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DECISION-MAKING WITH UNCERTAIN AGGREGATION OPERATORS USING THE DEMPSTER-SHAFER BELIEF STRUCTURE

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Abstract. *We develop a new decision-making model using the Dempster-Shafer (D-S) belief structure when available information is uncertain and can be assessed with interval numbers. We use a wide range of aggregation operators involving interval numbers such as the uncertain weighted average (UWA), the uncertain ordered weighted average (UOWA), the uncertain generalized weighted average (UGWA) and the uncertain generalized ordered weighted average (UGOWA). We present a new approach to using interval weights in these uncertain aggregation operators. By using these aggregation operators within a D-S framework, we obtain various belief structures (BS), including the UWA (BS-UWA), the BS-UOWA, the BS-UGWA and the BS-UGOWA. We also use more complete formulations by using induced, hybrid and quasi-arithmetic aggregation operators. We end the paper by applying these operators to a decision-making problem regarding strategic management.*

Keywords: Decision-making, OWA operator, Dempster-Shafer theory of evidence, Interval numbers

1. **Introduction.** The Dempster-Shafer (D-S) theory of evidence [1,2] provides a unifying framework for representing uncertainty because it includes situations of risk and ignorance (also known as uncertainty) as special cases. Since its appearance, D-S theory has been studied in a wide range of contexts [3-10].

Usually, when using D-S theory, it is assumed that the available information is clearly known and can be assessed with exact numbers [3-5,7-10]. However, this may not be the real-life situation found in the decision-making problems [11-23] because often times available information is vague or imprecise, or it is not possible to analyze the situation with exact numbers. In this case, a better approach may be the use of interval numbers because this approach considers the best and worst possible scenarios as well as the most likely results that will occur. When using interval numbers, we follow the ideas presented in [24].

To aggregate information in a decision-making process, we must use an aggregation operator such as the weighted average (WA), the ordered weighted average (OWA) and the hybrid average (HA). Note that the OWA operator [25] is an aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum values. For further research on the OWA operator, see [26-42]. The HA operator [43] is an aggregation operator that uses the WA and the OWA in the same formulation. Since its appearance, it has been studied by various authors [44-46].

In the context of uncertain environments, these aggregation operators become the uncertain WA (UWA), the uncertain OWA (UOWA) operator [37] and the uncertain HA

(UHA) operator [17]. Furthermore, it is possible to provide a more general formulation of the previous aggregation operators by using generalized and quasi-arithmetic means. Thus, we obtain the uncertain generalized WA (UGWA), the uncertain weighted quasi-arithmetic mean (Quasi-UWA) [33], the uncertain generalized OWA (UGOWA), the uncertain quasi-arithmetic OWA (Quasi-UOWA) operator [33], the uncertain generalized HA (UGHA) and the uncertain quasi-arithmetic HA (Quasi-UHA) operator [17].

The aim of this paper is to present a new decision-making model using the D-S belief structure that addresses uncertain environments with interval numbers. To do so, we use a wide range of uncertain aggregation operators such as those discussed above, obtaining various belief structures, including the UWA belief structure (BS-UWA), the UOWA belief structure (BS-UOWA) and the BS-UHA aggregation. We study some of the main properties of these structures as well as particular cases such as the uncertain average belief structure (BS-UA), the BS-step-UOWA, the BS-olympic-UOWA and the BS-centered-UOWA operator. The main advantage of this approach against the classical models is that we can assess complex environments where the information is very uncertain and cannot be treated with the usual exact numbers but it is possible to use interval numbers, By using interval numbers, we can consider the maximum and the minimum result that may occur. Thus, we can represent the uncertain information knowing at least the best and the worst situation that could happen. Sometimes, the interval numbers also provide the most possible results. That is, accepting that the information is uncertain, the exact result that we think is going to happen.

We also present a new approach to using interval weights. Our method is focused on the normalization process of the initial weighting vector so that final results are consistent with the usual aggregations. That is, the sum of the weighting vector must be equal to one. We also discuss some other potential improvements to this approach.

We further generalize this approach by using uncertain generalized aggregation operators, obtaining the BS-UGWA, the BS-UGOWA, the BS-UGHA, the BS-Quasi-UWA, the BS-Quasi-UOWA and the BS-Quasi-UHA operators. Furthermore, we use uncertain induced generalized aggregation operators, such as the uncertain induced generalized OWA (UIGOWA) operator [33], the uncertain induced quasi-arithmetic OWA (Quasi-UIOWA) operator [33], the uncertain induced generalized HA (UIGHA) [17] and the Quasi-UIHA operator [17]. Thus, we obtain the BS-UIGOWA, the BS-Quasi-UIOWA, the BS-UIGHA and the BS-Quasi-UIHA operators. The main advantage of these generalizations is that they include a wide range of particular cases such as the BS-UIOWA, the BS-UIOWQA, the BS-UIOWGA and the BS-UIOWHA operators. Thus, we can consider a wide range of scenarios and select the one that is in closest accordance with our interests.

We also apply the proposed decision-making approach to a problem in strategic management. We consider an enterprise that is looking for an optimal strategy and must develop a decision-making process to select a strategy. We see that depending on the particular type of aggregation operator used, the results and decisions may be different. Note that it is possible to develop a lot of other applications in problems where we use the probability or the weighted average.

This paper is organized as follows. In Section 2, we briefly describe some basic concepts regarding uncertain aggregation operators. Section 3 presents a new method for using interval weights. Section 4 presents the proposed decision-making approach with D-S theory. In Sections 5 and 6, we present further generalizations of the suggested approach by using hybrid and generalized aggregations. Finally, in Section 7, we apply the proposed model to a decision-making problem, and in Section 8, we summarize the main conclusions of the paper.

2. Preliminary Concepts.

In this section, we briefly review some basic concepts that will be used throughout the paper. We discuss interval numbers, the UWA, the UOWA operator, the UHA operator, the UGWA, the UGOWA operator and the UGHA operator.

2.1. Interval numbers.

Interval numbers [24] provide a very useful and simple technique for representing uncertainty. They have been used in an astonishingly wide range of applications and can be defined as follows.

Definition 2.1. *Let $a = [a_1, a_2] = \{x|a_1 \leq x \leq a_2\}$, then, a is called an interval number. Note that a is a real number if $a_1 = a_2$.*

The interval numbers can be expressed in different forms. For example, assume a 4-tuple $[a_1, a_2, a_3, a_4]$, that is to say, a quadruplet, and let a_1 and a_4 represent the minimum and the maximum of the interval number, respectively, and a_2 and a_3 represent the interval with the highest probability or possibility, depending on how we plan to use the interval numbers. Note that $a_1 \leq a_2 \leq a_3 \leq a_4$. If $a_1 = a_2 = a_3 = a_4$, then the interval number is an exact number. If $a_2 = a_3$, it is a 3-tuple known as triplet, and if $a_1 = a_2$ and $a_3 = a_4$, it is a simple 2-tuple interval number.

In the following section, we review some basic interval number operations. Let A and B be two triplets, where $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$. Then:

1. $A + B = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.
2. $A - B = [a_1 - b_3, a_2 - b_2, a_3 - b_1]$.
3. $A \times k = [k \times a_1, k \times a_2, k \times a_3]$, for $k > 0$.
4. $A \times B = [a_1 \times b_1, a_2 \times b_2, a_3 \times b_3]$, for R^+.
5. $A \times B = [\min(a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3), a_2 \times b_2, \max(a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3)]$, for R.
6. $A \div B = [a_1 \div b_3, a_2 \div b_2, a_3 \div b_1]$, for R^+.
7. $A \div B = [\min(a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3), a_2 \div b_2, \max(a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)]$, for R.

