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IDENTIFICATION OF OPERATORS IN SYSTEMS GOVERNED BY SECOND ORDER EVOLUTION INCLUSIONS WITH APPLICATIONS TO HEMIVARIATIONAL INEQUALITIES

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Abstract. *We consider the identification problem of three operators having different properties for the systems governed by nonlinear second order evolution inclusions with the Volterra integral term. For the abstract identification problem, we show the existence of optimal solutions. We provide applications to evolution hemivariational inequalities and to viscoelastic frictional contact problem of mechanics.*

Keywords: Identification, Evolution inclusion, Inverse problem, Hemivariational inequality, Pseudomonotone, Multifunction, Volterra memory operator

1. **Introduction.** In this paper, we study the problem of estimation of parameters in an abstract evolution inclusion of second order. We consider a nonlinear inclusion with the Volterra memory operator. Such inclusion serves as a mathematical model for several important problems arising in mechanics, physics and engineering science. For this reason the mathematical literature dedicated to identification problems is extensive, see, e.g., [1, 3, 9, 14, 15, 16, 17, 18, 19, 28]. The direct problem under consideration is formulated as the following Cauchy problem for evolution inclusion in the framework of evolution triple of spaces

$$\begin{cases} u''(t) + A(p,t,u'(t)) + B(p,t,u(t)) + \displaystyle\int_0^t C(p,t-s)u(s)ds + \\ \qquad\qquad\qquad +F(t,u(t),u'(t)) \ni f(t) \quad \text{a.e.} \quad t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \tag{$*$}$$

where A, $B\colon (0,T)\times V \to V^*$ are nonlinear operators, $C(t)$ is a bounded linear operator from V to its dual V^*, for $t \in (0,T)$, $F\colon (0,T)\times V\times V \to 2^{Z^*}$ stands for a multivalued mapping, $f \in L^2(0,T;V^*)$, $u_0 \in V$, $v_0 \in H$, V and Z are reflexive Banach spaces with $V \subset Z$ compactly, H is a Hilbert space such that $Z \subset H$ and $0 < T < \infty$. The operators A, B and C depend on some unknown (i.e., to be estimated) parameters p with values in an admissible family P of parameters.

The aim of this paper is to prove a new existence result for the identification problem for $(*)$ and to apply this result in the analysis of integrodifferential hemivariational inequality and in the study of parameters in a viscoelastic frictional contact problem. The trait of novelty of our paper arises in the special structure of the abstract problem $(*)$ which is governed by an operator depending on the history of the solution and which contains a special form of the multivalued term. The direct problem $(*)$ without the Volterra memory term and time independent operator B has been studied in [6] with $F\colon (0,T)\times H\times H \to$

2^H, [24] in the case B is linear, continuous, symmetric and coercive operator, and in [20] in the case B is linear, continuous, symmetric and monotone. None of the results on nonlinear evolution inclusions in [6, 11, 12, 20, 24] can be applied in the study of hemivariational inequalities because of their restrictive assumption on the multivalued term which was supposed to have values in H. For the hemivariational inequalities and the contact problems under consideration, the multivalued term has values in the space dual to Z which is larger than H.

The identification problem for the model $(*)$ is a new one and has not been considered in the literature. This problem is studied in the first part of the paper and it consists in finding parameters which appear in the operators A, B and C which give the best fit of the solutions to $(*)$ to the observation data. The problem is formulated as an optimal control one. This is a widely used approach to the identification problems which covers the estimation of the unknown parameters appearing in the system by minimizing a quadratic cost functional of the difference between observed value and desired value, the so-called output least-square identification problem. The well-posedness of the identification problem for systems governed by $(*)$ is established by using the direct method of the calculus of variations. To this end, we obtain a new result on the continuous dependence of the solution to $(*)$ on the parameters. It is assumed that the parameter-dependent operators A, B and C satisfy suitable continuity hypotheses uniformly in $p \in P$.

In the second part of the paper we present applications of our result to the dynamic hemivariational inequalities describing the frictional contact problems for viscoelastic materials with long memory. We mention that the notion of hemivariational inequality was introduced and investigated in the early 1980s by Panagiotopoulos [26, 27]. These inequalities are a natural generalization of variational inequality and they are derived from nonsmooth and nonconvex superpotentials by using the generalized gradient of Clarke, cf. [4]. In the mechanical problem under consideration the operators A, B and C correspond to the viscosity, elasticity and relaxation operators, respectively. The integrodifferential hemivariational inequality is derived from the evolution inclusion $(*)$ where the multivalued term is of the form of the Clarke subdifferential of a locally Lipschitz superpotential. By means of hemivariational inequality, many problems in nonsmooth contact mechanics involving multivalued and nonmonotone constitutive laws and boundary conditions can be treated mathematically. The real-world applications of hemivariational inequalities include models of tectonic plate movement, construction and exploitation of machines, metal forming, artificial limbs and joints, teeth implants, bone remodeling models, semipermeable membranes, ultrasonic transducers, etc. that can ultimately be used for the improvement of industrial applications of economic benefits.

The optimization, control and identification of systems described by evolution equations on Banach spaces have been studied in [1, 2]. The inverse problems for damped second order evolution systems can be found in [19, 28] while the applications to smart materials technology and control are investigated in [3]. On the other hand, the theory of hemivariational inequalities and their applications to mechanical problems are extensively studied in recent years, cf. [11, 12, 13, 16, 20, 21, 22, 23, 24, 25]. The identification and control problems for various classes of hemivariational inequalities have been considered in [14, 15, 17, 18].

The paper is organized as follows. The prelimary material is recalled in Section 2 and a result on the unique solvability of $(*)$ is given in Section 3. Section 4 is devoted to the existence of solutions to the identification problem for the evolution inclusion. The applications are given in Sections 5 and 6 where we provide results for hemivariational inequality and the frictional contact problem for viscoelastic materials with memory.

2. **Preliminaries.** In this section we recall the basic notation and definitions needed in the sequel.

Let V and Z be separable and reflexive Banach spaces with their topological duals V^* and Z^*, respectively. Let H denote a separable Hilbert space and we identify H with its dual. We assume that $V \subset H \subset V^*$ and $Z \subset H \subset Z^*$ are evolution triples of spaces where all embedings are continuous, dense and compact (see, e.g., Chapter 23.4 of [29], Chapter 3.4 of [6]). We also suppose that V is compactly embedded in Z. Let $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{V^*}$ denote the norms in V, H and V^*, respectively, and let $\langle\cdot,\cdot\rangle$ be the duality pairing between V^* and V. Given a finite interval $(0,T)$, we also introduce the following spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{Z} = L^2(0,T;Z)$, $\hat{\mathcal{H}} = L^2(0,T;H)$, $\mathcal{Z}^* = L^2(0,T;Z^*)$, $\mathcal{V}^* = L^2(0,T;V^*)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$, where the time derivative is understood in the sense of vector-valued distributions. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle\langle z, w\rangle\rangle = \int_0^T \langle z(t), w(t)\rangle dt \quad \text{for} \quad z \in \mathcal{V}^*,\ w \in \mathcal{V}.$$

It is well known (cf. [6]) that the space $\mathcal{W}$ is embedded continuously in $C(0,T;H)$ (the space of continuous functions on $[0,T]$ with values in H), i.e., every element of $\mathcal{W}$, after a possible modification on a set of measure zero, has a unique continuous representative in $C(0,T;H)$. Moreover, since V is embedded compactly in H, then so does $\mathcal{W}$ into $L^2(0,T;H)$ (cf. [6]).

Let (Ω,Σ) be a measure space, X be a separable Banach space and let 2^X be a family of all subsets of X. A multifunction $F\colon \Omega \to 2^X$ is called graph measurable if $GrF = \{(\omega,x) \in \Omega \times Y \mid x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-field of X. It is said to be measurable if for each closed set $C \subset X$, the set $F^-(C) = \{\omega \in \Omega \mid F(\omega) \cap C \neq \emptyset\} \in \Sigma$ (cf. Section 4.2 of [5]).

