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DESIGN OF LIFE INSURANCE PARTICIPATING POLICIES WITH VARIABLE GUARANTEES AND ANNUAL PREMIUMS

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Abstract. *This paper deals with the design of a life insurance participating policy with annual premiums. Participating policies guarantee a minimum interest rate at the end of each policy year and an additional interest rate may be credited to the policyholder according to the annual performance of the insurer's investment portfolio. Particularly, we consider the case in which the minimum interest guarantee is adjusted periodically. Moreover, the policy offers the policyholder the right to terminate the contract before maturity in exchange for its cash value. Differing from previous research, we consider a policy with features that make it more similar to real insurance products found in the market. Furthermore, with the use of optimization techniques, we put forward an approach for deciding an optimal set of contractual rates and annual premium for the participating policy under scrutiny. To the best of our knowledge, this is the first work to address the actual design of such participating policies.*

Keywords: Participating policy, Guarantee rates, Annual premiums, Fair valuation, Optimization

1. **Introduction.** Life insurance participating policies, also known as with profits policies, are important financial products that have become popular over the years as a means of achieving long-term growth of capital. These policies offer a basic benefit, which is determined at the inception of the contract, and guarantee that the value of policy will grow by at least a minimum rate each year, this rate is called guarantee rate. Additional interest, which is often referred to as participation rate, may be credited according to the performance of the company's investment portfolio. Most participating policies give the policyholder the right to terminate the policy before maturity in exchange of its cash value. This type of policies poses a great challenge to every insurance company, because products with such appealing features are difficult to design and price accurately, and any mistake in the process could have disastrous financial consequences.

There is rather abundant literature addressing the valuation of participating policies with guarantees, for example, [1-7]. In regard to the valuation of participating policies with guarantees and surrender option, we found valuable contribution in [8-10]. Although, in the literature, many important contributions have been reported, there are several important issues that have not been considered. The aforementioned works have not taken into account the fact that one of the insurers' goals is to make a profit. Previous research has solely focused on the fair valuation of the policy; that is, the premium of the policy is calculated so that it equals the market value of the benefits to the policyholder, which, in turn, represents the insurer's expenses.

There are other issues that have been left out in previous works. In reality, the guarantee rate offered by the policy is more likely to change throughout the life of the policy rather than remain constant; however, most authors have only considered the case where the minimum guarantee rate does not change throughout the policy. Furthermore, the policy's premium is often paid on installments, yet most authors have focused on the analysis of policies purchased by a single premium; only [9] has dealt with a policy where the premium is paid annually. In addition, despite the fact that one of the insurer's concerns is to select the right level for the premium and each contractual rate (guarantee and participation rate), none of the above-mentioned works proposed a way to determine an optimal set of these contractual rates and premium. This void was partially filled in [11] where the authors used optimization techniques to determine the optimal contractual rates and the single premium paid at the beginning of the contract; however, periodical premiums and variable guarantee rates were not considered in their approach.

The aim of this paper is to fill these gaps in the literature by determining the annual premium and optimal contractual rates of a policy in which the minimum interest guarantee may vary periodically. As far as we know, this is the first study to consider these issues. We first derive a formula to calculate the expected payments of the policy using the notion of fair value. We then formulate an optimization model which allows us to decide the optimal contractual rates and premium while taking into account restrictions on the demand of the policy and making sure that the insurer's profit is maximized. We solve the model using the soft approach put forward in [12]. Finally, we validate the model by conducting computational experiments.

The paper is organized as follows: Section 2 explains the features of the participating policy; Section 3 describes the approach for the fair valuation of a participating policy with variable guarantees and a surrender option; in Section 4, we first explain how to calculate the profit of the policy, then we propose an optimization model for the design of a policy with single premium and then suggest a way to calculate the annual premium; numerical results are shown in Section 5, and Section 6 concludes the paper.

2. Features of the Participating Policy.

Participating policies offer a basic benefit, which is fixed at the inception of the policy. The insurer guarantees that the value of the policy will increase at least by a minimum guarantee each year. We consider the case where this guarantee is not constant and it will vary periodically.

In these policies, the insurer shares a percentage of his profits with the policyholder; this percentage is commonly known as participating rate. The sharing mechanism is as follows: the insurer will allocate most of the premiums collected from participating policies in its investment portfolio (hereafter referred to as *reference portfolio*), whenever the return of the reference portfolio exceeds the minimum guarantee rate of that period an additional interest will be credited to the policy. This additional benefit is referred to as *bonus rate*.