Note that the ranking of the arguments is difficult because we are using interval numbers. In some cases, it is not clear which interval number is higher, so we must establish an additional criterion for ranking the interval numbers. For simplicity, we recommend the following criteria. For 2-tuples, calculate the arithmetic mean of the interval, with $(a_1 + a_2)/2$. For 3-tuples and above, calculate a weighted average that yields more importance to the central values. That is, for 3-tuples, $(a_1 + 2a_2 + a_3)/4$. For 4-tuples, we calculate: $(a_1 + 2a_2 + 2a_3 + a_4)/6$, and so on. In the case of a tie between the intervals, we select the interval with the lowest difference, i.e., $(a_2 - a_1)$. For 3-tuples and above odd-tuples, we select the interval with the highest central value. Note that for 4-tuples and above even-tuples, we must calculate the average of the central values following the initial criteria.

The main advantage of this method is that we can reduce the interval number into a representative and exact number of the interval. In this manner, we can always establish a ranking of the interval numbers. To understand the usefulness of this method, we present a simple example.

Example 2.1. *Assume we want to rank the following interval numbers: $A = (20, 40, 50)$, $B = (25, 35, 45)$ and $C = (32, 37, 43)$. Initially, it is not clear which is higher. Obviously, we can use a wide range of methods depending on the importance we want to give to each part of the interval. As explained before, we assume in this paper a ranking based on $(a_1 + 2a_2 + a_3)/4$. Thus, we assume that the central value is more important than the extreme values. This seems reasonable because the central value has the highest degree of* *occurrence. We convert the triplet to exact numbers using the method discussed above. Thus:*

$$A = (20 + 2 \times 40 + 50)/4 = 37.5.$$
$$B = (25 + 2 \times 35 + 45)/4 = 35.$$
$$C = (32 + 2 \times 37 + 43)/4 = 37.25.$$

With these results, we can reorder the interval numbers such that $A > C > B$.

Note that other operations and ranking methods could be studied [24] but in this paper, we focus on those discussed above.

2.2. **The uncertain weighted average.** The uncertain weighted average (UWA) is an extension of the weighted average for situations in which the available information is uncertain and can be assessed using interval numbers. Its main advantage is that it represents information in a more complete way because it considers the maximum and minimum results that may occur in an uncertain environment. The UWA can be defined as follows.

Definition 2.2. *Let Ω be the set of interval numbers. An UWA operator of dimension n is a mapping UWA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_i \in [0, 1]$ and $\sum_{i=1}^{n} \tilde{w}_i = 1$ such that:*

$$UWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{i=1}^{n} \tilde{w}_i \tilde{a}_i, \tag{1}$$

where $\tilde{a}_i$ is an interval number.

Note that in Section 3, we explain how to address uncertain weights $\tilde{w}_i$ in order that the sum of these weights is equal to 1. The same condition applies for the UOWA, UHA, UGWA, UGOWA, UGHA and UIGOWA operators explained in the following subsections. Note also that if $w_i = 1/n$, for all i, then, the UWA operator becomes the uncertain average (UA).

2.3. **The uncertain ordered weighted averaging operator.** The UOWA operator was introduced by Xu and Da [37] and it represents an extension of the OWA operator. The main difference is that it uses interval numbers in the arguments that will be aggregated. The reason for using this aggregation operator is that sometimes, an environment is very uncertain, and information is not clear. In this case, a better approach may be to use interval numbers. The UOWA operator provides a parameterized family of aggregation operators that include the uncertain maximum, the uncertain minimum and the UA, among others. It can be defined as follows.

Definition 2.3. *Let Ω be the set of interval numbers. An UOWA operator of dimension n is a mapping UOWA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with the following properties:*
1) $\sum_{j=1}^{n} \tilde{w}_j = 1$
2) $\tilde{w}_j \in [0, 1]$
and such that:

$$UOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j, \tag{2}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, and each $\tilde{a}_i$ is an interval number.

Observe that we denote the usual function f as UOWA to distinguish it from other types of OWA operators that are analyzed throughout this paper. Additionally, the function f for other types of OWA operators is also studied using a similar notation.

2.4. **The uncertain hybrid averaging operator.** The uncertain hybrid averaging (UHA) operator is an extension of the HA operator [43] that uses uncertain information represented in the form of interval numbers. It involves the same formulation as the uncertain weighted average (UWA) and the UOWA operator. With this operator, we can represent subjective probability and attitudinal character of a decision maker in the same problem. The main advantage of the operator is that it can represent uncertain situations that cannot be assessed with exact numbers or singletons, as it is possible to use interval numbers. The decision maker can thus obtain a more complete view of the decision problem.

Definition 2.4. *Let Ω be the set of interval numbers. An UHA operator of dimension n is a mapping UHA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_j \in [0,1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ such that:*

$$UHA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j, \tag{3}$$

where $\tilde{b}_j$ is the jth largest of the $\hat{a}_i$, where $\hat{a}_i = n\tilde{\omega}_i\tilde{a}_i$, $i = 1, 2, \cdots, n$, $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)$ is the weighting vector of the $\tilde{a}_i$, where $\tilde{\omega}_i \in [0,1]$ and the sum of the weights is 1, and each $\tilde{a}_i$ is an interval number.

Note that it is possible to distinguish between the descending UHA (DUHA) and the ascending UHA (AUHA) operators. The UHA operator is monotonic. It is not bounded by the maximum and the minimum because there may be situations in which aggregation yields higher and lower results than the maximum and the minimum, respectively. Therefore, neither it is idempotent. We can prove that hybrid aggregations are not idempotent nor bounded by the minimum and maximum using the following example. We consider a simple example with exact numbers.

Example 2.2. *Assume $w_1 = 1$, $w_2 = 0$, $\omega_1 = 1$, $\omega_2 = 0$ and $a_1 = a_2 = 20$. If this example fulfils the idempotency property, we should obtain a final result of 20. But instead, we obtain the following:*

$$HA = 1 \times (1 \times 2 \times 20) + 0 \times (0 \times 2 \times 20) = 40.$$

Thus, it is clear that in general, hybrid aggregations are not idempotent and bounded by the minimum and the maximum.

Different families of UHA operators use different manifestations of the weighting vector, including the UA, the UWA, the UOWA, the step-UHA, the olympic-UHA, the median-UHA, the centered-UHA and so on. For more information, see [17,33,38].

2.5. **The uncertain generalized weighted average.** The uncertain generalized weighted average (UGWA) is an aggregation operator that generalizes the UWA operator by using generalized means. Thus, it includes a wide range of particular cases, such as the UWA, the uncertain weighted geometric average (UWGA), the uncertain weighted quadratic average (UWQA) and the uncertain weighted harmonic average (UWHA).

Definition 2.5. *Let Ω be the set of interval numbers. An UGWA operator of dimension n is a mapping UGWA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n with $\sum_{i=1}^{n} \tilde{w}_i = 1$ and $\tilde{w}_i \in [0,1]$ such that:*

$$UGWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\sum_{i=1}^{n} \tilde{w}_i \tilde{a}_i^{\lambda} \right)^{1/\lambda}, \tag{4}$$

where each $\tilde{a}_i$ is an interval number, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

If $\lambda \leq 0$, we can only use positive numbers in R^+; otherwise, we cannot obtain consistent results. Furthermore, it is also possible to use interval numbers for λ in an UGWA operator.