Let X and Y be Banach spaces. A multifunction $F\colon X \to 2^Y \setminus \{\emptyset\}$ is lower semicontinuous (upper semicontinuous, respectively) if for $C \subset Y$ closed, the set $F^+(C) = \{x \in X \mid F(x) \subset C\}$ ($F^-(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\}$, respectively) is closed in X. F is bounded on bounded sets if $F(B) = \cup_{x\in B} F(x)$ is a bounded subset of Y for all bounded sets B in X.

Let Y be a reflexive Banach space and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Y and its dual. An operator $T\colon Y \to Y^*$ is called to be monotone if $\langle Ty - Tz, y - z\rangle \geq 0$ for all y, $z \in Y$. It is said to be pseudomonotone if $y_n \to y_0$ weakly in Y and $\limsup\langle Ty_n, y_n - y_0\rangle \leq 0$ imply that $\langle Ty_0, y_0 - y\rangle \leq \liminf\langle Ty_n, y_n - y\rangle$ for all $y \in Y$. It is said to be demicontinuous if $y_n \to y_0$ in Y implies $Ty_n \to Ty_0$ weakly in Y^*. It is hemicontinuous if the real-valued function $t \to \langle T(y + tv), w\rangle$ is continuous on $[0,1]$ for all y, v, $w \in Y$.

A multivalued mapping $T\colon Y \to 2^{Y^*}$ is said to be pseudomonotone, if it satisfies

(a) for every $y \in Y$, Ty is a nonempty, convex, and weakly compact set in Y^*;
(b) T is upper semicontinuous from every finite dimensional subspace of Y into Y^* endowed with the weak topology;
(c) if $y_n \to y$ weakly in Y, $y_n^* \in Ty_n$, and $\limsup\langle y_n^*, y_n - y\rangle \leq 0$, then for each $z \in Y$ there exists $y^*(z) \in Ty$ such that $\langle y^*(z), y - z\rangle \leq \liminf\langle y_n^*, y_n - z\rangle$.

Let $L\colon D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator. A mapping $T\colon Y \to 2^{Y^*}$ is said to be L-pseudomonotone (pseudomonotone with respect to $D(L)$) if and only if (a), (b) and the following hold:

(d) if $\{y_n\} \subset D(L)$ is such that $y_n \to y$ weakly in Y, $y \in D(L)$, $Ly_n \to Ly$ weakly in Y^*, $y_n^* \in Ty_n$, $y_n^* \to y^*$ weakly in Y^* and $\limsup\langle y_n^*, y_n - y\rangle \leq 0$, then $y^* \in Ty$ and $\langle y_n^*, y_n\rangle \to \langle y^*, y\rangle$.

Given a Banach space $(X, \|\cdot\|_X)$, the symbol w–X is always used to denote the space X endowed with the weak topology. By $\mathcal{L}(X, X^*)$ we denote the class of linear and bounded operators from X to X^*. If $U \subset X$, then we write $\|U\|_X = \sup\{\|x\|_X \mid x \in U\}$. Furthermore, we will use the following notation

$$\mathcal{P}_{f(c)}(X) = \{A \subseteq X \mid A \text{ is nonempty, closed, (convex)}\};$$
$$\mathcal{P}_{(w)k(c)}(X) = \{A \subseteq X \mid A \text{ is nonempty, (weakly) compact, (convex)}\}.$$

3. **Evolution Inclusion.** In this section we formulate the second order evolution inclusion which is a direct problem in the identification problem under consideration. We recall the existence and uniqueness result obtained recently in [21].

Problem $\mathcal{P}$: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + \displaystyle\int_0^t C(t-s)u(s)ds + \\ \qquad\qquad\qquad\qquad + F(t, u(t), u'(t)) \ni f(t) \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

Definition 3.1. *A function $u \in \mathcal{V}$ is a solution to Problem $\mathcal{P}$, if $u' \in \mathcal{W}$ and there exists $z \in \mathcal{Z}^*$ such that*

$$u''(t) + A(t, u'(t)) + B(t, u(t)) + \int_0^t C(t-s)u(s)ds + z(t) = f(t) \quad \text{a.e. } t \in (0,T)$$
$$z(t) \in F(t, u(t), u'(t)) \quad \text{a.e. } t \in (0,T)$$
$$u(0) = u_0, \quad u'(0) = v_0.$$

Remark 3.1. *We observe that the statement "$u \in \mathcal{V}$ is such that $u' \in \mathcal{W}$" is equivalent to "$u \in C(0,T;V)$ is such that $u' \in \mathcal{W}$".*

We need the following hypotheses on the data of Problem $\mathcal{P}$.

$\underline{H(A)}$: $A\colon (0,T) \times V \to V^*$ is such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for every $v \in V$;
(ii) $A(t, \cdot)$ is hemicontinuous for a.e. $t \in (0,T)$;
(iii) $A(t, \cdot)$ is strongly monotone for a.e. $t \in (0,T)$, i.e., there exists $m_1 > 0$ such that $\langle A(t,v) - A(t,u), v-u\rangle \geq m_1\|v-u\|^2$ for a.e. $t \in (0,T)$, all v, $u \in V$;
(iv) $\|A(t,v)\|_{V^*} \leq a_0(t) + a_1\|v\|$ for a.e. $t \in (0,T)$, all $v \in V$ with $a_0 \in L^2(0,T)$, $a_0 \geq 0$ and $a_1 > 0$;
(v) $\langle A(t,v), v\rangle \geq \alpha\|v\|^2$ for a.e. $t \in (0,T)$, all $v \in V$ with $\alpha > 0$.

$\underline{H(B)}$: $B\colon (0,T) \times V \to V^*$ is such that

(i) $B(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;
(ii) $B(t, \cdot)$ is Lipschitz continuous for a.e. $t \in (0,T)$, i.e., $\|B(t,u) - B(t,v)\|_{V^*} \leq L_B\|u - v\|$ for all u, $v \in V$, a.e. $t \in (0,T)$ with $L_B > 0$;
(iii) $\|B(t,v)\|_{V^*} \leq b_0(t) + b_1\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$ with $b_0 \in L^2(0,T)$ and b_0, $b_1 \geq 0$.

$\underline{H(C)}$: C is such that $C \in L^2(0,T;\mathcal{L}(V,V^*))$.

$\underline{H(F)}$: $F\colon (0,T) \times V \times V \to \mathcal{P}_{fc}(Z^*)$ is such that

(i) $F(\cdot, u, v)$ is measurable on $(0,T)$ for all u, $v \in V$;
(ii) $F(t, \cdot, \cdot)$ is upper semicontinuous from $V \times V$ into w–Z^* for a.e. $t \in (0,T)$, where $V \times V$ is endowed with $(Z \times Z)$-topology;

(iii) $\|F(t,u,v)\|_{Z^*} \le d_0(t) + d_1\|u\| + d_2\|v\|$ for all u, $v \in V$, a.e. $t \in (0,T)$ with $d_0 \in L^2(0,T)$ and d_0, d_1, $d_2 \ge 0$;

(iv) $\langle F(t,u_1,v_1) - F(t,u_2,v_2), v_1 - v_2\rangle_{Z^*\times Z} \ge -m_2\|v_1-v_2\|^2 - m_3\|v_1-v_2\|\|u_1-u_2\|$ for all u_i, $v_i \in V$, $i = 1$, 2, a.e. $t \in (0,T)$
with m_2, $m_3 \ge 0$.

$\underline{(H_0)}$: $\quad f \in \mathcal{V}^*$, $u_0 \in V$, $v_0 \in H$.

$\underline{(H_1)}$: $\quad \alpha > 2\sqrt{3}c_e(d_1T + d_2)$, where $c_e > 0$ is the embedding constant of V into Z, i.e., $\|\cdot\|_Z \le c_e\|\cdot\|$.