Due to the uncertainty in the financial markets, the performance of the reference portfolio is variable, and the chances of producing high returns may be the same as incurring in heavy loses. However, the losses do not affect the benefits to the policyholder since the minimum interest guarantee rate serves as protection against periods of bad performance and once the interest rates have been credited they cannot be taken away.

The total benefit of the policy is payable upon the death of the policyholder or at maturity of the contract. However, in our framework the policy also offers the policyholder the right to cancel the policy before maturity and receive its cash value. This feature is often called *surrender option* and the sum received upon cancellation is referred to as *surrender value*. The contract determines, for each possible cancellation date, the related surrender value. Typically, the surrender value is calculated as a percentage of the benefit accumulated by the time of surrender and increases over time, therefore, the longer the policy is in effect the larger the benefit. Since insurers are adverse to the idea of buying back the policy, the surrender values are kept low so that policyholders are discouraged from terminating the policy before maturity.

3. **Fair Valuation of the Participating Policies.**

In this section, we first describe the basic assumptions underlying our approach and we then derive a formula to calculate the fair value of the policy.

3.1. **Assumptions.**

Life insurance participating policies are affected by both financial and mortality risk. We assume these two risk are independent of each other; moreover, the insurer is assumed to be risk neutral with respect to mortality. Based on [1], we justify this assumption on the grounds that in actuarial practice life tables used in valuation are generally adjusted in a way that the risk aversion of the insurer is reflected. Hence, the actuarial life tables can be considered to be risk-neutral.

We follow standard practice in the finance literature and assume financial markets are perfect, frictionless and free of arbitrage opportunities in order to avoid imperfections such as transaction costs, taxes, divisibility. The annual compounded risk-free rate, represented by r, is deterministic and constant. The reference portfolio is assumed to be well-diversified, and evolves according to the following geometric Brownian motion:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \quad t \in [0, T], \tag{1}$$

where σ is constant and represents the volatility parameter. W is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathfrak{F}, Q)$ on the time interval $[0, T]$ where discounted prices are martingales under the equivalent risk-neutral probability measure, Q.

The stochastic differential equation has a well-known analytical solution.

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}, \quad t \in [0, T], \tag{2}$$

where S_0 is an arbitrary initial value.

Assuming that the continuously compounded annual rates of return, represented by s_t, are given by

$$s_t = \frac{S_t}{S_t - 1} - 1, \quad t \in [0, T], \tag{3}$$

allows us to define

$$1 + s_t = e^{(r-\sigma^2/2)+\sigma(W_t - W_{t-1})}, \tag{4}$$

which are stochastically independent and identically distributed for $t = 1, 2, \ldots, R$ [1]. Moreover, their logarithms follow a normal distribution with mean $r - \sigma^2/2$ and variance σ^2, i.e., $ln(1 + s_t) \sim N(r - \sigma^2/2, \sigma^2)$.

3.2. **Fair valuation of the policy.**

In this section, we describe how to calculate the fair value of a participating policy purchased with a single-sum at the inception of the contract, we address the calculation of annual premiums in Section 4.3.

Basically, the fair value is equal to the sum of the present value at issue of all the expected benefits to the policyholder, weighted with the respective life, death and surrender probabilities.

Consider that the policy is issued at time zero, matures after T years and has in initial basic benefit denoted by C_0. Let x be the age of the policyholder at the inception of the contract and let α^x be the participation rate, which is assumed to be constant in time and takes values within [0, 1]. Further, let g_t^x be the minimum guarantee rate relevant at time t. As explained in Section 2, each period the policyholder may receive a bonus, this means that the policyholder participates in any surplus profit of the reference portfolio at the rate α_x, provided this amount is greater than the benefit guaranteed on that period. From this, we can express the bonus rate in the t-the year, denoted by b_t, as follows:

$$b_t = \max\{\alpha^x s_t - g_t^x, 0\}. \tag{5}$$

The evolution of the benefit paid at time t, denoted by C_t, can be expressed as:

$$C_t = C_{t-1}\left(1 + g_t^x + b_t\right), \quad t = 1, 2, \cdots, T. \tag{6}$$