Moreover, we can also develop a further generalization of the UGWA operator by using quasi-arithmetic means. The result is the Quasi-UWA operator, and it is defined as follows.

Definition 2.6. *Let Ω be the set of interval numbers. A Quasi-UWA operator of dimension n is a mapping QUWA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n with $\sum_{i=1}^{n} \tilde{w}_i = 1$ and $\tilde{w}_i \in [0,1]$ such that:*

$$QUWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = g^{-1} \left(\sum_{i=1}^{n} \tilde{w}_i g(\tilde{a}_i) \right), \tag{5}$$

where each $\tilde{a}_i$ is an interval number, and $g(\tilde{a})$ is a strictly continuous monotone function such that g: $I \rightarrow R$.

Note that the function g can also be studied by using interval numbers (Ω), fuzzy numbers (Ψ), linguistic variables (S) and more complex structures. For simplicity, we focus on the usual notation explained in Definition 2.6.

2.6. **The uncertain generalized OWA operator.** The uncertain generalized OWA (UGOWA) operator [17,33] is an extension of the GOWA operator that uses uncertain information in aggregation in the form of interval numbers. By using interval numbers, we obtain a more complete aggregation operator that considers the maximum and minimum values that could occur in a problem. Moreover, by using the UGOWA, we obtain a generalization that includes a wide range of aggregation operators, such as the uncertain maximum, the uncertain minimum, the UA and so on. It can be defined as follows.

Definition 2.7. *Let Ω be the set of interval numbers. An UGOWA operator of dimension n is a mapping UGOWA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} \tilde{w}_j = 1$ and $\tilde{w}_j \in [0,1]$ and such that:*

$$UGOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j^{\lambda} \right)^{1/\lambda}, \tag{6}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, each $\tilde{a}_i$ is an interval number, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

If $\lambda \leq 0$, we can only use positive numbers in R^+ to obtain consistent results, and it is possible to distinguish between the descending UGOWA and the ascending UGOWA operators.

The UGOWA can be further generalized by using quasi-arithmetic means, thereby obtaining the Quasi-UOWA operator [17,33]. It can be defined as follows.

Definition 2.8. *Let Ω be the set of interval numbers. A Quasi-UOWA operator of dimension n is a mapping QUOWA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} \tilde{w}_j = 1$ and $\tilde{w}_j \in [0,1]$ and such that:*

$$QUOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = g^{-1}\left(\sum_{j=1}^{n} \tilde{w}_j g(\tilde{b}_j)\right), \tag{7}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, each $\tilde{a}_i$ is an interval number and $g(b)$ is a strictly continuous monotone function such that g: $I \to R$.

2.7. **The uncertain generalized hybrid averaging operator.** The uncertain generalized hybrid averaging (UGHA) operator [17] is an extension of the GHA operator [47] that uses uncertain information represented in terms of interval numbers. It uses the same formulation as the UGWA and UGOWA operators. With this operator, we can represent subjective probability and attitudinal character of a decision maker in the same problem. The main advantage is that it can represent uncertain situations that cannot be assessed with exact numbers or singletons, as it is possible to use interval numbers. Thus, the decision maker obtains a more complete view of the decision problem.

Definition 2.9. *Let Ω be the set of interval numbers. An UGHA operator of dimension n is a mapping UGHA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_j \in [0,1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ such that:*

$$UGHA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j^{\lambda}\right)^{1/\lambda}, \tag{8}$$

where $\tilde{b}_j$ is the jth largest of the $\hat{a}_i$, where $\hat{a}_i = n\tilde{\omega}_i\tilde{a}_i$, $i = 1, 2, \cdots, n$, $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)$ is the weighting vector of the $\tilde{a}_i$, with $\tilde{\omega}_i \in [0,1]$, the sum of the weights is 1, each $\tilde{a}_i$ is an interval number, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that it is possible to distinguish between the descending UGHA (DUGHA) and the ascending UGHA (AUGHA) operators.

The UGHA operator can be further generalized by using quasi-arithmetic means. The result is the Quasi-UHA operator. It can be defined as follows.

Definition 2.10. *Let Ω be the set of interval numbers. A Quasi-UHA operator of dimension n is a mapping QUHA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_j \in [0,1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ such that:*

$$QUHA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = g^{-1}\left(\sum_{j=1}^{n} \tilde{w}_j g(\tilde{b}_j)\right), \tag{9}$$

where $\tilde{b}_j$ is the jth largest of the $\hat{a}_i$, with $\hat{a}_i = n\tilde{\omega}_i\tilde{a}_i$, $i = 1, 2, \cdots, n$, $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)$ is the weighting vector of the $\hat{a}_i$, with $\tilde{\omega}_i \in [0,1]$, the sum of the weights is 1, each $\hat{a}_i$ is an interval number, and $g(b)$ is a strictly continuous monotone function such that g: $I \to R$.

2.8. **Uncertain induced generalized aggregation operators.** The uncertain induced generalized OWA (UIGOWA) operator [33] is an extension of the IGOWA operator [35] that uses uncertain information in aggregation in the form of interval numbers. The reason for using this operator is that sometimes, the uncertain factors that affect decisions are not clearly known. To assess the problem at hand, we must use interval numbers. This operator uses a reordering process based on order-inducing variables to assess complex reordering processes. It can be defined as follows.

Definition 2.11. *Let Ω be the set of interval numbers. An UIGOWA operator of dimension n is a mapping UIGOWA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_j \in [0,1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ such that:*

$$UIGOWA(\langle u_1, \tilde{a}_1\rangle, \langle u_2, \tilde{a}_2\rangle, \cdots, \langle u_n, \tilde{a}_n\rangle) = \left(\sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j^{\lambda}\right)^{1/\lambda}, \tag{10}$$

where $\tilde{b}_j$ equals $\tilde{a}_i$ in the UIGOWA pair $\langle u_i, \tilde{a}_i\rangle$ having the jth largest u_i, u_i is the order-inducing variable, $\tilde{a}_i$ is the argument variable represented terms of interval numbers, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

The UIGOWA can be further generalized by using quasi-arithmetic means. The result is the Quasi-UIOWA operator [33]. It can be defined as follows.

Definition 2.12. *Let Ω be the set of interval numbers. A Quasi-UIOWA operator of dimension n is a mapping QUIOWA: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\tilde{w}_j \in [0,1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ such that:*

$$QUIOWA(\langle u_1, \tilde{a}_1\rangle, \langle u_2, \tilde{a}_2\rangle, \cdots, \langle u_n, \tilde{a}_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} \tilde{w}_j g(\tilde{b}_j)\right), \tag{11}$$

where $\tilde{b}_j$ equals $\tilde{a}_i$ in the Quasi-UIOWA pair $\langle u_i, \tilde{a}_i\rangle$ with the jth largest u_i, u_i is the order-inducing variable, $\tilde{a}_i$ is the argument variable represented in terms of interval numbers, and $g(b)$ is a strictly continuous monotone function such that g: $I \to R$.