$\underline{(H_2)}$: $\quad m_1 > m_2 + \dfrac{1}{\sqrt{2}}m_3T$.

We shortly comment on the above hypotheses.

Remark 3.2. *i) The hypothesis $H(A)$(ii) and (iii) imply that $A(t,\cdot)$ is pseudomonotone for a.e. $t \in (0,T)$, cf. Proposition 27.6(a) of* [29] *and Remark 1.1.13 of* [6].
ii) If the condition $H(B)$(ii) holds and $B(\cdot,0) \in L^2(0,T;V^*)$, then $\|B(t,v)\|_{V^*} \le b(t) + L_B\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$, where $b(t) = \|B(t,0)\|_{V^*}$, $b \in L^2(0,T)$, $b \ge 0$. Moreover, it is clear that if $B \in L^\infty(0,T;\mathcal{L}(V,V^*))$, then $H(B)$(ii) holds.
iii) The conditions (H_1) and (H_2) provide a restriction on the length of time interval T unless $d_1 = m_3 = 0$. This means that under (H_1) and (H_2), the existence and uniqueness result of Theorem 3.1 below is local and holds for a sufficiently small time interval. On the other hand, if the data satisfy (H_1) and (H_2) with $d_1 = m_3 = 0$, then this result is global in time. For example, we observe that if the multifunction $F(t,u,\cdot)$ is monotone for all $u \in V$, a.e. $t \in (0,T)$, i.e., $\langle F(t,u,v_1) - F(t,u,v_2), v_1 - v_2\rangle_{Z^*\times Z} \ge 0$ for all u, $v_i \in V$, $i = 1$, 2, a.e. $t \in (0,T)$, then the hypothesis (H_2) clearly holds with $m_2 = m_3 = 0$ and every $m_1 > 0$.
iv) It follows from Lemma 5 of [21] *that under the hypothesis $H(F)$, the multifunction $G\colon W^{1,2}(0,T;V) \to 2^{\mathcal{Z}^*}$ defined by*

$$G(u) = \{z \in \mathcal{Z}^* \mid z(t) \in F(t,u(t),u'(t)) \text{ a.e. on } (0,T)\}$$

for $u \in W^{1,2}(0,T;V)$ is $\mathcal{P}_{wkc}(\mathcal{Z}^*)$-valued. Hence, the multifunction $t \mapsto F(t,u(t),u'(t))$ has a measurable $\mathcal{Z}^*$ selection and Definition 3.1 makes sense.

The following is the main result on Problem $\mathcal{P}$.

Theorem 3.1. *Under hypotheses $H(A)$, $H(B)$, $H(C)$, $H(F)$, (H_0), (H_1) and (H_2), Problem $\mathcal{P}$ admits a unique solution.*

We shortly comment on the proof of Theorem 3.1. In the first step we consider the evolution inclusion without the Volterra term and the operator B, i.e.,

$$\begin{cases} u''(t) + A(t,u'(t)) + F(t,u(t),u'(t)) \ni f(t) & \text{a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \tag{1}$$

We formulate it as follows

$$\begin{cases} z'(t) + A(t,z(t)) + F(t,Kz(t),z(t)) \ni f(t) & \text{a.e. } t \in (0,T) \\ z(0) = v_0, \end{cases} \tag{2}$$

where $(Kv)(t) = \int_0^t v(s)ds + u_0$. Then, z solves (2) if and only if $u = Kz$ is a solution to (1). Next, we rewrite (2) as an operator inclusion $(L + \mathcal{F})z \ni f$, where $Lz = z'$ denotes the generalized time derivative, $\mathcal{F} = \mathcal{A}_1 + F_1$ with $(\mathcal{A}_1 z)(t) = A(t,z(t) + v_0)$ and

$$F_1 z = \{z^* \in \mathcal{Z}^* \mid z^*(t) \in F(t,K(z(t)+v_0),z(t)+v_0) \text{ a.e. } t \in (0,T)\}.$$

We are able to prove that $\mathcal{F}$ is bounded, coercive and pseudomonotone with respect to the graph norm topology of the domain of L. By exploting the fact that L is closed, densely defined and maximal monotone operator, from Theorem 1.3.73 of [6], we obtain that $L + \mathcal{F}$ is surjective which implies that (1) is solvable. Subsequently, we show that the solution to (1) is unique. In the second step we consider the operator Λ defined by

$$(\Lambda\eta)(t) = B(t, u_\eta(t)) + \int_0^t C(t-s)u_\eta(s)ds,$$

where u_η is the unique solution to the following inclusion

$$\begin{cases} u''(t) + A(t, u'(t)) + F(t, u(t), u'(t)) \ni f(t) - \eta(t) \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \tag{3}$$

Applying the Banach Contraction Principle, we show that Λ has a unique fixed point η^*. The solution of (3) corresponding to η^* is the unique solution to Problem $\mathcal{P}$. For the detailed proof we refer to [10, 21].

4. **Identification Problem.** The goal of this section is to provide the main result of the paper on the existence of solutions to the identification.

The identification problem consists in finding parameters which give the best fit of the solutions to Problem $\mathcal{P}$ to the observation data. Let P denote the set of admissible parameters. For $p \in P$ we consider the following

Problem $\mathcal{P}_p$: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(p, t, u'(t)) + B(p, t, u(t)) + \displaystyle\int_0^t C(p, t-s)u(s)ds + \\ \qquad\qquad\qquad\qquad + F(t, u(t), u'(t)) \ni f(t) \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

Let $\mathcal{F}\colon P \to \mathbb{R}$ be the functional defined by

$$\mathcal{F}(p) = l(u(T), u'(T)) + \int_0^T L(t, u(t), u'(t))dt \quad \text{for } p \in P, \tag{4}$$

where $u = u(t) = u(t;p)$ denotes the solution of Problem $\mathcal{P}_p$ corresponding to $p \in P$.

The identification problem under consideration is formulated as an optimal control one. It consists in finding $p^* \in P$ that imparts a minimum to the functional $\mathcal{F}$ given by (4) subject to the dynamics $\mathcal{P}_p$:

$$\mathcal{F}(p^*) = \min_{p \in P} \mathcal{F}(p). \tag{5}$$

Our goal is to show that the identification problem (5) is solvable. The existence of solutions to problem (5) is obtained by applying the direct method of the calculus of variations. To this end, we establish a result on the continuous dependence, in suitable topologies, of solution to Problem $\mathcal{P}_p$ on the parameter.

We admit the following hypotheses.

$\underline{H(P)}$: P is a compact subset of a metric spaces of parameters $\widetilde{P}$.

$\underline{H(A)_1}$: The family of operators $\{A(p, \cdot, \cdot), p \in P\}$ satisfy $H(A)$ uniformly in $p \in P$ and the mapping $p \mapsto A(p, t, v)$ is continuous in the sense that

$$A(p_n, \cdot, w(\cdot)) \to A(p, \cdot, w(\cdot)) \quad \text{in } \mathcal{V}^* \quad \text{for all } w \in \mathcal{W}$$

whenever $p_n \to p$ in P.

$\underline{H(B)_1}$: The family of operators $\{B(p,\cdot,\cdot), p \in P\}$ satisfy $H(B)$ uniformly in $p \in P$ and the mapping $p \mapsto B(p,t,v)$ is continuous in the sense that

$$B(p_n,\cdot,v(\cdot)) \to B(p,\cdot,v(\cdot)) \ \text{ in } \mathcal{V}^* \ \text{ for all } v \in \mathcal{W}$$

whenever $p_n \to p$ in P.

$\underline{H(C)_1}$: The family of operators $\{C(p,\cdot), p \in P\}$ is such that $C(p,\cdot) \in L^2(0,T;\mathcal{L}(V,V^*))$ for all $p \in P$ and the mapping $p \mapsto C(p,t)$ is continuous in the sense that

$$C(p_n,\cdot) \to C(p,\cdot) \ \text{ in } L^2(0,T;\mathcal{L}(V,V^*))$$

whenever $p_n \to p$ in P.