This means that the benefit at any time t is equal to the benefit of the previous period (C_{t-1}) plus the amounts related to the guarantee and bonus rate relevant to time t. Using relations (5) and (6), we can also express C_t in terms of the basic benefit, as shown below:

$$C_t = C_0 \prod_{j=1}^{t} (1 + g_j^x)(1 + \theta_j), \quad t = 1, 2, \cdots, T, \tag{7}$$

where

$$\theta_t = \max\left\{\frac{\alpha^x s_t - g_t^x}{1 + g_t^x}, 0\right\}. \tag{8}$$

The above equation determines the value of the benefit at a given time t. However, to value the policy we need to estimate the value at time 0 of the benefit C_t, let us denote it by $\pi(C_t)$. In order to estimate this value we will make use of martingale theory introduced in [13-15]. Martingale theory is widely used in financial engineering problems because it provides a useful framework for estimating asset prices such as stock and option prices and, as is the case in this paper, contingents claims. According to the martingale approach, $\pi(C_t)$ can be expressed as:

$$\pi(C_t) = E^Q\left[e^{-rt} C_t\right], \quad t = 1, 2, \cdots, T, \tag{9}$$

where $E^Q[\cdot]$ denotes expectation under the risk-neutral measure Q. Substituting Equation (7) into Equation (9) gives

$$\pi(C_t) = E^Q\left[e^{-rt} C_0 \prod_{j=1}^{t} \left(1 + g_j^x\right)\left(1 + \theta_j\right)\right], \quad t = 1, 2, \cdots, T. \tag{10}$$

Substituting Equation (8) into (10), and after performing algebraic operations taking into account the properties of the expectation operator, leads to:

$$\pi(C_t) = C_0 \prod_{j=1}^{t} \left(\left(1 + g_j^x\right) e^{-r} + \alpha^x * E^Q\left[e^{-r} \max\left\{(1 + s_j) - \left(1 + g_j^x/\alpha^x\right), 0\right\}\right]\right). \tag{11}$$

From the set of assumptions about the evolution of the reference portfolio, $1 + s_j$ are identically, independently and log-normally distributed. This is the same distribution followed by the returns of the underlying security in the well-known Black-Scholes model (see [16]). Therefore, the expectation value in Equation (11) can be viewed as the price, at time 0, of an European call option on a non-dividend paying stock, with initial price equal to 1 and exercise price equal to $1 + g_j^x/\alpha^x$. Let c_j represent this price, then we have

$$\pi(C_t) = C_0 \prod_{j=1}^{t} \left(\left(1 + g_j^x\right) e^{-r} + \frac{\alpha^x}{1 + g_t^x} c_j\right), \quad t = 1, 2, \cdots, T, \tag{12}$$

where c_t is defined by the Black-Scholes model

$$\begin{aligned} c_t &= \Phi(d_1) - (1 + g_t^x/\alpha^x)\, e^{-r}\Phi(d_2), \\ d_1 &= \frac{r + (1/2)\sigma^2 - ln\,(1 + g_t^x/\alpha^x)}{\sigma}, \quad d_2 = d_1 - \sigma, \end{aligned} \tag{13}$$

where Φ denotes the standard normal cumulative distribution function.

Thus far we have defined the dynamics of the annual benefits which are connected to the minimum guarantee rate and participating rate. However, the policy under scrutiny also offers a surrender option and we need to estimate the present value of any income the policyholder might receive from exercising it. When a policy is terminated before maturity, the policyholder is entitled to receive the surrender value of the policy. This value is usually defined as a percentage of the accumulated benefit at the time of surrender, and its value increases the longer the policy is held for. Let β_t represent this percentage. We can express the surrender value, denoted by SV_t, as follows:

$$SV_t = \beta_t C_t, \quad t = 1, 2, \cdots, T. \tag{14}$$

Based on martingale theory, the value at time 0 of SV_t, denoted by $\pi(SV_t)$ is given by

$$\pi(SV_t) = E^Q\left[e^{-rt}SV_t\right], \quad t = 1, 2, \cdots, T. \tag{15}$$

Substituting Equation (14) into (15) and performing some algebraic operations gives

$$\pi(SV_t) = \beta_t \pi(\ C_t), \quad t = 1, 2, \cdots, T. \tag{16}$$

As we mentioned before, we need to adjust all future benefits with the probabilities of death, survival and surrender. Regarding the surrender probability, it is not unussual for insurers to make use of their historical data to obtain information about the policyholder's likelihood of surrender, believing that it reflects faithfully likely experience on the policy. In this paper, we follow this practice, and assume that insurers make use of historical information to estimate the surrender probabilities of the policyholders. Now, let us define the following probabilities for a policyholder of age x at the inception of the contract:

