subsequent analysis, we shall drop the subscript m from all transformation matrices, however, in numerical simulations, the order of Haar transformation m will be increased as appropriate for better numerical accuracy.

3. **Optimal Control Computation via Wavelets.**

Next we consider minimization of the cost

$$J(u) = \frac{1}{2}\langle x(T), Sx(T)\rangle + \frac{1}{2}\int_0^T \{\langle x, Wx\rangle + \langle u, Ru\rangle\}dt \tag{16}$$

where R is positive definite, and S and W are positive semidefinite. The system is assumed to be linear and time invariant:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_0 \end{aligned} \tag{17}$$

where $x \in R^n$ is the state vector, and $u \in R^s$ is the control input. Without any loss of generality, we shall assume that $T = 1$. It is known that the optimal solution satisfies the two-point-boundary-value problem

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}B'\psi, \quad x(0) = x_0 \\ \dot{\psi} &= -Wx - A'\psi, \qquad \psi(T) = Sx(T) \end{aligned} \tag{18}$$

For notational simplicity, we denote the above TPBVP as

$$\begin{aligned} \dot{x} &= C_{11}x + C_{12}\psi, \quad x(0) = x_0 \\ \dot{\psi} &= C_{21}x + C_{22}\psi, \quad \psi(T) = Sx(T) \end{aligned} \tag{19}$$

Since the derivative $\dot{x}$ is an L_2 function, we can express

$$\dot{x}(t) = \sum_{i=0}^{m} f_i h_i(t) = FH(t) \tag{20}$$

where f_i are wavelet coefficients and h_i are the Haar wavelets. In matrix form, F is the $n \times m$ matrix of wavelet coefficients, and H is the Haar transformation ($m \times 1$) matrix. Then we have

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau = x_0 + F\int_0^t H(\tau)d\tau \tag{21}$$

and the corresponding discrete time representation

$$\widehat{x} = X_0 + F\widehat{H}Q \tag{22}$$

where $\widehat{x}$ denotes the $n \times m$ matrix for values of $x(t)$ at various collocation points, and $X_0 = \begin{bmatrix} x_0 & x_0 & \cdots x_0 \end{bmatrix}_{n\times m}$. Similarly, for the adjoint variable, we have

$$\begin{aligned} \psi(t) &= \psi(T) - \int_t^T \dot{\psi}(\tau)d\tau \\ &= Sx(T) - \int_t^T \dot{\psi}(\tau)d\tau \\ &= S\left(x_0 + \int_0^T \dot{x}(\tau)d\tau\right) - \int_t^T \dot{\psi}(\tau)d\tau \end{aligned} \tag{23}$$

and the corresponding discrete time equation

$$\widehat{\psi} = S(X_0 + FD) - G\widehat{H}Q' \tag{24}$$

where $\widehat{\psi}$ denotes the $n \times m$ matrix for values of $\psi(t)$ at various collocation points. For notational simplicity, denote

$$\begin{aligned} F\widehat{H} &\simeq Y \\ G\widehat{H} &\simeq Z \end{aligned} \tag{25}$$

Substituting the wavelet expansion (22) and (24) in (19), the vectorized matrix equation for $\widehat{\dot{x}}$ and $\widehat{\dot{\psi}}$ is obtained as

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Z) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(C_{11}x_0) + \mathrm{vec}(C_{12}Sx_0) \\ \mathrm{vec}(C_{21}x_0) + \mathrm{vec}(C_{22}Sx_0) \end{bmatrix} \tag{26}$$

where

$$\begin{aligned} V_{11} &= I - Q' \otimes C_{11} - (\widehat{H}^{-1}D)' \otimes C_{12}S \\ V_{12} &= Q \otimes C_{12} \\ V_{21} &= -Q' \otimes C_{21} - (\widehat{H}^{-1}D)' \otimes C_{22}S \\ V_{22} &= I + Q \otimes C_{22} \end{aligned}$$

The matrix D in the above equation is defined in (14), except that it is of dimension $m \times m$. The solution of the two-point-boundary-value problem is then constructed using (25) in (22) and (24).