$\underline{H(l)}$: $l \colon V \times H \to \mathbb{R}$ is sequentially lower semicontinuous on $V \times H$.

$\underline{H(L)}$: $L \colon (0,T) \times V \times H \to \mathbb{R} \cup \{+\infty\}$ is such that

(i) $L(\cdot,u,v)$ is measurable on $(0,T)$ for every $u \in V$, $v \in H$;
(ii) $L(t,u,v) > -\infty$ for a.e. $t \in (0,T)$ and all $u \in V$, $v \in H$;
(iii) $L(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $V \times H$ for a.e. $t \in (0,T)$.

Theorem 4.1. *Under hypotheses $H(A)_1$, $H(B)_1$, $H(C)_1$, $H(F)$, (H_0), (H_1), (H_2), $H(l)$ and $H(L)$, the functional $\mathcal{F} \colon P \to \mathbb{R}$ defined by (4) is sequentially lower semicontionuous on P.*

Proof: Let p_n, $p \in P$, $p_n \to p$ in P. Let $u_n = u(t;p_n)$, $u = u(t;p)$ denote the solutions of Problem $\mathcal{P}_p$ corresponding to the parameters p_n and p, respectively. From Theorem 3.1 we know that u_n and u are uniquely determined. Everywhere in the proof, we denote by c a positive generic constant which may depend on A, B, C, u and T but is independent of n, and whose value may change from place to place. We have u_n, $u \in \mathcal{V}$ with u_n', $u' \in \mathcal{W}$ and

$$u_n''(t) + A(p_n,t,u_n'(t)) + \eta_n(t) + z_n(t) = f(t) \qquad \text{a.e. } t \in (0,T), \tag{6}$$

$$u''(t) + A(p,t,u'(t)) + \eta(t) + z(t) = f(t) \qquad \text{a.e. } t \in (0,T), \tag{7}$$

where

$$\eta_n(t) = B(p_n,t,u_n(t)) + \int_0^t C(p_n,t-s)u_n(s)ds \ \ \text{a.e. } t \in (0,T),$$

$$\eta(t) = B(p,t,u(t)) + \int_0^t C(p,t-s)u(s)ds \qquad \text{a.e. } t \in (0,T)$$

and

$$z_n(t) \in F(t,u_n(t),u_n'(t)), \quad z(t) \in F(t,u(t),u'(t)) \quad \text{a.e. } t \in (0,T)$$

with $u_n(0) = u(0) = u_0$ and $u_n'(0) = u'(0) = v_0$. We will show that $\{u_n\}$ converges to u in the following sense

$$\lim_{\varepsilon \to 0} \left(\|u_n - u\|_{C(0,T;V)} + \|u_n' - u'\|_{C(0,T;H)} + \|u_n' - u'\|_{\mathcal{V}} \right) = 0. \tag{8}$$

From (6) and (7), we get

$$\int_0^t \langle u_n''(s) - u''(s), u_n'(s) - u'(s)\rangle ds + \int_0^t \langle A(p_n,s,u_n'(s)) - A(p,s,u'(s)), u_n'(s) - u'(s)\rangle ds$$
$$+ \int_0^t \langle \eta_n(s) - \eta(s), u_n'(s) - u'(s)\rangle ds + \int_0^t \langle z_n(s) - z(s), u_n'(s) - u'(s)\rangle_{Z^* \times Z} ds = 0$$

for all $t \in [0,T]$. Since u_n, $u \in W^{1,2}(0,T;V)$ and V is reflexive, by Theorem 3.4.11 and Remark 3.4.9 of [5], we know that u_n and u may be identified with absolutely continuous functions with values in V and

$$u_n(t) = u_n(0) + \int_0^t u_n'(s)ds, \quad u(t) = u(0) + \int_0^t u'(s)ds \text{ for all } t \in [0,T].$$

Hence,

$$\|u_n(t) - u(t)\| \leq \int_0^t \|u_n'(s) - u'(s)\| ds \tag{9}$$

and by the Jensen inequality, we obtain

$$\begin{aligned}
\int_0^t \|u_n(s) - u(s)\|^2 ds &\leq \int_0^t \left(\int_0^s \|u_n'(\tau) - u'(\tau)\| d\tau \right)^2 ds \\
&\leq \int_0^t s \left(\int_0^s \|u_n'(\tau) - u'(\tau)\|^2 d\tau \right) ds \\
&\leq \int_0^t s \|u_n' - u'\|^2_{L^2(0,t;V)} ds \leq \frac{T^2}{2} \|u_n' - u'\|^2_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in [0,T]$. Exploiting $H(F)$(iv) and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t \langle z_n(s) - z(s), u_n'(s) - u'(s) \rangle_{Z^* \times Z} ds \\
&\geq - m_2 \int_0^t \|u_n'(s) - u'(s)\|^2 ds - m_3 \int_0^t \|u_n'(s) - u'(s)\| \|u_n(s) - u(s)\| ds \\
&\geq - m_2 \|u_n' - u'\|^2_{L^2(0,t;V)} - m_3 \|u_n' - u'\|_{L^2(0,t;V)} \left(\int_0^t \|u_n(s) - u(s)\|^2 ds \right)^{1/2} \\
&\geq - m_2 \|u_n' - u'\|^2_{L^2(0,t;V)} - m_3 \|u_n' - u'\|_{L^2(0,t;V)} \frac{T}{\sqrt{2}} \|u_n' - u'\|_{L^2(0,t;V)} \\
&= - \left(m_2 + \frac{m_3 T}{\sqrt{2}} \right) \|u_n' - u'\|^2_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in [0,T]$. Hence, and from the integration by parts formula, we have

$$\begin{aligned}
&\frac{1}{2} |u_n'(t) - u'(t)|^2 + \int_0^t \langle A(p_n, s, u_n'(s)) - A(p_n, s, u'(s)), u_n'(s) - u'(s) \rangle ds \\
&\quad + \int_0^t \langle A(p_n, s, u'(s)) - A(p, s, u'(s)), u_n'(s) - u'(s) \rangle ds - \left(m_2 + \frac{m_3 T}{\sqrt{2}} \right) \|u_n' - u'\|^2_{L^2(0,t;V)} \\
&\leq - \int_0^t \langle \eta_n(s) - \eta(s), u_n'(s) - u'(s) \rangle ds \text{ for all } t \in [0,T].
\end{aligned}$$

Since $A(p,t,\cdot)$ is strongly monotone, uniformly in $p \in P$, for $t \in [0,T]$, we deduce

$$\begin{aligned}
&\frac{1}{2} |u_n'(t) - u'(t)|^2 + \left(m_1 - m_2 - \frac{m_3 T}{\sqrt{2}} \right) \|u_n' - u'\|^2_{L^2(0,t;V)} \\
&\leq \Big(\|A(p_n, \cdot, u'(\cdot)) - A(p, \cdot, u'(\cdot))\|_{L^2(0,t;V^*)} + \|\eta_n - \eta\|_{L^2(0,t;V^*)} \Big) \|u_n' - u'\|_{L^2(0,t;V)}
\end{aligned} \tag{10}$$

for all $t \in [0,T]$. On the other hand, using the fact that $B(p,t,\cdot)$ is uniformly in $p \in P$ Lipschitz continuous, for $t \in [0,T]$, we have

$$\begin{aligned}
&\|\eta_n(s) - \eta(s)\|_{V^*} \\
&\le \|B(p_n, s, u_n(s)) - B(p_n, s, u(s))\|_{V^*} + \|B(p_n, s, u(s)) - B(p, s, u(s))\|_{V^*} \\
&\quad + \left\| \int_0^s C(p_n, s-\tau)(u_n(\tau) - u(\tau))d\tau \right\|_{V^*} + \left\| \int_0^s (C(p_n, s-\tau) - C(p, s-\tau))u(\tau)d\tau \right\|_{V^*} \\
&\le L_B \|u_n(s) - u(s)\| + \|B(p_n, s, u(s)) - B(p, s, u(s))\|_{V^*} \\
&\quad + \|C(p_n,\cdot)\|_{L^2(0,t;\mathcal{L}(V,V^*))} \|u_n - u\|_{L^2(0,t;V)} + \|C(p_n,\cdot) - C(p,\cdot)\|_{L^2(0,t;\mathcal{L}(V,V^*))} \|u\|_{L^2(0,t;V)}
\end{aligned}$$