- $\lambda_x(t, t-1)$ denotes the probability that the policyholder surrenders the policy during the t-th year,
- $\kappa_x(T-1)$ denotes the probability that the policyholder has not surrendered the policy at year $T-1$,
- $q_x(t, t-1)$ denotes the probability the policyholder dies the t-th year,
- $p_x(T-1)$ denotes the probability that the policyholder is alive at year $T-1$.

Finally, the fair value of a participating policy with surrender option, denoted by F_x^s, can be obtained by summing up the values of C_t and SV_t at time 0 weighted with the above-mentioned probabilities. Consequently, we have

$$\begin{aligned} F_s^x = \sum_{t=1}^{T-1} q_x(t, t-1)\pi(C_t) + \sum_{t=1}^{T-1} p_x(t, t-1)\lambda_x(t, t-1)\beta_t\pi(C_t) \\ + p_x(T-1)\kappa_x(T-1)\pi(C_T). \end{aligned} \tag{17}$$

From the point of view of the insurer, the first summation term represents the expenses amount that may arise when the policyholder dies before maturity weighted with the probability of paying such an amount (i.e., the probability that the policyholder dies). The second summation term, represents the expenses that might result when the policyholder surrenders the policy before maturity multiplied by the probability of paying that amount. Since the policy can be surrendered at a given time only if the policyholder is alive, the probability of paying the surrender value at a given period is obtained by multiplying the survival probability and the surrender probability of that period. Finally, the third summation term represents the expenses amount that may be payed when the policy reach maturity provided that the policyholder is still alive and has not surrendered the policy.

4. **Optimization Approach for the Design of Participating Policies with Annual Premiums.** This section begins with the outline of the dynamics of the policy's profit. Then, we explain how to calculate the annual premiums of the policy, introduce our optimization model, and explain our approach to identify the parameters needed for the model. Finally, the section concludes with an overview of the solution method applied in our model.

4.1. **Profit of the participating policy.** We consider that the profit of a participating policy equals the aggregated value of the profits produced from groups of policyholders who are the same age.

Let pf^x be the profit from policyholders whose age are x at the beginning of the policy. Then, the profit of a policy, denoted by pf, is given by

$$pf = \sum^{x} pf^x. \tag{18}$$

Consequently, the profit produced by one group is independent from the other groups, therefore, for the remaining of the paper, we will focus only in the profit from a group of individuals of age x at the inception of the policy.

Recalling that our analysis is from the point of view of the insurer, the profit is the difference between the income, which is composed of the total earned premiums and profits from investment, and expenses. To formulate this, let us define the following notation: EP^x denotes the total premiums collected from policyholders with age x, II^x, the profit from investment, and L^x, the expenses. Then, we can represent the profit as:

$$pf^x = EP^x + II^x - L^x. \tag{19}$$

Let n^x denote the number of policyholders who acquired the policy at age x, and let P^x denote the premium paid for the policy. The total earned premiums are easily calculated by multiplying the premium for one policy by the number of policyholders.

$$EP^x = n^x P^x. \tag{20}$$

Most of these earned premiums are invested in the insurer's reference portfolio. Consider that the invested amount is calculated as a percentage of the earned premiums. Let γ represent this percentage. It follows that the income from investment is equal to the invested amount multiplied by the the expected return of the reference portfolio. This can be expressed as:

$$II^x = \gamma EP^x \times R, \tag{21}$$

where R represents the expected return of the reference portfolio.