It is worthwhile to note that unlike conventional methods, there is no need to compute the wavelet coefficient matrix F explicitly. The solution of the TPBVP at collocation points is constructed directly from (22) and (24) as $\widehat{x} = X_0 + YQ$ and $\widehat{\psi} = S(X_0 + Y\widehat{H}^{-1}D) - ZQ'$ which significantly reduces computer memory and computation time.

Example 3.1. *In order to facilitate evaluation of accuracy of wavelet solution, we use an example with known analytical solution, see for example* [34]*. The system model is given by*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{27}$$

The cost function is taken as

$$J(u) = \frac{1}{2}\langle x(T), Sx(T)\rangle + \frac{1}{2}\int_0^T \{\langle x, Wx\rangle + \langle u, Ru\rangle\}dt$$

with

$$W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1$$

The TPBVP developed above was solved using the wavelet method discussed above. The optimal control u was computed using $u(t) = -R^{-1}B'\psi(t)$. The results for wavelet solution with $m = 32$ are shown in Figures 2 and 3 along with its analytical solution of the two-point-boundary-value problem:

$$x_1(t) = \frac{3}{8}t^2 - \frac{3}{24}t^3 - 1, \quad x_2(t) = \frac{3}{4}t - \frac{3}{8}t^2, \quad u(t) = \frac{3}{4}(1 - t) \tag{28}$$

As seen from these figures, the analytical solution and the wavelet solution are practically indistinguishable. Nevertheless, we compute the L_2 error between the analytical solution and the numerical solution for various values of m as shown in the following table.