for a.e. $s \in (0,t)$. Hence, we obtain

$$\begin{aligned}
\|\eta_n - \eta\|^2_{L^2(0,t;V^*)} &\le c\Big(\|u_n - u\|^2_{L^2(0,t;V)} + \|B(p_n,\cdot,u(\cdot)) - B(p,\cdot,u(\cdot))\|^2_{L^2(0,t;V^*)} \\
&\qquad + \|C(p_n,\cdot)\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \|u_n - u\|^2_{L^2(0,t;V)} \\
&\qquad + \|C(p_n,\cdot) - C(p,\cdot)\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \|u\|^2_{L^2(0,t;V)} \Big) \\
&\le c\Big(\|u_n - u\|^2_{L^2(0,t;V)} + \|B(p_n,\cdot,u(\cdot)) - B(p,\cdot,u(\cdot))\|^2_{L^2(0,t;V^*)} \\
&\qquad + \|C(p_n,\cdot) - C(p,\cdot)\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \Big)
\end{aligned}$$

which implies

$$\begin{aligned}
\|\eta_n - \eta\|_{L^2(0,t;V^*)} \le c\Big(&\|u_n - u\|_{L^2(0,t;V)} + \|B(p_n,\cdot,u(\cdot)) - B(p,\cdot,u(\cdot))\|_{L^2(0,t;V^*)} \\
&+ \|C(p_n,\cdot) - C(p,\cdot)\|_{L^2(0,t;\mathcal{L}(V,V^*))} \Big)
\end{aligned}$$

for all $t \in [0,T]$. Substituting this inequality into (10), it follows

$$\begin{aligned}
&\frac{1}{2}|u_n'(t) - u'(t)|^2 + \left(m_1 - m_2 - \frac{m_3 T}{\sqrt{2}} \right) \|u_n' - u'\|^2_{L^2(0,t;V)} \qquad (11) \\
&\le c\Big(\|A(p_n,\cdot,u'(\cdot)) - A(p,\cdot,u'(\cdot))\|_{L^2(0,t;V^*)} + \|u_n - u\|_{L^2(0,t;V)} \\
&\quad + \|B(p_n,\cdot,u(\cdot)) - B(p,\cdot,u(\cdot))\|_{L^2(0,t;V^*)} \\
&\quad + \|C(p_n,\cdot) - C(p,\cdot)\|_{L^2(0,t;\mathcal{L}(V,V^*))} \Big) \|u_n' - u'\|_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in [0,T]$. Omitting the first term on the left hand side, by (H_2), we deduce

$$\|u_n' - u'\|_{L^2(0,t;V)} \le c\left(\|u_n - u\|_{L^2(0,t;V)} + r_n \right) \qquad (12)$$

where

$$\begin{aligned}
r_n = &\|A(p_n,\cdot,u'(\cdot)) - A(p,\cdot,u'(\cdot))\|_{\mathcal{V}^*} + \|B(p_n,\cdot,u(\cdot)) - B(p,\cdot,u(\cdot))\|_{\mathcal{V}^*} \\
&+ \|C(p_n,\cdot) - C(p,\cdot)\|_{L^2(0,T;\mathcal{L}(V,V^*))}.
\end{aligned}$$

Using (9), we have

$$\|u_n(t) - u(t)\| \le \int_0^t \|u_n'(s) - u'(s)\| ds \le \sqrt{T} \|u_n' - u'\|_{L^2(0,t;V)}$$

for all $t \in [0,T]$ which together with (12) implies

$$\|u_n(t) - u(t)\| \le c\left(\|u_n - u\|_{L^2(0,t;V)} + r_n \right) \quad \text{for all } t \in [0,T]$$

and

$$\|u_n(t) - u(t)\|^2 \le c \left(\int_0^t \|u_n(s) - u(s)\|^2 ds + r_n^2 \right) \quad \text{for all } t \in [0, T].$$

Applying now the Gronwall inequality, we have $\|u_n(t) - u(t)\| \le c r_n^2$ which, by hypotheses, entails

$$\lim_{n\to\infty} \|u_n - u\|_{C(0,T;V)} = 0.$$

Next, from (12), we have $\|u_n' - u'\|_{L^2(0,t;V)} \le c \left(\|u_n - u\|_{C(0,T;V)} + r_n \right)$ which implies

$$\lim_{n\to\infty} \|u_n' - u'\|_{\mathcal{V}} = 0.$$

Finally, from (11), after omitting the second term on the left hand side, we obtain

$$\frac{1}{2} |u_n'(t) - u'(t)|^2 \le c \left(\|u_n - u\|_{C(0,T;V)} + r_n \right) \|u_n' - u'\|_{\mathcal{V}}.$$

Hence, we deduce

$$\lim_{n\to\infty} \|u_n' - u'\|_{C(0,T;H)} = 0.$$

This completes the proof of (8).

From (8) and the hypothesis $H(L)$, we have

$$L(t, u(t), u'(t)) \le \liminf_{n\to\infty} L(t, u_n(t), u_n'(t)) \quad \text{for a.e. } t \in (0, T)$$

and consequently, by Fatou's lemma

$$\int_0^T L(t, u(t), u'(t)) dt \le \liminf_{n\to\infty} \int_0^T L(t, u_n(t), u_n'(t)) dt. \tag{13}$$

Also from $H(l)$, we obtain

$$l(u(T), u'(T)) \le \liminf_{n\to\infty} l(u_n(T), u_n'(T)). \tag{14}$$

Clearly (13) and (14) imply $\mathcal{F}(p) \le \liminf_{n\to\infty} \mathcal{F}(p_n)$. This completes the proof of the theorem.

Applying the direct method of the calculus of variations, from $H(P)$ and Theorem 4.1, we have the following.

Theorem 4.2. *Let the hypotheses $H(P)$, $H(A)_1$, $H(B)_1$, $H(C)_1$, $H(F)$, (H_0), (H_1), (H_2), $H(l)$ and $H(L)$ hold. Then the identification problem (5) admits at least one solution.*

5. **Integrodifferential Hemivariational Inequalities.** In this section we apply Theorems 3.1 and 4.2 in the study of a class of second order hemivariational inequalities.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let Γ_C be a measurable part of Γ, $\Gamma_C \subseteq \Gamma$. The direct problem we are interested in is the following problem called a hemivariational inequality.

Problem (HVI): find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} \left\langle u''(t) + A(t, u'(t)) + B(t, u(t)) + \displaystyle\int_0^t C(t-s) u(s) ds, v \right\rangle \\ \qquad\qquad + \displaystyle\int_{\Gamma_C} j^0(x, t, \gamma u'(t); \gamma v) d\Gamma \ge \langle f(t), v \rangle \quad \text{for all } v \in V \text{ and a.e. } t \in (0, T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

In the study of Problem (HVI) we consider the following additional hypothesis.

$\underline{H(j)}$: $\quad j \colon \Gamma_C \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot,\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}$ and $j(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$;
(ii) $j(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $|\partial j(x,t,\xi)| \le \widetilde{c}(1+\|\xi\|_{\mathbb{R}^d})$ for all $\xi \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\widetilde{c} > 0$;
(iv) $(\eta_1 - \eta_2, \xi_1 - \xi_2)_{\mathbb{R}^d} \ge -\widetilde{m}_2 \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$ for all $\eta_i \in \partial j(x,t,\xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\widetilde{m}_2 \ge 0$,

where j^0 and ∂j denote the directional derivative and the Clarke generalized gradient of $j(x,t,\cdot)$, respectively.