The expenses are the total amount expected to be paid out in benefits to the policyholder. Since this amount is given by the notion of fair value, we can calculate the total expenses by multiplying the number of people that purchased the policy by the fair value of a single policy:

$$L^x = n^x F_s^x. \tag{22}$$

Finally, using Equations (20) – (22), it follows that

$$\begin{aligned} pf^x &= EP^x + II^x - L^x \\ &= n^x P^x + \gamma \left(n^x P^x\right) \times R - n^x F_s^x. \end{aligned} \tag{23}$$

4.2. **Optimization model.** We are now ready to introduce our optimization approach for designing a participating policy. Our goal is to design the policy by maximizing its profit, we formalize this by the following model:

$$\boldsymbol{Max}_{(P^x,\alpha^x,g^x_t)} \quad pf^x : P^x \in [F^x_s, P^x_u], \quad \alpha^x \in [\alpha^x_l, \alpha^x_u], \quad g^x_t \in [g^x_{t,l}, g^x_{t,u}]. \tag{24}$$

Here, F^x_s and P^x_u represent the endpoints of the set containing P^x, α^x_l and α^x_u represent these for the set of participating rates while $g^x_{t,l}$ and $g^x_{t,u}$ represent these for the set containing the guarantee rate relevant at time t.

Note that the lower endpoint of the set for the premium is given by the fair value of the policy. This is because insurers are not willing to receive less than the expenses incurred and, as explained in the previous section, these expenses are given by the fair value.

Finally, by using (23), the optimization model becomes

$$\begin{aligned} &\boldsymbol{Max}_{(P^x,\alpha^x,g^x_t)} \quad n^x(P^x + \gamma P^x \times R - F^x_s) \\ &\text{s.t.} \quad P^x \in [F^x_s, P^x_u], \alpha^x \in [\alpha^x_l, \alpha^x_u], g^x \in [g^x_{t,l}, g^x_{t,u}]. \end{aligned} \tag{25}$$

4.3. **Estimation of the annual premium.** In the previous section, we formalized the optimization model for the design of a policy where the premium is paid in a single installment at issuance. However, the policyholder usually does not pay for the policy by a single premium but rather in a series of periodic premiums. We now turn to the determination of these periodical premiums.

We consider the case where the premiums are paid annually and remain constant throughout the duration of the policy. Moreover, we consider that the single premium obtained by solving (25) equals the discounted value of the annual premiums with respect to the surviving probabilities. Denoting by P^A_x the constant annual premium, we have

$$P^x = \sum_{t=1}^{T-1} p_x(t, t-1)\pi\left(P^A_x\right) \tag{26}$$

where, $\pi(P^A_x) = E^Q\left[e^{rt}P^A_x\right]$. Taking into account the properties of the expectation operator, and after performing algebraic operations, the above equation can be rewritten as:

$$P^x = P^A_x \sum_{t=1}^{T-1} p_x(t, t-1)e^{rt}. \tag{27}$$

Finally, by solving Equation (23) for the annual premium P^A_x we get

$$P^A_x = \frac{P^x}{\sum_{t=1}^{T-1} p_x(t, t-1)e^{rt}}. \tag{28}$$

4.4. **Parameters identification.** We now explain our approach to estimating the values of the parameters for our model. The parameters are determined by several factors such as financial market conditions and the preferences of insurers and policyholders.

We consider that the following elements are determined at the discretion of the insurer: the age of the group of people to which the company wants to offer the policy (x) and the discount rates that the company will use to calculate the surrender value (SV_t). As for the rate of return and volatility of the reference portfolio, these are affected by the condition of the financial market and the portfolio's asset allocation, which in turn depend on the insurer's risk preference. The insurer can estimate these parameters based on his own financial statements and other available financial information.

Further, we consider that the initial benefit (C_0) and the maturity time of the contract (T) depend on preferences of the policyholder and insurer, and usually they are negotiated at the inception of the policy.

As for the risk-free rate, in practice most companies and academics use short-dated government bonds as the risk-free rate (r); consequently, the appropriate rate used in the model depends on the relevant currency. For example, for USD investments, US Treasury Bills are typically used, while a common choice for EUR investments are German government bills or Euribor rates.

Another important parameter we need to determine in our model is the number of policyholders (n^x), which shows the demand for the policy. Usually, the demand depends on the overall features of the policy; however, to simplify our analysis we assume that the demand follows a linear decreasing function of the premium, i.e., the higher the premium, the less people will buy the policy. Accordingly, let the demand for a policy be expressed by

$$n^x = a^x - b^x P^x, \tag{29}$$

where a^x and b^x are positive numbers.

Suppose that the company can estimate the percentage of the target market that will buy the policy when the premium is very competitive, denote this percentage by δ^x.