When $g(b) = b^{\lambda}$, we obtain the UIGOWA operator. Moreover, both the UIGOWA and the Quasi-UIOWA operators include a wide range of particular cases such as the UIOWA operator, the uncertain induced ordered weighted geometric averaging (UIOWGA) operator, the uncertain induced ordered weighted quadratic averaging (UIOWQA) operator and the uncertain induced ordered weighted harmonic averaging (UIOWHA) operator.

Furthermore, it is also possible to consider a hybrid version of the UIGOWA and Quasi-UIOWA operators. The result is the UIGHA and Quasi-UIHA operators introduced in [17].

3. **A Method for Addressing Uncertain Weights in Uncertain Aggregation Operators.** In this section, we introduce a new method for addressing uncertain weights in aggregation problems in which information is uncertain. Note that this approach is just a method. Therefore, it is useful in some specific situations when the assumptions given below are accomplished. However, other methods could be considered but in this paper we simply want to introduce a method that is useful when we make some specific assumptions. This method is a particular case of a more general approach in which we assume a neutral attitude towards uncertainty. Note that in a more general framework,

we could consider methods that give more importance to the higher values or to the lower ones. The main characteristics addressed here are as follows:

- The weighting vector must be normal; that is, it must sum to 1. If not, we must normalize it to be consistent with usual aggregation methods that sum to 1. However, in other methods it is possible to relax this condition, but then we do not have usual aggregation but rather heavy aggregation [32] in which the sum of the weighting vector is not 1.
- In order to assess the weighting vector, we must have a method for converting the interval numbers into exact numbers. Note that we employ the method discussed in Section 2.1. although it is possible to use other approaches. If we do not use a conversion method, we are not able to prove that the final result obtained for the weighting vector sums to 1.
- We always address interval weights (and interval arguments) to provide the most complete information during the aggregation process. We only reduce the interval weights into exact numbers to normalize the information and to obtain the final result, if necessary.
  - Note that another possibility that we could consider is to use the method explained in Section 2.1. to convert interval numbers into exact numbers at the beginning of the aggregation process so that only exact numbers are used during the aggregation process. However, if we choose to do so, then we are not providing the most complete information in the aggregation process because exact numbers are less informative.
- We assume that the initial interval weights are an initial representation. However, during the normalization process, it is possible to be less than or greater than the bounds because the weights were not normalized in the beginning due to uncertainty.
  - The other alternative is to assume that interval weights were already strictly analyzed to ensure that they will not be below or above their bounds. This approach will be studied in future research.

The procedure that we suggest in this paper for interval weights in uncertain aggregation operators can be summarized as follows.

Step 1: Calculate the sum of all weights.

Step 2: Convert the result into a representative exact number. In this paper, we use $a = (a_1 + 2a_2 + a_3)/4$. The result obtained will be the coefficient used for normalizing the initial weighting vector.

Step 3: If a is 1, the initial weights are already normalized. If not, divide all initial weights $\tilde{w}_i$ (for the UWA) or $\tilde{w}_j$ (for the UOWA) by a, that is, $\dfrac{\tilde{w}_i}{a}$ or $\dfrac{\tilde{w}_j}{a}$.

Step 4: Use the weight $\dfrac{\tilde{w}_i}{a}$ or $\dfrac{\tilde{w}_j}{a}$ in the uncertain aggregation process. For example, apply them to Definitions 2.2-2.12.

Step 5: The result obtained in the aggregation process is the final uncertain aggregated result presented in the form of an interval number.

Additionally, it is possible to convert this final result into a representative exact number by setting $a = (a_1 + 2a_2 + a_3)/4$ or another approach for converting interval numbers into exact numbers. A general way to do this that includes the previous method is by using a weighted average $a = \sum_{i=1}^{n} w_i a_i$, where a_i is the ith value of the interval number, and w_i is the weight given to each value of the interval. Note that in the previous method, we have $a = 0.25 \times a_1 + 0.5 \times a_2 + 0.25 \times a_3$. Note also that this weighted average is a general expression that can be used in *Step* 2.

Example 3.1. *Assume the following interval weights for use in an aggregation process:* $W = (w_1 = (0.1, 0.2, 0.3), w_2 = (0.3, 0.4, 0.5), w_3 = (0.4, 0.5, 0.6))$.

Step 1: We calculate the sum of the weighting vector W.

$$W = (0.1, 0.2, 0.3) + (0.3, 0.4, 0.5) + (0.4, 0.5, 0.6) = (0.8, 1.1, 1.4).$$

Step 2: We convert the interval weighting vector into a representative exact number.

$$W = \frac{0.8 + 2 \times 1.1 + 1.4}{4} = 1.1.$$

Step 3: As the sum is not 1, we normalize all initial weights by dividing them by 1.1.

$$w_1 = \frac{(0.1, 0.2, 0.3)}{1.1} = (0.0909, 0.1818, 0.2727).$$
$$w_2 = \frac{(0.3, 0.4, 0.5)}{1.1} = (0.2727, 0.3636, 0.4545).$$
$$w_3 = \frac{(0.4, 0.5, 0.6)}{1.1} = (0.3636, 0.4545, 0.5454).$$

Step 4: Now we use these weights in the uncertain aggregation process.

To prove that these weights are consistent with usual aggregation, we prove that the sum of the new weighting vector is equal to 1. Thus, we calculate the sum of the new weighting vector.

$$\begin{aligned} W^* &= (0.0909, 0.1818, 0.2727) + (0.2727, 0.3636, 0.4545) + (0.3636, 0.4545, 0.5454) \\ &= (0.7272, 1, 1.2727). \end{aligned}$$

Now we convert the interval obtained into a representative exact number, with $(0.7272 + 2 \times 1 + 1.2727)/4 = 1$. As we can see, the sum of the weighting vector is equal to 1 in the particular case that we use a representative exact number.

Moreover, it is worth noting that there are many other methods that could be considered depending on the assumptions we make in this analysis. Especially when the results obtained in the previous approach do not sum to 1 in the central value, we may prefer to consider another approach.

For example, we could use a method that normalizes the weighting vector by considering the value(s) with the highest degree of possibility for an interval distinct from the upper and lower bounds. Thus, the normalization process consists in normalizing the weights w_{2h} using $\dfrac{w_{2h}}{\sum_{h=1}^{m} w_{2h}}$, where w_{2h} is the hth central value of the weighting vector, and the w_{1h} and w_{3h} using $\dfrac{w_{1h}}{\sum_{h=1}^{m}\left(\dfrac{w_{1h}+w_{3h}}{2}\right)}$ and $\dfrac{w_{3h}}{\sum_{h=1}^{m}\left(\dfrac{w_{1h}+w_{3h}}{2}\right)}$. Note that according to this method, we always assume that the sum of the central values of the weighting vector is equal to 1.

These and other methods will be considered in more detail in future research. Note that it is straightforward to implement these methods when using interval probabilities because the probability can be treated as a weighted average.

4. **Uncertain Decision-Making with Dempster-Shafer Theory.** In this section, we present a new decision-making model. First, we briefly describe the main concepts behind the D-S theory. Second, we develop the decision-making process. Third, we study the use of uncertain aggregation operators in a D-S framework. Finally, we study some particular cases for use in analysis.

4.1. **The Dempster-Shafer theory of evidence.** D-S theory provides a unifying framework for representing uncertainty, as it can address situations of risk and ignorance as special cases. Note that the case of certainty is also included, as it can be seen as a particular case of risk and ignorance.