We consider the functional $J\colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(t,v) = \int_{\Gamma_C} j(x,t,v(x))d\Gamma \quad \text{a.e. } t \in (0,T) \text{ and } v \in L^2(\Gamma_C;\mathbb{R}^d). \tag{15}$$

We recall the following result, cf. Lemma 3.1 of [25].

Lemma 5.1. *Assume that $H(j)$ holds. Then the functional J given by (15) satisfies the following properties.*

(i) $J(\cdot,v)$ is measurable for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$ and $J(\cdot,0) \in L^1(0,T)$;
(ii) $J(t,\cdot)$ is locally Lipschitz for a.e. $t \in (0,T)$;
(iii) $\|\partial J(t,v)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le c_0\left(1+\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with $c_0 > 0$;
(iv) $(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \ge -\widetilde{m}_2\|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$ for all $z_i \in \partial J(t,w_i)$, $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i = 1,2$, a.e. $t \in (0,T)$ with $\widetilde{m}_2 \ge 0$;
(v) for all u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$, we have

$$J^0(t,u;v) \le \int_{\Gamma_C} j^0(x,t,u(x);v(x))d\Gamma, \tag{16}$$

where $J^0(t,u;v)$ denotes the directional derivative of $J(t,\cdot)$ at a point $u \in L^2(\Gamma_C;\mathbb{R}^d)$ in the direction $v \in L^2(\Gamma_C;\mathbb{R}^d)$.

We now use Theorem 3.1 and Lemma 5.1 to obtain the following existence result.

Corollary 5.1. *(A) Assume that $H(A)$, $H(B)$, $H(C)$, $H(j)$, (H_0) hold and*

$$\alpha > 2\sqrt{3}c_0c_e^2\|\gamma\|^2, \tag{17}$$

$$m_1 > \widetilde{m}_2c_e^2\|\gamma\|^2. \tag{18}$$

Then Problem (HVI) has at least one solution.

(B) If, in addition to the hypotheses in (A), either $j(x,t,\cdot)$ or $-j(x,t,\cdot)$ is regular for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then Problem (HVI) admits a unique solution.

Proof: (A) Define $F\colon (0,T) \times V \times V \to \mathcal{P}_{fc}(Z^*)$ by

$$F(t,u,v) = \gamma^*\partial J(t,\gamma v) \quad \text{for } t \in (0,T), u, v \in V,$$

where J is defined by (15), $\gamma\colon Z \to L^2(\Gamma_C;\mathbb{R}^d)$ is the trace operator and $\gamma^*\colon L^2(\Gamma_C;\mathbb{R}^d) \to Z^*$ denotes its adjoint. Using the linearity and continuity of the trace operator, the properties of the Clarke subdifferential (cf. Propositions 5.6.9 and 5.6.10 of [5]) and Lemma 5.1, we obtain that F satisfies $H(F)$, cf. [10, 21] for details. Hence, by Theorem 3.1, we know that there exists a unique solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ of the evolution inclusion

$$\begin{cases} u''(t) + A(t,u'(t)) + B(t,u(t)) + \displaystyle\int_0^t C(t-s)u(s)ds + \\ \qquad\qquad\qquad\qquad + F(t,u(t),u'(t)) \ni f(t) \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

According to Definition 3.1, we have

$$u''(t) + A(t, u'(t)) + B(t, u(t)) + \int_0^t C(t-s)u(s)ds + \zeta(t) = f(t), \tag{19}$$

for a.e. $t \in (0, T)$ with $\zeta(t) = \gamma^* z(t)$ and $z(t) \in \partial J(t, \gamma u'(t))$ for a.e. $t \in (0, T)$. The latter is equivalent to $(z(t), w)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t, \gamma u'(t); w)$ for all $w \in L^2(\Gamma_C; \mathbb{R}^d)$ and a.e. $t \in (0, T)$. Hence, using (19) and (16), we deduce

$$\left\langle f(t) - u''(t) - A(t, u'(t)) - Bu(t) - \int_0^t C(t-s)u(s)ds, v \right\rangle = \langle \zeta(t), v\rangle_{Z^*\times Z} =$$

$$= (z(t), \gamma v)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t, \gamma u'(t); \gamma v) \le \int_{\Gamma_C} j^0(x, t, \gamma u'(t); \gamma v) d\Gamma,$$

for all $v \in V$ and a.e. $t \in (0, T)$. This means that u is a solution to Problem (HVI).

(B) Let u be a solution to Problem (HVI) obtained in (A). It follows from Theorem 5.6.38 of [5] that if either $j(x, t, \cdot)$ or $-j(x, t, \cdot)$ is regular for a.e. $(x, t) \in \Gamma_C \times (0, T)$, then either $J(t, \cdot)$ or $-J(t, \cdot)$ is regular for a.e. $t \in (0, T)$, respectively, and (16) holds with equality. Using the equality in (16), it follows

$$\left\langle u''(t) + A(t, u'(t)) + B(t, u(t)) + \int_0^t C(t-s)u(s)ds - f(t), v \right\rangle + J^0(t, \gamma u'(t); \gamma v) \ge 0$$

for all $v \in V$ and a.e. $t \in (0, T)$. From Proposition 2(i) of [22], we have

$$\left\langle f(t) - u''(t) - A(t, u'(t)) - B(t, u(t)) - \int_0^t C(t-s)u(s)ds, v \right\rangle \le (J \circ \gamma)^0(t, u'(t); v)$$

for all $v \in V$ and a.e. $t \in (0, T)$. Hence, by Proposition 2(ii) of [22] and the definition of the subdifferential, we obtain

$$f(t) - u''(t) - A(t, u'(t)) - B(t, u(t)) - \int_0^t C(t-s)u(s)ds \in$$
$$\in \partial(J \circ \gamma)(t, u'(t)) = \gamma^* \partial J(t, \gamma u'(t)) = F(t, u(t), u'(t))$$

for a.e. $t \in (0, T)$. Thus u is a solution to the evolution inclusion in Problem $\mathcal{P}$. The uniqueness of solution to Problem (HVI) follows now from the uniqueness result of Theorem 3.1. It completes the proof.

The identification problem for the hemivariational inequality (HVI) reads as follows: find the solution $p^* \in P$ of the minimization problem

$$\mathcal{F}(p^*) = \min_{p\in P} \mathcal{F}(p), \tag{20}$$

where the cost functional is defined by (4) and the dynamics is described by the following inequality.

Problem $(HVI)_p$: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} \left\langle u''(t) + A(p, t, u'(t)) + B(p, t, u(t)) + \int_0^t C(p, t-s)u(s)ds, v \right\rangle + \\ \qquad + \int_{\Gamma_C} j^0(x, t, \gamma u'(t); \gamma v) d\Gamma \ge \langle f(t), v\rangle \text{ for all } v \in V \text{ and a.e. } t \in (0, T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

From Theorem 4.2, we obtain the following

Corollary 5.2. *Let the hypotheses $H(P)$, $H(A)_1$, $H(B)_1$, $H(C)_1$, $H(j)$, (H_0), (17), (18), $H(l)$ and $H(L)$ hold. Then the identification problem (20) admits at least one solution.*

6. **Viscoelastic Frictional Contact Problem.** In this section, we study the problem of identification of viscosity, elasticity and relaxation operators in a dynamic viscoelastic frictional contact problem of mechanics. This contact problem leads to a hemivariational inequality of the form (HVI) for the displacement field.

We shortly describe the mechanical frictional contact problem, for details we refer to [25]. We suppose that a viscoelastic body occupies a subset Ω of $\mathbb{R}^d$, $d = 2, 3$ in applications. The body is acted upon by volume forces and surface tractions and, as a result, its state is evolving. We are interested in dynamic evolution process of the mechanical state of the body on the time interval $[0, T]$ with $0 < T < \infty$. The boundary Γ of Ω is supposed to be Lipschitz continuous and therefore the unit outward normal vector ν exists a.e. on Γ. It is assumed that Γ is divided into three mutually disjoint parts Γ_D, Γ_N and Γ_C such that the measure of Γ_D is positive. We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Volume forces of density f_1 act in Ω and surface tractions of density f_2 are applied on Γ_N. The body may come in contact with an obstacle over the potential contact surface Γ_C.