Furthermore, suppose that if the premium is equal or greater than the value of the expected benefits at maturity, i.e., C_T, discounted at the risk free rate (r), the demand for the policy will be 0. If we denote such a premium by P_u^x then

$$P_u^x = \frac{C_T}{(1+r)^T}, \tag{30}$$

where C_T is given by (7).

Let the number of individuals in the target market be N^x. Then, since the fair value of a policy is the lowest premium the company is willing to receive, we can assume that n^x equals $\delta^x N^x$ when P^x is set to F_s^x.

Based on these information, we can estimate a^x and b^x from solving the following equations:

$$0 = a^x - b^x P_u^x, \tag{31}$$

$$\delta^x N^x = a^x - b^x F_s^x, \tag{32}$$

consequently,

$$b^x = \frac{\delta^x N^x}{P_u^x - F_s^x}, \quad a^x = b^x \times P_u^x.$$

Example 4.1. *A life insurance company is designing a policy for individuals of age 30, i.e.,* $x = 30$. *Then, by conducting a survey of a group of people, representative of the population that is 30 year old, the company observes that when the premium is equal to the fair value of a policy, say 100, the percentage of the group that would buy it is 90%, that is* $\delta^{30} = 90\%$. *Furthermore, suppose that when buying the policy the policyholder estimates that the present value of the benefit he will receive at maturity is 175, i.e.,* $P_u^{30} = 175$.

Now, with these results the analyst is able to set the coefficients of the demand in Equation (20), $b^{30} = N^{30} * 90\%/75 = 0.012N^{30}$, $a^{30} = 175 \times 0.012N^{30} = 2.1N^{30}$, *and consequently,*

$$n^{30} = (2.1 - 0.012P^{30})N^{30}.$$

4.5. **Solution of the model.** The model proposed in Section 4.2 is a complicated non-linear optimization model, difficult to solve using conventional optimization techniques. Therefore, we make use the soft approach put forward in [12] to find the solution of this model.

The soft approach was proposed for solving complicated optimization models where obtaining an optimal solution is practically impossible, it has been applied successfully in solving complicated portfolio optimization problems [17], portfolio rebalancing problems [18] and in the design of life insurance policies [11].

The idea behind the soft approach is that a model is solved when a good enough solution is obtained with a high probability, i.e., a solution is acceptable if it is highly likely to be good enough. In order to define a good enough solution, the soft approach does not use cardinal performance, but rather the order of a solution in the feasible set. Furthermore, the solution of the model depends on the specifications of the model.

Example 4.2. *Suppose that the requirements of the optimization model defines that the top* 1% *solutions are good enough and* 98% *is a high probability. Therefore, the model is solved when a solution is found in the top* 1% *with a probability higher than* 98%.

In order to produce a solution the soft approach follows two stages:

- Stage 1: Sample the feasible set in order to generate a finite subset S, which contains with a high probability at least one good enough condition. Define $G \subset Z$ as the set of good enough solutions, and k as a high probability, then we have
$$Pr\{\mid S \cap G \mid \geq 1\} \geq k\%. \tag{33}$$
Therefore, the best alternative in S will be a good enough solution with a probability greater than $k\%$.
- Stage 2: Calculate the performance of each sample. Then, the sample with the highest performance is the solution we are looking for.

Stage 2 can become time consuming if the number of samples needed to generate S is too high. However, in order to satisfy the requirements the model need to satisfy, there is no need to generate a large number. We illustrate this with the following example.

Example 4.3. *Generate* S *by taking* $10,000$ *uniform samples from the feasible set* Z*. The probability that* S *does not contain one of the top* 0.1% *good enough solutions is* $p = (1 - 0.1\%)^{10,000} = 0.0045\% < 0.1\%$. *Consequently,* $Pr\{|S \cap G| \geq 1\} = 1 - p > 99.9\%$. *This means that a set with 10,000 uniform samples satisfies (33) when G is the set of top 0.1% solutions and* 99.9% *is taken as high probability.*

Suppose that producing a top 0.1% solution with a probability greater than 99.9% is adequate when designing a participating policy. Then, from example it follows that we need only 10,000 uniform samples in Stage 1.