Definition 4.1. *D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for $j = 1, \cdots, n$, called focal elements, and a mapping m, called the basic probability assignment, defined as m: $2^X \to [0, 1]$ such that:*

$$\begin{array}{l} \text{1) } m(B_j) \in [0,1]. \\ \text{2) } \sum_{j=1}^{n} m(B_j) = 1. \\ \text{3) } m(A) = 0,\ \forall A \neq B_j. \end{array} \tag{12}$$

As discussed above, risk and ignorance are included as special cases of belief structure in the D-S framework. For risk, a belief structure is called a Bayesian belief structure if it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. We note that we are in a situation of decision-making under risk environment if $m(B_j) = P_j = \mathrm{Prob}\{x_j\}$.

The case of ignorance occurs when the belief structure consists of only one focal element B, where $m(B)$ comprises all possible states of nature. Thus, $m(B) = 1$. Other special cases of belief structures, such as a consonant belief structure or a simple support function, are studied in [2].

Note that we can use interval numbers in $m(B_j)$. In order to deal with these situations, we recommend the methodology explained in Section 3 for addressing interval weights.

Note also that two important evidential functions associated with these belief structures are the measures for plausibility and belief [2]. We provide definitions of these two measures as developed by Shafer.

Definition 4.2. *The plausibility measure Pl is defined as Pl: $2^X \to [0, 1]$ such that:*

$$\mathrm{Pl}(A) = \sum_{A \cap B_j \neq \varnothing} m(B_j). \tag{13}$$

Definition 4.3. *The belief measure Bel is defined as Bel: $2^X \to [0, 1]$ such that:*

$$\mathrm{Bel}(A) = \sum_{B_j \subseteq A} m(B_j). \tag{14}$$

$\mathrm{Bel}(A)$ represents the exact support to A, and $\mathrm{Pl}(A)$ represents the possible support to A. With these two measures, we can form the interval of support to A as $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$. This interval can be seen as the lower and upper bounds of the probability that A is supported such that $\mathrm{Bel}(A) \leq \mathrm{Prob}(A) \leq \mathrm{Pl}(A)$. From this, we note that $\mathrm{Pl}(A) \geq \mathrm{Bel}(A)$ for all A. Another interesting feature of these two measures is that they are related insofar as either $\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\bar{A})$ or $\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\bar{A})$, where $\bar{A}$ is the complement of A.

4.2. **A decision-making approach.** A new method for decision-making using D-S theory is possible by employing interval numbers (or uncertain aggregation operators) in a given problem. The main advantage of this approach is that we can represent in a more complete way the decision problem because the use of interval numbers allows us to consider the maximum and the minimum values that may occur in the uncertain environment. Note that in the classical approach [3,5,9], we can only consider an exact number, which is usually not enough to represent the environment. Moreover, by using triplets, quadruplets and more complex structures, we can also consider internal results of the interval that represents most probable results in the particular problem under analysis.

The decision process can be summarized as follows. Note that we assume the use of either the UWA or UOWA operator.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \cdots, A_p\}$ with states of nature $\{S_1, \cdots, S_n\}$. $\tilde{a}_{ih}$ is the uncertain payoff, known in the form of interval numbers to the decision maker if he/she selects alternative A_i under state of nature S_h. The knowledge regarding the state of nature is captured in terms of a belief structure m with focal elements $B_1, \cdots, B_r$. Associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative that yields the best result to the decision maker. To do so, we observe the following steps:

Step 1: Express (or calculate) the results of the payoff matrix using interval numbers. In this paper, we assume that these results are given. However, note that to formulate these results, we must analyze each alternative with each state of nature and analyze the result obtained by taking into account all the variables that affect the results under the given alternative.

Step 2: Calculate the belief function m for the states of nature to obtain probabilistic information regarding the problem.

Step 3: Calculate the attitudinal character (or degree of orness) of the decision maker $\alpha(W)$ [25].

Step 4: Calculate the collection of weights w to be used in UWA or UOWA aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the preferences of the decision maker [5,35,38]. Note that for UWA aggregation, we must calculate the weights according to a degree of importance (or subjective probability) of each state of nature. This can be carried out by using the opinion of a group of experts that has some information about the probability that each state of nature will occur. Moreover, note that the weights may be represented in the form of interval numbers. To address these uncertain weights, we recommend using the method explained in Section 3.

Step 5: Determine the uncertain results of the collection M_{ik} if we select alternative A_i and focal element B_k occurs for all values of i and k. Hence, $M_{ik} = \{\tilde{a}_{ih} | S_h \in B_k\}$.

Step 6: Calculate the uncertain aggregated results $V_{ik} = \text{UWA}(M_{ik})$ using Equation (1), and $V_{ik} = \text{UOWA}(M_{ik})$ using Equation (2) for all values of i and k. Note that the UWA operator (or the UOWA operator) aggregates all elements of the set $M_{ik} = \{\tilde{a}_{ih} | S_h \in B_k\}$.

Step 7: For each alternative, calculate the generalized expected value C_i by using the belief function m and the uncertain aggregated results V_{ik}, where:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k). \tag{15}$$

Step 8: Select the alternative with the largest C_i as optimal. Note that in a minimization problem, the optimal choice is the lowest result. Moreover, sometimes the ranking is not clear because we are comparing interval numbers. When ranking interval numbers, we recommend the method explained in Section 2.1 for simplicity.

In terms of the generalized reordering step, it is possible to distinguish between ascending and descending orders in UOWA aggregation.

The procedure under the AUOWA operator during aggregation is the same as the procedure used for the UOWA or DUOWA operator with the following differences.

In *Step 3*, when calculating the degree of optimism (or degree of orness) of the decision maker, the reordering of the arguments is carried out in an ascending way.

In *Step 4*, when calculating the collection of weights, the reordering will be different such that we may be able to associate each weight correctly with its corresponding position.

In *Step 6*, when calculating the aggregated payoff, we let $V_{ik} = \text{AUOWA}(M_{ik})$ for all the values of i and k.

4.3. **Belief structures with uncertain aggregation operators.** By analyzing aggregation in *Steps* 6 and 7 presented in the previous subsection, it is possible to formulate the entire aggregation process in one sentence. We will call this process the belief structure BS-UWA for the UWA operator and the belief structure BS-UOWA for the UOWA operator. They can be defined as follows.

Definition 4.4. *A BS-UWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{h=1}^{q} m(B_k)\tilde{w}_{kh}\tilde{a}_{kh}, \tag{16}$$

where $\tilde{w}_{kh}$ is the weight of the hth state of nature of the kth focal element such that $\frac{1}{r}\sum_{h=1}^{q} \tilde{w}_{kh} = 1$ and $\tilde{w}_{kh} \in [0, 1]$, each $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element and represented in terms of interval numbers, and $m(B_k)$ is the basic probability assignment.

Definition 4.5. *A BS-UOWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{q} m(B_k)\tilde{w}_{kj}\tilde{b}_{kj}, \tag{17}$$

where $\tilde{b}_{kj}$ is the kjth largest of the $\tilde{a}_{kh}$, each $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element and represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q} \tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0, 1]$, and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element, and r is the total number of focal elements.

The BS-UOWA operator is monotonic, commutative, bounded and idempotent. We can prove these properties with the following theorems.

Theorem 4.1 (Commutativity)**.** *Assume f is the BS-UOWA operator. Then,*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr}), \tag{18}$$

where $(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr})$ is any permutation of $(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr})$ for each focal element k.