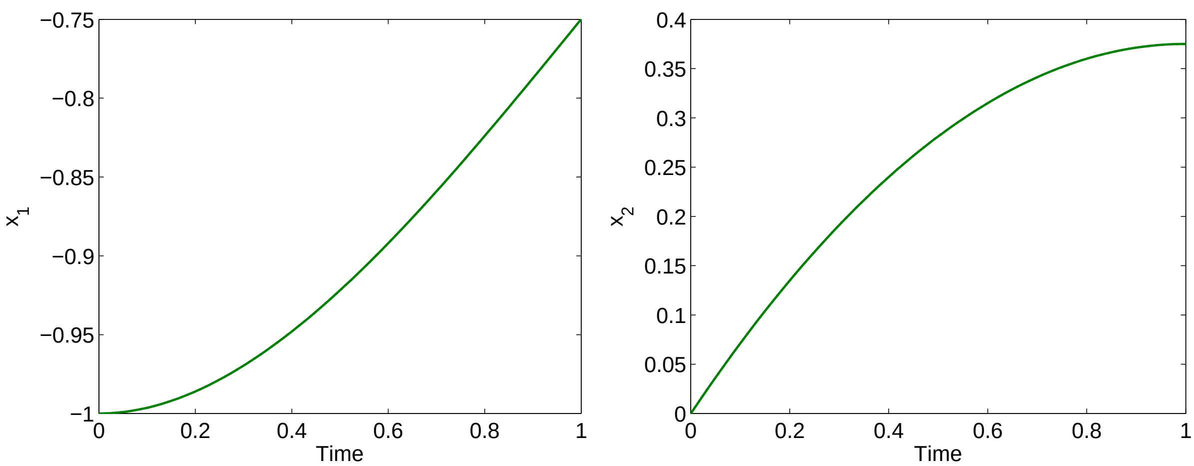

FIGURE 2. Optimal solution, x_1 and x_2

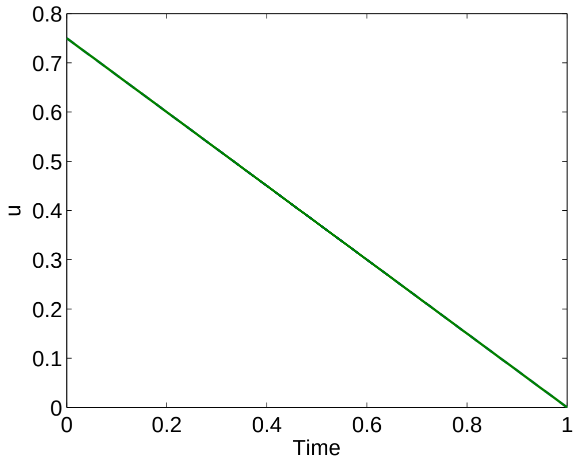

FIGURE 3. Optimal control, u

Error Analysis	
Wavelet order m	*Error*
8	0.2093×10^{-5}
16	0.1284×10^{-6}
32	0.7943×10^{-8}
64	0.4939×10^{-9}

The proposed method can also be used for solving Riccati equation. One of the methods of solution [33] of the Riccati equation involves transformation of the Riccati equation to a final value problem. The resulting set of differential equations can then be easily solved backward in time using the proposed wavelet method. We skip the details for brevity.

4. **Optimal Solution of Singular Systems.** Singular differential equations arise naturally as dynamic models of many physical systems. In what follows, we use the Haar wavelet transformation for solving two-point-boundary-value problems arising from optimal control of singular linear time invariant systems.

We consider the singular dynamic system

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ Ex(0) &= Ex_0 \end{aligned} \tag{29}$$

where $x(t) \in R^n$ is the state vector, and $u(t) \in R^m$ is the control input. The matrices E, A and B are assumed to be time invariant, and the matrix E is singular with $rank(E) = r < n$. Without any loss of generality, we can assume that the matrix E has the form

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

where I_r is the identity matrix. We also assume that the system (29) is *regular*, i.e.,

$$|sE - A| \neq 0 \tag{31}$$

so that there exists a unique solution $x(t)$ of the system (29) for all admissible initial conditions x_0. Controllability of the singular system has been investigated in [21].

We now consider the optimization problem of minimizing the cost function

$$J(u) = \frac{1}{2}\langle Ex(T), SEx(T)\rangle + \frac{1}{2}\int_0^T \{\langle x, Wx\rangle + \langle u, Ru\rangle\}dt \tag{32}$$

We assume that the matrices W and S are positive semidefinite, and R is positive definite. Existence of solution of this optimization problem has been proved in [21]. It can be shown that the optimal solution of the singular optimal control problem is equivalent to the two-point-boundary-value problem:

$$\begin{array}{ll} E\dot{x} = Ax - BR^{-1}B'\psi, & Ex(0) = Ex_0 \\ E'\dot{\psi} = -Wx - A'\psi, & E'\psi(T) = E'SEx(T) \end{array} \tag{33}$$

We assume that the matrix S has the same block diagonal structure as E, i.e., $S = \text{diag}(S_r, 0)$ where S_r is of dimension $r \times r$. The matrix W is also assumed to have the same block diagonal structure. Since E is a singular matrix, it is clear from (33) that the boundary condition $\psi(T)$ is undefined for some of its components. We partition ψ and x as $\psi = [\psi_1 \quad \psi_2]'$ and $x = [x_1 \quad x_2]'$ with ψ_1, $x_1 \in R^r$ and ψ_2, $x_2 \in R^{n-r}$. Then it follows from (33) that

$$\psi_1(T) = S_r x_1(T) \tag{34}$$

To find $\psi_2(T)$, we use the adjoint differential equation from (33) to obtain

$$0 = -A'_{12}\psi_1(T) - A'_{22}\psi_2(T) \tag{35}$$

where $A_{12} \in R^{r\times(n-r)}$ and $A_{22} \in R^{(n-r)\times(n-r)}$ are partitions of the matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Combining (34) and (35), we obtain

$$\psi(T) = S_1 Ex(T), \quad \text{where} \quad S_1 = \begin{bmatrix} S_r & 0 \\ -[A'_{22}]^{-1}A'_{12}S_r & 0 \end{bmatrix} \tag{36}$$

where we have assumed that A_{22} is invertible. Note that nonsingularity of A_{22} is a required condition [18, 21] so that the pencil $(sE - A)$ is regular, i.e., $|sE - A| \neq 0$, and that the system does not have any impulsive modes.