Define the feasible set of model (25) as:

$$Z_x^f = \left\{ (P^x, g_t^x, \alpha^x) \mid F_s^x \leq P^x \leq P_u^x, g_{t,l}^x \leq g_t^x \leq g_{t,u}^x, \alpha_l^x \leq \alpha^x \leq \alpha_u^x \right\}. \tag{34}$$

Let θ_1^i, θ_2^i, θ_3^i be numbers generated uniformly between 0 and 1. Then, by using interpolation

$$\begin{aligned} P_i^x &= F_s^x + \theta_1^i \times (P_u^x - F_s^x) \\ g_{t,i}^x &= g_{t,l}^x + \theta_2^i \times \left(g_{t,u}^x - g_{t,l}^x\right) \\ \alpha_i^x &= \alpha_l^x + \theta_3^i \times (\alpha_u^x - \alpha_l^x) \end{aligned}$$

we can get a uniform sample $\left(P_i^x, g_{t,i}^x, \alpha_i^x\right)$ in set Z_x^f.

Once all the samples are generated, we calculate the profit of each sample. Finally, the solution of the model is the sample that produced the highest profit.

5. **Computational Experiments.** In this section, we present the results from the computational experiments of the model.

Let us first define the parameters we used. We fixed, $x = 40$, $C_0 = 10$, $T = 6$, $r = 3\%$, $R = 15\%$, $\gamma = 85\%$, $N^{40} = 10,000$, $\delta^{40} = 85\%$, $\sigma = 15\%$. We use data extracted from the U.S. actuarial life tables for males (as for 2002) provided in the official website of the United States social security administration.

β_t is given by

$$\beta_t = \begin{cases} 0\%, & t \in [0,2) \\ 60\%, & t \in [2,4) \\ 80\%, & t \in [4,6] \end{cases}$$

This function is consistent with insurer's desire to discourage the policyholder from terminating the policy, nothing is paid back to the policyholder for at least two years, then gradually the surrender value will increase the longer that the policy is in effect.

As for the surrender probabilities, we consider the following function:

$$\lambda_{40}(t, t-1) = \begin{cases} 0, & t \in [1,2) \\ 10\%, & t \in [2,6] \end{cases}$$

therefore, $\kappa_x(T-1) = \kappa_{40}(5) = 1 - \lambda_{40}(6,5) = 90\%$.

With respect to the guarantee rates, we assume that the minimum guarantee rate will be adjusted every 2 years. In our experiments, we consider two cases. In case 1, the guarantee rate is defined by

$$g_t^{40} = \begin{cases} i_1, & t \in [0,2) \\ i_2 = 2i_1, & t \in [2,4) \\ i_3 = 3i_1, & t \in [4,6] \end{cases}$$

where i_1, i_2, $i_3 \in [0, 10\%]$.

In case 2, guarantee rates are given by

$$g_t^{40} = \begin{cases} i_1, & t \in [0,2) \\ i_2, & t \in [2,4) \\ i_3, & t \in [4,6] \end{cases}$$

where $i_1 \in [0, 10\%]$, $i_2 \in [i_1, 10\%]$ and $i_3 \in [i_2, 10\%]$. Note that in both cases the upper endpoint of the guarantees is set to 10%. We consider that this is an appropriate bound since insurance policies usually offer guarantees lesser than 10%.

Table 1. Values of the guarantee rates in case 1

g_t^{40}	
$t \in [1,2)$	0.47%
$t \in [2,4)$	0.95%
$t \in [4,6)$	1.42%

Table 2. Values of the guarantee rates in case 2

g_t^{40}	
$t \in [1,2)$	0.73%
$t \in [2,4)$	2.53%
$t \in [4,6)$	2.54%

Following the soft approach stages, we first generate 10,000 samples and estimate P_u^x for each sample as discussed in Section 4.4. Then, we calculate the fair value and profit produced by each sample. Finally, we solve the model by choosing the sample that produced the highest profit. The results are reported in Tables 1 to 4. Table 1 and Table 2 show the values obtained for the guarantees in case 1 and case 2, respectively. In Tables 3 and 4, we present the values of the rest of the variables obtained in case 1 and case 2, respectively.

From the results reported in Table 1 and Table 2, it may seem that those in case 2 are better than those obtained in case 1. However, in Table 3 and Table 4, we observe that the participating rate in case 2 is lower than that in case 1 by 2.37%. Further, note that P_t^A in case 2 is approximately lower than that in case 1 by 2%. Based on this, we can infer that, higher guarantees will be offset by lower participating rates, however, the premium of the policy will not be significantly different.