Proof: Let

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k)\tilde{w}_{jk}\tilde{b}_{jk}, \tag{19}$$

$$f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k)\tilde{w}_{jk}\tilde{b}^*_{jk}. \tag{20}$$

Since $(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{qr})$ is a permutation of $(\tilde{a}_{11}, \cdots, \tilde{a}_{qr})$ for each focal element k, we have $\tilde{b}_{jk} = \tilde{b}^*_{jk}$, and then

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{qr})$$

.

Theorem 4.2 (Monotonicity)**.** *Assume f is the BS-UOWA operator. If $\tilde{a}_{kh} \geq \tilde{a}^*_{kh}$, then*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) \geq f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr}). \tag{21}$$

Proof: Let

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b}_{jk}, \tag{22}$$

$$f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b}^*_{jk}. \tag{23}$$

Since $\tilde{a}_{ik} \geq \tilde{a}^*_{ik}$, $\forall i$, it follows that $b_{jk} \geq \tilde{b}^*_{jk}$, and then

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) \geq f(\tilde{a}^*_{11}, \cdots, \tilde{a}^*_{q1}, \cdots, \tilde{a}^*_{qr}).$$

Theorem 4.3 (Boundedness)**.** *Assume f is the BS-UOWA operator. Then,*

$$\min\{\tilde{a}_i\} \leq f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) \leq \max\{\tilde{a}_i\}. \tag{24}$$

Proof: Let $\max\{\tilde{a}_i\} = b$ and $\min\{\tilde{a}_i\} = a$, then

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b}_{jk} \leq \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b} = \tilde{b} \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk}, \tag{25}$$

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b}_{jk} \geq \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{a} = \tilde{a} \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk}. \tag{26}$$

Since $\sum_{j=1}^{q_k} \tilde{w}_{jk} = 1$ for each focal element and $\sum_{k=1}^{r} m(B_k) = 1$, we get

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = b, \tag{27}$$

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = a. \tag{28}$$

Therefore,

$$\min\{\tilde{a}_i\} \leq f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) \leq \max\{\tilde{a}_i\}.$$

Theorem 4.4 (Idempotency)**.** *Assume f is the BS-UOWA operator. If $\tilde{a}_{kh} = \tilde{a}$ for all $i \in N$, then*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \tilde{a}. \tag{29}$$

Proof: Since $\tilde{a}_i = \tilde{a}, \forall i \in N$, we have

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{b}_{jk} = \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk} \tilde{a} = \tilde{a} \sum_{k=1}^{r} \sum_{j=1}^{qk} m(B_k) \tilde{w}_{jk}. \tag{30}$$

Since $\sum_{j=1}^{q_k} \tilde{w}_{jk} = 1$ for each focal element and $\sum_{k=1}^{r} m(B_k) = 1$, we get

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{qr}) = \tilde{a}.$$

Based on the generalized reordering step, it is possible to distinguish between descending and ascending orders by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DUOWA and w^*_{n-j+1} is the jth weight of the AUOWA operator. Thus, we obtain the BS-DUOWA and BS-AUOWA operators.

4.4. **Families of BS-UOWA operators.** By choosing a different manifestation of the weighting vector of the UOWA operator, we can develop different families of UOWA and BS-UOWA operators. As it can be seen in Definition 4.5, each focal element uses a different weighting vector in the aggregation step under the UOWA operator. Therefore, these different vectors must be analyzed separately.

Remark 4.1. *For example, it is possible to obtain the uncertain maximum, the uncertain minimum and the UA operator.*

- *The uncertain maximum is obtained if* $w_p = 1$ *and* $w_j = 0$ *for all* $j \neq p$ *and* $u_p = \max\{\tilde{a}_i\}$.
- *The uncertain minimum is obtained if* $w_p = 1$ *and* $w_j = 0$ *for all* $j \neq p$ *and* $u_p = \min\{\tilde{a}_i\}$.
- *The UA is found when* $w_j = 1/n$ *for all* $\tilde{a}_i$.

Other families of UOWA operators can be used in the BS-UOWA operator such as the step-UOWA, the olympic-UOWA and the centered-UOWA operator, among others. In the literature, we find a wide range of methods for determining OWA weights that could be applied to the UOWA. However, in this subsection, we simply provide a general overview of some basic cases that are applicable to the UOWA operator. Note that these families provide a more complete picture of the aggregation process and each of them is useful in some particular situations depending on the interests of the specific problem considered. For more information on these families, see [5,35,38].

Remark 4.2. *The step-UOWA operator is obtained when* $w_k = 1$ *and* $w_j = 0$*, for all* $j \neq k$*. Note that the median-UOWA can be understood as a particular case of the step-UOWA operator that occurs when the number of arguments is odd.*

Remark 4.3. *The olympic-UOWA operator is obtained if* $w_1 = w_n = 0$ *and* $w_j = 1/(n-2)$ *for all others weights. A general form of the olympic-UOWA can be employed by considering that* $w_j = 0$ *for* $j = 1, 2, \cdots, k, n, n-1, \cdots, n-k+1$ *and* $w_{j^*} = 1/(n-2k)$ *for all others weights, where* $k < n/2$*. Note that if* $k = 1$*, then this general form becomes the usual olympic aggregation.*

Remark 4.4. *Another interesting family is the S-UOWA operator. In this case, we can distinguish between three types, namely, the "orlike", the "andlike" and the "generalized" S-UOWA operator. The generalized S-UOWA operator is obtained when* $w_1 = (1/n)(1-(\alpha+\beta))+\alpha$*,* $w_n = (1/n)(1-(\alpha+\beta))+\beta$ *and* $w_j = (1/n)(1-(\alpha+\beta))$ *for all* $j = 2$ *to* $n-1$*, where* $\alpha, \beta \in [0, 1]$ *and* $\alpha + \beta \leq 1$*. Note that if* $\alpha = 0$*, we obtain the andlike S-UOWA operator, and if* $\beta = 0$*, we obtain the orlike S-UOWA operator. Also, note that if* $\alpha + \beta = 1$*, we obtain the uncertain Hurwicz criteria.*

Remark 4.5. *The centered-UOWA operator is obtained if the aggregation is symmetric, strongly decaying and inclusive. It is symmetric if* $w_j = w_{j+n-1}$*. It is strongly decaying when* $i < j \leq (n+1)/2$*,* $w_i < w_j$ *and* $i > j \geq (n+1)/2$ *so that* $w_i < w_j$*. It is inclusive if* $w_j > 0$*. Note that it is possible to consider different particular cases of this operator by softening the second condition to* $w_i \leq w_j$ *instead of* $w_i < w_j$ *and by removing the third condition.*

Remark 4.6. *Finally, if we assume that all focal elements use the same weighting vector (which is a very special case), we can refer to families such as the BS-uncertain maximum, the BS-uncertain minimum, the BS-UA, the BS-UWA, the BS-S-UOWA, the BS-olympic-UOWA and the BS-centered-UOWA.*

5. **Uncertain Hybrid Aggregation Operators in D-S Theory.** In some situations, the decision maker may prefer to use another type of uncertain aggregation operator such as the UHA operator. The main advantage of this operator is that it uses the characteristics of the UWA and UOWA operators in the same aggregation. Thus, if we introduce this operator in decision-making with D-S theory, we are able to develop a unifying framework that includes characteristics of the UWA and UOWA operators in the same formulation and probabilities.