For notational simplicity, we rewrite the TPBVP as

$$\begin{array}{ll} E\dot{x} = C_{11}x + C_{12}\psi, & Ex(0) = Ex_0 \\ E'\dot{\psi} = C_{21}x + C_{22}\psi, & \psi(T) = S_1Ex(T) \end{array} \tag{37}$$

Since the derivatives $\dot{x}(t)$ and $\dot{\psi}(t)$ are L_2 functions, following the same approach as in Section 3, let

$$\dot{x}(t) = FH(t) \tag{38}$$

$$\dot{\psi}(t) = GH(t) \tag{39}$$

and integrating the above, we obtain the discrete time representation of x and ψ as

$$\begin{array}{ll} \widehat{x} = X_0 + F\widehat{H}Q = X_0 + YQ, & \text{with } Y \simeq F\widehat{H} \\ \widehat{\psi} = \Psi_T - G\widehat{H}Q' = \Psi_T - ZQ', & \text{with } Z \simeq G\widehat{H} \end{array} \tag{40}$$

where $\widehat{x}$ and $\widehat{\psi}$ denote the $n \times m$ matrix for values of $x(t)$ and $\psi(t)$, respectively at various collocation points, and $X_0 = \begin{bmatrix} x_0 & x_0 & \cdots x_0 \end{bmatrix}_{n\times m}$, and $\Psi_T = [\ \psi(T) \quad \psi(T) \quad \cdots \quad \psi(T)\ ]_{n\times m}$. Furthermore, using the boundary condition (37), we verify that

$$\begin{aligned} \psi(T) &= S_1Ex(T) \\ &= S_1E\left(x(0) + \int_0^T \dot{x}(\tau)d\tau\right) \end{aligned} \tag{41}$$

and the corresponding discrete time equation

$$\widehat{\psi}_T = S_1 E(X_0 + FD) = S_1 E(X_0 + Y\widehat{H}^{-1}D) \tag{42}$$

Substituting the above equations and the wavelet expansion (40) in (37), the vectorized matrix equations for $\widehat{x}$ and $\widehat{\psi}$ in discrete time are obtained as

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Z) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(C_{11}X_0) + \mathrm{vec}(C_{12}S_1 E X_0) \\ \mathrm{vec}(C_{21}X_0) + \mathrm{vec}(C_{22}S_1 E X_0) \end{bmatrix} \tag{43}$$

where

$$\begin{aligned} V_{11} &= I \otimes E - Q' \otimes C_{11} - (\widehat{H}^{-1}D)' \otimes C_{12}S_1 E \\ V_{12} &= Q \otimes C_{12} \\ V_{21} &= -Q' \otimes C_{21} - (\widehat{H}^{-1}D)' \otimes C_{22}S_1 E \\ V_{22} &= I \otimes E' + Q \otimes C_{22} \end{aligned}$$

The solution of the TPBVP is then constructed using (40). Note that the solution is constructed without computing the wavelet coefficients explicitly, thereby reducing computation overhead.

Example 4.1. *As an example, consider the system*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -25 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

and the cost function

$$J(u) = \frac{1}{2}\langle Ex(T), SEx(T)\rangle + \frac{1}{2}\int_0^T \{\langle x, Wx\rangle + \langle u, Ru\rangle\} dt$$

where $T = 1$ and

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = 1$$

Note that this simple example that can be easily modified to a standard optimization problem, and solved using various numerical methods. Thus the above example facilitates a comparison of results obtained by the proposed method and the standard optimization method.

The TPBVP (43) was solved using the wavelet method discussed above, and the results are shown in Figures 4 and 5; we skip the plot of x_3 since $x_1 = x_3$. For the wavelet solution we used $m = 32$, i.e., 32 collocation points for describing the wavelets. This also means that the Haar transformation matrix H is of the order 32×32, and the corresponding operational matrix Q is of the same order. The optimal control u was computed using $u(t) = -R^{-1}B'\psi(t)$. Our results coincide with those obtained by other methods; we omit the details for brevity.