Let $\mathbb{S}^d$ be the linear space of second order symmetric tensors on $\mathbb{R}^d$ (equivalently, the space $\mathbb{R}^{d\times d}_s$ of symmetric matrices of order d) and let $Q = \Omega \times (0, T)$. For simplicity we skip the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. The frictional contact problem under consideration can be stated as follows:
find the displacement field $u\colon Q \to \mathbb{R}^d$ and the stress tensor $\sigma\colon Q \to \mathbb{S}^d$ such that

$$u''(t) - \operatorname{div} \sigma(t) = f_1(t) \quad \text{in} \quad Q \tag{21}$$

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\varepsilon(u(s))ds \quad \text{in} \quad Q \tag{22}$$

$$u(t) = 0 \quad \text{on} \quad \Gamma_D \times (0, T) \tag{23}$$

$$\sigma(t)\nu = f_2(t) \quad \text{on} \quad \Gamma_N \times (0, T) \tag{24}$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, u'_\nu(t)), \quad -\sigma_\tau(t) \in \partial j_\tau(t, u'_\tau(t)) \quad \text{on} \quad \Gamma_C \times (0, T) \tag{25}$$

$$u(0) = u_0, \quad u'(0) = v_0 \quad \text{in} \quad \Omega. \tag{26}$$

Conditions (25) represent the frictional contact condition in which j_ν and j_τ are given functions and the subscripts ν and τ for σ and u' indicate normal and tangential components of tensors and vectors. The symbol ∂j denotes the Clarke subdifferential of j with respect to the last variable. Concrete examples of frictional conditions which lead to subdifferential boundary conditions of the form (25) with the functions j_ν and j_τ satisfying assumptions $H(j_\nu)$ and $H(j_\tau)$ below can be found in [23]. We only remark that these examples include the viscous contact and the contact with nonmonotone normal damped response, associated to a nonmonotone friction law, to Tresca's friction law or to a power-law friction.

Equation (22) describes the constitutive law, where $\mathcal{A}$ is a nonlinear operator describing the purely viscous properties of the material, while $\mathcal{B}$ and $\mathcal{C}$ are the nonlinear elasticity and the linear relaxation operators, respectively. Note that the the operators $\mathcal{A}$ and $\mathcal{B}$ may depend explicitly on the time variable and this is the case when the viscosity properties of the material depend on the temperature field which plays the role of a parameter and which evolution in time is prescribed. In the inverse problem formulated below we consider these three operators to depend on a parameter to be identified. One-dimensional constitutive laws of the form (22) can be constructed by using rheological arguments, cf. [7], Chapter 6 of [8, 25]. For a detailed description of the model (21)-(26), we refer to [25].

In order to give the variational formulation of the problem (21)-(26), we recall the following notation. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \quad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

Summation convention over repeated indices running from 1 to d is used and the index that follows a comma indicates a partial derivative. We also introduce the spaces $H = L^2(\Omega;\mathbb{R}^d)$, $\mathcal{H} = L^2(\Omega;\mathbb{S}^d)$, $H_1 = \{u \in H \mid \varepsilon(u) \in \mathcal{H}\}$, $\mathcal{H}_1 = \{\tau \in \mathcal{H} \mid \operatorname{div}\tau \in H\}$, where $\varepsilon\colon H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{S}^d)$ and $\operatorname{div}\colon \mathcal{H}_1 \to L^2(\Omega;\mathbb{R}^d)$ denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{div}\sigma = \{\sigma_{ij,j}\}.$$

Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_ν and v_τ the usual normal and the tangential components of v on the boundary Γ, $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$. Similarily, for a smooth tensor field $\sigma\colon \Omega \to \mathbb{S}^d$, we define its normal and tangential components by $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

Let V be the closed subspace of $H^1(\Omega;\mathbb{R}^d)$ given by

$$V = \left\{ v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \ \text{ on } \Gamma_D \right\}.$$

On the space V we consider the inner product and the corresponding norm defined by

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}} \ \text{ for } \ u, v \in V.$$

It follows from Korn's inequality that $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|$ are the equivalent norms on V.

In the study of problem (21)-(26) we consider the following assumptions on the viscosity operator $\mathcal{A}$, on the elasticity operator $\mathcal{B}$ and on the relaxation operator $\mathcal{C}$.

$\underline{H(\mathcal{A})}$: $\mathcal{A}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathcal{A}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon \in \mathbb{S}^d$;
(ii) $\mathcal{A}(x,t,\cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $(x,t) \in Q$;
(iii) $\|\mathcal{A}(x,t,\varepsilon)\|_{\mathbb{S}^d} \le c_1 (b(x,t) + \|\varepsilon\|_{\mathbb{S}^d})$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x,t) \in Q$ with $b \in L^2(Q)$, $b \ge 0$ and $c_1 > 0$;
(iv) $(\mathcal{A}(x,t,\varepsilon_1) - \mathcal{A}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge m_1 \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x,t) \in Q$ with $m_1 > 0$;
(v) $\mathcal{A}(x,t,\varepsilon) : \varepsilon \ge c_2 \|\varepsilon\|^2_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x,t) \in Q$ with $c_2 > 0$.

$\underline{H(\mathcal{B})}$: $\mathcal{B}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathcal{B}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon \in \mathbb{S}^d$;
(ii) $\|\mathcal{B}(x,t,\varepsilon)\|_{\mathbb{S}^d} \le \widetilde{b_1}(x,t) + \widetilde{b_2}\|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x,t) \in Q$ with $\widetilde{b_1} \in L^2(Q)$, $\widetilde{b_1}$, $\widetilde{b_2} \ge 0$;
(iii) $\|\mathcal{B}(x,t,\varepsilon_1) - \mathcal{B}(x,t,\varepsilon_2)\|_{\mathbb{S}^d} \le L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x,t) \in Q$ with $L_{\mathcal{B}} > 0$.

$\underline{H(\mathcal{C})}$: $\mathcal{C}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that $\mathcal{C}(x,t,\varepsilon) = c(x,t)\varepsilon$ and $c(x,t) = \{c_{ijkl}(x,t)\}$ with $c_{ijkl} = c_{jikl} = c_{lkij} \in L^2(0,T;L^\infty(\Omega))$.

The contact and frictional potentials j_ν and j_τ satisfy the following hypotheses.

$\underline{H(j_\nu)}$: $j_\nu\colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $j_\nu(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$ and $j_\nu(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$;
(ii) $j_\nu(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $|\partial j_\nu(x,t,r)| \le c_\nu (1 + |r|)$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r \in \mathbb{R}$ with $c_\nu > 0$;

(iv) $(\eta_1 - \eta_2)(r_1 - r_2) \geq -m_\nu |r_1 - r_2|^2$ for all $\eta_i \in \partial j_\nu(x,t,r_i)$, $r_i \in \mathbb{R}$, $i = 1, 2$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $m_\nu \geq 0$.

$\underline{H(j_\tau)}$: $j_\tau \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$ satisfies

(i) $j_\tau(\cdot,\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $j_\tau(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$;
(ii) $j_\tau(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $\|\partial j_\tau(x,t,\xi)\|_{\mathbb{R}^d} \leq c_\tau (1 + \|\xi\|_{\mathbb{R}^d})$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $\xi \in \mathbb{R}^d$ with $c_\tau > 0$;
(iv) $(\eta_1 - \eta_2, \xi_1 - \xi_2)_{\mathbb{R}^d} \geq -m_\tau \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$ for all $\eta_i \in \partial j_\tau(x,t,\xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $m_\tau \geq 0$.

The volume force and traction densities satisfy

$\underline{H(f)}$: $f_1 \in L^2(0,T;H)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$

and the initial data have the regularity

$\underline{H(0)}$: $u_0 \in V$, $v_0 \in H$.