To use this type of aggregation in a D-S framework, in *Step* 3, we use two weighting vectors when calculating the collection of weights for the aggregation because we are mixing in the same problem the UWA and UOWA operators to address the same problem.

In *Step* 5, we employ the UHA operator instead of the UOWA operator by using Equation (3) when calculating the uncertain aggregated results.

Note that it is also possible to formulate in one equation the whole aggregation process. We call this the BS-UHA operator.

Definition 5.1. *A BS-UHA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} \sum_{j=1}^{q} m(B_k) \tilde{w}_{kj} \tilde{b}_{kj}, \tag{31}$$

where $\tilde{b}_{kj}$ *is the* kj*th largest of the* $\hat{a}_{kh}$*, where* $\hat{a}_i = n\tilde{\omega}_i \tilde{a}_i$*,* $i = (1, 2, \cdots, n)$*,* $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)$ *is the weighting vector of the* $\hat{a}_{kh}$*, with* $\tilde{\omega}_i \in [0, 1]$*, the sum of the weights is 1,* $\tilde{a}_{kh}$ *is an argument of the* h*th state of nature of the* k*th focal element represented in terms of interval numbers,* $\tilde{w}_{kj}$ *is the weight of the* j*th largest argument of the* k*th focal element such that* $\frac{1}{r} \sum_{j=1}^{q} \tilde{w}_{kj} = 1$ *and* $\tilde{w}_{kj} \in [0, 1]$*, and* $m(B_k)$ *is the basic probability assignment.*

As we can see, the focal weights aggregate the results obtained by using the UHA operator. Note that if $\omega_i = 1/n$ for all i, then Equation (22) is transformed into Equation (17), and if $w_j = 1/n$ for all j, then Equation (22) becomes Equation (16).

In this case, we could also study different properties and particular cases of the BS-UHA operator, such as the distinction between descending (BS-DUHA) and ascending (BS-AUHA) orders.

When aggregating the collection of uncertain results for each focal element, it is also possible to consider a wide range of UHA operator families. For example, we could analyze the uncertain hybrid maximum, the uncertain hybrid minimum, the UA, the UWA and the UOWA operator. These operators are obtained in a similar way as the method discussed in Section 4.4 with the exception of the UWA and UOWA operators. Note that the UWA operator is obtained when $w_j = 1/n$ for all j, and the UOWA operator is obtained when $\omega_i = 1/n$, for all i, respectively.

Finally, if we use the same family of UHA operators for all focal elements, then we can aggregate according to operators such as the BS-uncertain hybrid maximum, the BS-uncertain hybrid minimum, the BS-uncertain hybrid Hurwicz criteria, the BS-step-UHA, the BS-olympic-UHA, the BS-S-UHA and the BS-centered-UHA.

6. **Uncertain Generalized Aggregation Operators Using D-S Theory.** In this section, we analyze the use of uncertain generalized aggregation operators under D-S theory. First, we consider the use of uncertain generalized weighted aggregation operators. Second, the use of the uncertain generalized OWA is discussed. Third, the use of uncertain generalized hybrid aggregations is analyzed, and fourth, we consider the use of uncertain induced generalized aggregation operators.

6.1. **Uncertain generalized weighted aggregation operators using D-S theory.** A more general formulation can be formulated by using uncertain generalized aggregation operators. For example, we could use the uncertain generalized weighted average (UGWA) and the uncertain quasi-arithmetic weighted average (Quasi-UWA). The main advantage of this approach is that we can consider a wide range of UWA aggregations in the analysis such as quadratic and geometric ones. By using these procedures, we must make the following changes to the previous decision-making processes discussed in Sections 4.2 and 5.

In *Step* 3, when calculating the collection of weights for the aggregation, we must adapt the problem because we are using uncertain weighted averages.

In *Step* 5, when calculating the uncertain results, we must employ the UGWA and Quasi-UWA operators by using Equation (4) and Equation (5), respectively.

Note that it is also possible to formulate the entire aggregation process in one equation for each operator. We call these equations the BS-UGWA operator and the BS-Quasi-UWA operator.

Definition 6.1. *A BS-UGWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k) \left(\sum_{h=1}^{q} \tilde{w}_{kh} \tilde{a}_{kh}^{\lambda} \right)^{1/\lambda}, \tag{32}$$

where $\tilde{w}_{kh}$ is the weight of the hth state of nature of the kth focal element such that $\frac{1}{r}\sum_{h=1}^{q} \tilde{w}_{kh} = 1$ and $\tilde{w}_{kh} \in [0,1]$, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $m(B_k)$ is the basic probability assignment, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Definition 6.2. *A BS-Quasi-UWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k) g^{-1} \left(\sum_{h=1}^{q} \tilde{w}_{kh} g(\tilde{a}_{kh}) \right), \tag{33}$$

where $\tilde{w}_{kh}$ is the weight of the hth state of nature of the kth focal element such that $\frac{1}{r}\sum_{h=1}^{q} \tilde{w}_{kh} = 1$ and $\tilde{w}_{kh} \in [0,1]$, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $m(B_k)$ is the basic probability assignment, and $g(\tilde{a})$ is a strictly continuous monotonic function such that g: $I \to R$.

6.2. **Uncertain generalized OWA operator using D-S theory.** If we prefer to aggregate by considering the attitudinal character (or degree of orness) of the decision maker, we could use the uncertain generalized OWA operator and the Quasi-UOWA operator. The main advantage of this approach is that we can consider a parameterized family of aggregation operators between the uncertain minimum and maximum and a wide range of particular cases such as the uncertain geometric OWA operator and the uncertain quadratic OWA operator. In these cases, we make the following changes.

In *Step* 3, we use weights that are in accordance with the decision maker's preferences when calculating the collection of weights for the aggregation. This is obtained by using one of the methods to derive OWA weights, as discussed in previous literature.

In *Step* 5, when calculating the uncertain results, we use the UGOWA and the Quasi-UOWA operators by employing Equation (6) and Equation (7), respectively. Note that it is also possible to formulate the entire aggregation process in one equation for each operator. We call these equations the BS-UGOWA operator and the BS-Quasi-UOWA operator.

Definition 6.3. *A BS-UGOWA operator is defined by*

$$f(\tilde{a}_{11},\cdots,\tilde{a}_{q1},\cdots,\tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k)\left(\sum_{j=1}^{q} \tilde{w}_{kj}\tilde{b}_{kj}^{\lambda}\right)^{1/\lambda}, \tag{34}$$

where $\tilde{b}_{kj}$ is the kjth largest of the $\tilde{a}_{kh}$, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and λ is a parameter such that $\lambda \in (-\infty,\infty)$.

Definition 6.4. *A BS-Quasi-UOWA operator is defined by*

$$f(\tilde{a}_{11},\cdots,\tilde{a}_{q1},\cdots,\tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k)g^{-1}\left(\sum_{j=1}^{q} \tilde{w}_{kj}g(\tilde{b}_{kj})\right), \tag{35}$$

where $\tilde{b}_{kj}$ is the kjth largest of the $\tilde{a}_{kh}$, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and $g(\tilde{b})$ is a strictly continuous monotonic function such that g: $I \to R$.