Table 3. Values of the participating policy in case 1

Variable	Value
α^{40}	98.6%
P_{40}^A	2.33
F_S^{40}	12.03
P_u^{40}	19.37

Table 4. Values of the participating policy in case 2

Variable	Value
α^{40}	96.23%
P_{40}^A	2.28
F_S^{40}	12.07
P_u^{40}	18.81

Furthermore, our results show that, in order to maximize the profit and keep the policy attractive to the market, it is optimal for the insurer to keep high levels for the participation rate and low levels for the minimum guaranteed rates, specially in the first years of the policy. This is because higher guarantees are more difficult to meet and represent higher risk for the company. However, in order to keep the policy attractive, the company needs to compensate the policyholder by offering higher participation rates. This is in line with what we have seen in the insurance industry in recent years.

With low interest rates being predominant over the last decades, insurers across the world have been ordered to reduce their maximum guaranteed interest rates for new contracts, in an effort to reduce their exposure to the threat of insolvency due to interest guarantees. For example, in Belgium the maximum guaranteed interest rates were reduced from 4.74% before January 1999 to 3.75% in July 2000, and have remained constant as of 2007. In Netherlands, guarantees before 1998 were 4.0%, and after July 2000 they were reduced to 3% and remained unchanged as of 2007. Sweden guaranteed interest rate was 4% before 1998, 3% until April 2004 and 2.75% from February 2005. Table 5 reports the evolution over the past years of the maximum guaranteed interest rates in some member countries of the European Union.

The results of our experiments confirm that this tendency of offering low guarantee rates it is best for insurers in order to avoid the risk of not being able to meet them. Furthermore, the results suggest that, when dealing with variable guaranteed rates, insurers could reduce even more the guarantees particularly at the beginning of the policy. Our computations are reasonable and consistent with the insurance practice, which show that the proposed optimization model and solution method is suitable for the design of the life insurance participating policy.

TABLE 5. Maximum guaranteed rates

Country	Maximum Rate		Country	Maximum Rate	
Austria	3.0%	until January 1, 1995	Spain	3.0%	until June 21, 1997
	4%	until July 1, 2000		3.15%	until July 1, 2000
	3.25%	until January 1, 2004		2.68%	as of January 1, 2004
	2.75%	until January 1, 2006		2.42%	as of February 15, 2005
	2.25%	as of January 1, 2007		2.14%	as of 2006
Belgium	4.75%	until January 1, 1999	Netherlands	4%	before 1998
	3.75%	as of January 1, 2007		3%	as of January 1, 2007
Denmark	5%	until July 1, 1994	Norway	4%	until November, 1993
	3%	until January 1, 1999		3%	as of April, 2004
	2%	as of January 1, 2007		2.75%	from 2006
Germany	4%	until July 1, 2000	Luxembourg	3.75%	before 1998
	3.25%	until January 1, 2004		2.75%	as of July 1, 2000
	2.75%	before January 1, 2007		2.5%	as of April 15, 2004
	2.25%	as of January 1, 2007		2.25%	from April 1, 2005
Italy	2.5%	from September 1, 1999	Sweden	4%	before 1998
	3%	from May 1, 2000		3%	as of April 15, 2004
	2.5%	from December 1, 2003		2.75%	from February 15, 2005
	2%	from January 1, 2006			

6. **Conclusions and Remarks.** In this paper, we have proposed an approach for designing a participating life insurance policy with annual premiums, minimum guarantees that vary over time, and a surrender option. Our approach presents two main differences as compared to past works. Firstly, the features of the policy under scrutiny are different (i.e., annual premiums, variable guarantees and surrender option), which make the policy more similar to real insurance products. Secondly, while most of previous research deals with the evaluation of the value of an insurance policy for different levels of contractual rates, in this paper we take a step forward and design a policy by determining one set of optimal contractual rates and annual premium that maximizes the insurer's profit, while considering the restrictions on the demand of the policy.

By addressing these issues, this paper has contributed to fill some important gaps in the literature regarding the design of insurance policies. We hope that the approach proposed will help practitioners in the insurance business to identify the best design of a participating policy that satisfy their own interest as well as meet market limitations.

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