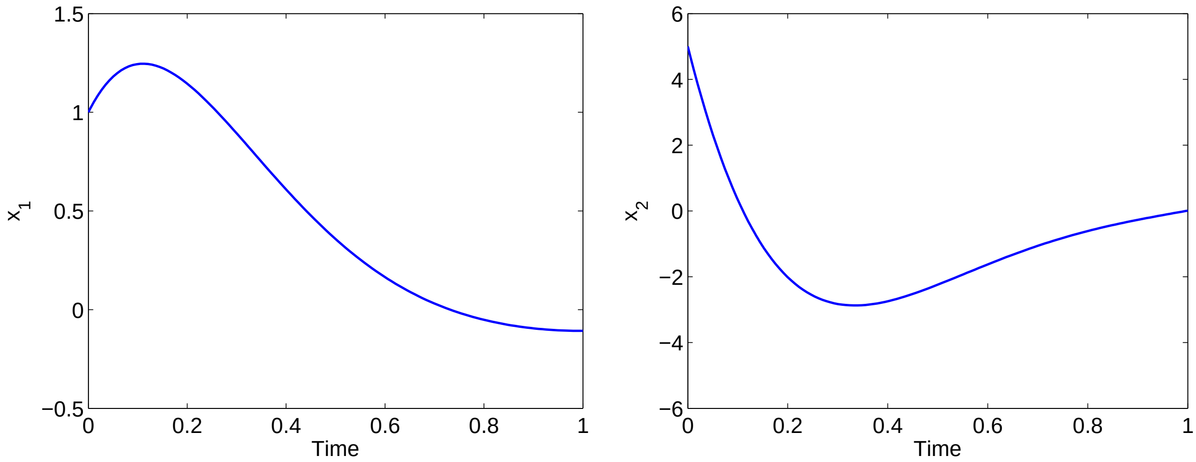

FIGURE 4. Optimal solution, x_1 and x_2

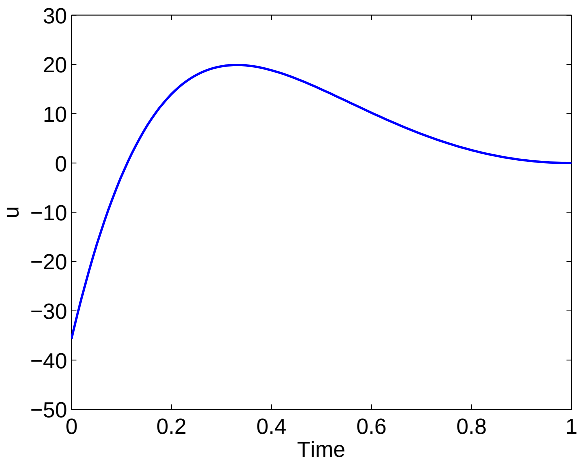

FIGURE 5. Optimal control, u

5. **Conclusions.** This paper presents a novel method for numerical solution of optimal control problems of linear differential equations using Haar wavelets. Compared with other methods in the open literature, there is no need to explicitly compute the wavelet coefficients, which reduces the need for computer memory and minimizes computation time. Additionally, note that other methods require recursive computation of the Haar transformation matrix, whereas in the proposed method, the transformation matrix has a very simple structure which is easily obtained for any order. Our simulations show that results obtained using the proposed wavelet method are very accurate with moderate resolution of wavelet functions. Accuracy of the solution depends on the number of collocation points of the wavelet representation, and improves with an increase of the order of transformation. The proposed method can also be extended for solution of time varying systems. One of the special features of the proposed method is that it is suitable for a parallel computation environment. This is evident as it computes the solution of the problem at all collocation points simultaneously, rather than sequentially, as in conventional numerical methods of computation of optimal control.

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