For examples of superpotentials j_ν and j_τ which satisfy $H(j_\nu)$ and $H(j_\tau)$, we refer to Example 5.1 of [25].

We introduce the operators $A \colon (0,T) \times V \to V^*$, $B \colon (0,T) \times V \to V^*$ and $C \colon (0,T) \times V \to V^*$ defined by

$$\langle A(t,u), v \rangle = \langle \mathcal{A}(t,\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} \tag{27}$$

$$\langle B(t,u), v \rangle = \langle \mathcal{B}(t,\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} \tag{28}$$

$$\langle C(t)u, v \rangle = \langle \mathcal{C}(t,\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} \tag{29}$$

for u, $v \in V$ and $t \in (0,T)$. We also consider the function $f \colon (0,T) \to V^*$ given by

$$\langle f(t), v \rangle = \langle f_1(t), v \rangle_H + (f_2(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \text{for } u, v \in V, \text{ a.e. } t \in (0,T). \tag{30}$$

The variational formulation of the problem (21)-(26) (cf. [25]) is the following:

$$\begin{cases} \text{find } u \colon (0,T) \to V \text{ such that } u \in \mathcal{V}, u' \in \mathcal{W} \text{ and} \\ \left\langle u''(t) + A(t,u'(t)) + B(t,u(t)) + \displaystyle\int_0^t C(t-s)u(s)ds, v \right\rangle + \\ \qquad + \displaystyle\int_{\Gamma_C} \left(j_\nu^0(x,t,u'_\nu(x,t);v_\nu(x)) + j_\tau^0(x,t,u'_\tau(x,t);v_\tau(x)) \right) d\Gamma \geq \\ \qquad\qquad \geq \langle f(t), v \rangle \quad \text{for all} \quad v \in V \text{ and a.e.} t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \tag{31}$$

The unique solvability of the problem (31) is given by the following result.

Theorem 6.1. *Assume that $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{C})$, $H(j_\nu)$, $H(j_\tau)$, $H(f)$, $H(0)$ hold, $c_2 > 2\sqrt{3}c_0 c_e^2 \|\gamma\|^2$ and $m_1 > (m_\nu + m_\tau)c_e^2\|\gamma\|^2$. Then problem (31) admits at least one solution. If, in addition,*

$$\begin{cases} \textit{either } j_\nu(x,t,\cdot) \textit{ and } j_\tau(x,t,\cdot) \textit{ are regular} \\ \textit{or } -j_\nu(x,t,\cdot) \textit{ and } -j_\tau(x,t,\cdot) \textit{ are regular} \end{cases} \tag{32}$$

for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then problem (31) has a unique solution.

Proof: The proof is based on arguments used in [23] and thus we skip the details. The main steps of the proof are the following.

a) Under the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$ and $H(\mathcal{C})$, the operators A, B and C defined by (27), (28) and (29) satisfy hypotheses $H(A)$, $H(B)$ and $H(C)$, respectively.

b) Let $j \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$j(x,t,\xi) = j_\nu(x,t,\xi_\nu) + j_\tau(x,t,\xi_\tau) \quad \text{a.e. } (x,t) \in \Gamma_C \times (0,T), \text{ all } \xi \in \mathbb{R}^d.$$

It can be shown that, under the hypotheses $H(j_\nu)$ and $H(j_\tau)$, the function j satisfies $H(j)$ with $\widetilde{c} = \max\{c_\nu, c_\tau\}$ and $\widetilde{m}_2 = m_\nu + m_\tau$.

c) The assumptions $H(f)$ and $H(0)$ combined with (30) imply that (H_0) holds. It is clear that (H_1) also is satisfied.

The steps above allow us to apply Corollary 5.1 to obtain the existence of a solution to the hemivariational inequality (31). It can be easily observed that the regularity hypotheses on j_ν, j_τ or $-j_\nu$, $-j_\tau$ imply the regularity of j or $-j$, respectively. In this case by Corollary 5.1, we deduce the uniqueness of a solution to (31).

The result of Theorem 6.1 extends a result of Theorem 5.1 of [25].

Finally, we consider the identification problem for the hemivariational inequality (31). We suppose that the operators $H(\mathcal{A})$, $H(\mathcal{B})$ and $H(\mathcal{C})$ depend on a parameter $p \in P$, P being a subset of a metric space, and we consider the following direct problem:

$$\begin{cases} \text{find } u \in \mathcal{V},\ u' \in \mathcal{W} \text{ and} \\ \left\langle u''(t) + A(p,t,u'(t)) + B(p,t,u(t)) + \displaystyle\int_0^t C(p,t-s)u(s)ds, v \right\rangle + \\ \qquad + \displaystyle\int_{\Gamma_C} \left(j_\nu^0(x,t,u_\nu'(x,t); v_\nu(x)) + j_\tau^0(x,t,u_\tau'(x,t); v_\tau(x)) \right) d\Gamma \\ \qquad\qquad \geq \langle f(t), v\rangle \ \text{ for all } \ v \in V \text{ and a.e. } t \in (0,T) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \tag{33}$$

The identification problem for the mechanical problem (31) is formulated as follows: find the solution $p^* \in P$ of the problem

$$\mathcal{F}(p^*) = \min_{p \in P} \mathcal{F}(p), \tag{34}$$

where the cost functional is defined by (4) and the dynamics is described by (33).

We need the following hypotheses on the data of problem (33).

$\underline{H(\mathcal{A})_1}$: The family of operators $\{\mathcal{A}(p,\cdot,\cdot,\cdot), p \in P\}$ satisfy $H(\mathcal{A})$ uniformly in $p \in P$ and the mapping $p \mapsto \mathcal{A}(p,t,x,\varepsilon)$ is continuous in the sense that

$$\mathcal{A}(p_n,x,t,\varepsilon) \to \mathcal{A}(p,x,t,\varepsilon) \ \text{ in } \mathbb{S}^d \text{ for a.e. } (x,t) \in Q, \text{ all } \varepsilon \in \mathbb{S}^d$$

whenever $p_n \to p$ in P.

$\underline{H(\mathcal{B})_1}$: The family of operators $\{\mathcal{B}(p,\cdot,\cdot,\cdot), p \in P\}$ satisfy $H(\mathcal{B})$ uniformly in $p \in P$ and the mapping $p \mapsto \mathcal{B}(p,t,x,\varepsilon)$ is continuous in the sense that

$$\mathcal{B}(p_n,x,t,\varepsilon) \to \mathcal{B}(p,x,t,\varepsilon) \ \text{ in } \mathbb{S}^d \text{ for a.e. } (x,t) \in Q, \text{ all } \varepsilon \in \mathbb{S}^d$$

whenever $p_n \to p$ in P.

$\underline{H(\mathcal{C})_1}$: The family of operators $\{\mathcal{C}(p,\cdot,\cdot,\cdot), p \in P\}$ satisfy $H(\mathcal{C})$ uniformly in $p \in P$, and if $c(p_n,\cdot,\cdot)$, $c(p,\cdot,\cdot)$ are the corresponding coefficients, then

$$c(p_n,\cdot,\cdot) \to c(p,\cdot,\cdot) \ \text{ in } L^2(0,T;L^\infty(\Omega))$$

whenever $p_n \to p$ in P.

Directly from Corollary 5.2 and Theorem 6.1, we deduce the solvability of the problem (34).

Corollary 6.1. *Let the hypotheses $H(P)$, $H(\mathcal{A})_1$, $H(\mathcal{B})_1$, $H(\mathcal{C})_1$, $H(j_\nu)$, $H(j_\tau)$, $H(f)$, $H(0)$, $H(l)$ and $H(L)$ hold, $c_2 > 2\sqrt{3}c_0c_e^2\|\gamma\|^2$ and $m_1 > (m_\nu + m_\tau)c_e^2\|\gamma\|^2$. Then the identification problem (34) admits a solution.*

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