6.3. **Uncertain generalized hybrid aggregation operators using D-S theory.** Another formulation that can be introduced into the model involves the use of hybrid aggregations. The main advantage of hybrid aggregations is that we can consider the UWA and UOWA operators in the same formulation under a general framework that is assessed with generalized aggregation operators. In these cases, we should also revise *Steps* 3 and 5 by using both the UGHA and Quasi-UHA operators. If we want to formulate the entire aggregation process in one equation for each operator, we obtain the BS-UGHA and BS-Quasi-UHA operators.

Definition 6.5. *A BS-UGHA operator is defined by*

$$f(\tilde{a}_{11},\cdots,\tilde{a}_{q1},\cdots,\tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k)\left(\sum_{j=1}^{q} \tilde{w}_{kj}\tilde{b}_{kj}^{\lambda}\right)^{1/\lambda}, \tag{36}$$

where $\tilde{b}_{kj}$ is the kjth largest of the $\hat{a}_{kh}$, with $\hat{a}_i = n\tilde{\omega}_i\tilde{a}_i$, $i = (1,2,\cdots,n)$, $\tilde{\omega} = (\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)$ is the weighting vector of the $\tilde{a}_{kh}$, with $\tilde{\omega}_i \in [0,1]$, the sum of the weights is 1, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and λ is a parameter such that $\lambda \in (-\infty,\infty)$.

Definition 6.6. *A BS-Quasi-UHA operator is defined by*

$$f(\tilde{a}_{11},\cdots,\tilde{a}_{q1},\cdots,\tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k)g^{-1}\left(\sum_{j=1}^{q} \tilde{w}_{kj}g(\tilde{b}_{kj})\right), \tag{37}$$

where $\tilde{b}_{kj}$ is the kjth largest of the $\hat{a}_{kh}$, with $\hat{a}_i = n\tilde{\omega}_i\tilde{a}_i$, $i = (1,2,\cdots,n)$, $\tilde{\omega} = (\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)$ is the weighting vector of the $\tilde{a}_{kh}$, with $\tilde{\omega}_i \in [0,1]$, the sum of the weights is 1, $\hat{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of *interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and $g(\tilde{b})$ is a strictly continuous monotonic function such that g: $I \to R$.*

6.4. **Uncertain induced generalized aggregation operators using D-S theory.** A more general formulation of the previous methods can be developed by using uncertain induced generalized aggregation operators [17,33]. This means that we would employ the UIGOWA, the Quasi-UIOWA, UIHA and Quasi-UIHA operators. Thus, we can deal with complex reordering processes in the aggregation phase that represent complex attitudinal characters (or degree of orness) by considering psychological or personal factors in the analysis. If we introduce these uncertain induced generalized aggregation operators in the decision-making process under a D-S belief structure, we make the following changes.

To calculate the collection of weights for the aggregation in *Step* 3, we adapt the problem of calculating the OWA weights based on the particular type of aggregation operator that we are using.

When calculating the uncertain aggregated payoff in *Step* 5, we employ the UIGOWA, Quasi-UIOWA, UIGHA and Quasi-UIHA operators by using Equation (10) and Equation (11), respectively. Note that it is also possible to formulate the entire aggregation process in one equation for each operator. We call these equations the BS-UIGOWA and BS-Quasi-UIOWA operators.

Definition 6.7. *A BS-UIGOWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k)\left(\sum_{j=1}^{q}\tilde{w}_{kj}\tilde{b}_{kj}^{\lambda}\right)^{1/\lambda}, \tag{38}$$

where $\tilde{b}_{kj}$ equals $\hat{a}_{kh}$ from the UIGOWA pair $\langle u_{kh}, \tilde{a}_{kh}\rangle$ having the kjth largest u_{kh}, u_{kh} is the order-inducing variable, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Definition 6.8. *A BS-Quasi-UIOWA operator is defined by*

$$f(\tilde{a}_{11}, \cdots, \tilde{a}_{q1}, \cdots, \tilde{a}_{qr}) = \sum_{k=1}^{r} m(B_k) g^{-1}\left(\sum_{j=1}^{q}\tilde{w}_{kj} g(\tilde{b}_{kj})\right), \tag{39}$$

where $\tilde{b}_{kj}$ equals $\tilde{a}_{kh}$ from the Quasi-UIOWA pair $\langle u_{kh},\ \tilde{a}_{kh}\rangle$ having the kjth largest u_{kh}, u_{kh} is the order-inducing variable, $\tilde{a}_{kh}$ is an argument of the hth state of nature of the kth focal element represented in terms of interval numbers, $\tilde{w}_{kj}$ is the weight of the jth largest argument of the kth focal element such that $\frac{1}{r}\sum_{j=1}^{q}\tilde{w}_{kj} = 1$ and $\tilde{w}_{kj} \in [0,1]$, $m(B_k)$ is the basic probability assignment, and $g(\tilde{b})$ is a strictly continuous monotonic function such that g: $I \to R$.

Note that the BS-Quasi-UIGHA operator and the BS-Quasi-UIHA operator have the same definition with the only difference being that now we use the formulation of the hybrid aggregations with order-inducing variables.

Note that for these operators, we could study a wide range of particular cases of the UIGOWA, Quasi-UIOWA, UIGHA and Quasi-UIHA operators. In particular, it is worth noting the uncertain induced ordered weighted geometric averaging belief structure (BS-UIOWGA), the uncertain induced ordered weighted quadratic averaging belief structure (BS-UIOWQA) and the uncertain induced hybrid quadratic averaging belief structure (BS-UIHQA) operators.

7. **Illustrative Example.** This approach can be applied in a wide range of fields. It is very useful in decision making problems because we can assess complex environments in a more complete way [3,5,6,8]. For example, we can use it in portfolio management, human resource selection, financial decision making and political management. We can also use it in a wide range of problems in statistics such as linear regression and descriptive statistics. In general, all the previous studies that use the probability and the weighted average can be revised and extended with this new approach because we can always reduce it to the classical model.

In the following section, we develop an application of the proposed approach to a decision-making problem. We analyze a problem regarding the selection of strategies in which an enterprise searches for an optimal strategy for the coming year. We develop our analysis by considering a wide range of particular cases of uncertain aggregation operators, such as the UA, UWA, UOWA, UHA and UIOWA operators. Note that in the literature, there are a wide range of decision making methods [48-54].

Assume an enterprise that operates in Europe and North America is planning its strategy for the next year by considering five possible strategies as follows.

- A_1 = Expand to the Asian market.
- A_2 = Expand to the South American market.
- A_3 = Expand to the African market.
- A_4 = Expand to all three continents.
- A_5 = Do not expand.

To evaluate these strategies, a group of experts from the enterprise evaluates the economic situation of the company for the coming year. After careful analysis, the experts develop five possible situations that could occur in the future.

- S_1 = Very bad economic situation.
- S_2 = Bad economic situation.
- S_3 = Regular economic situation.
- S_4 = Good economic situation.
- S_5 = Very good economic situation.

Using these uncertain situations the experts establish the payoff matrix. As the future states of nature are very imprecise, the experts cannot determine exact numbers in the payoff matrix. Instead, they use interval numbers to calculate the future benefits of the enterprise depending on the state of nature that may occur in the future and the strategy selected. The results are shown in Table 1.

Table 1. Payoff matrix

	S_1	S_2	S_3	S_4	S_5
A_1	(20,30,40)	(50,60,70)	(30,40,50)	(60,70,80)	(60,70,80)
A_2	(60,70,80)	(40,50,60)	(30,40,50)	(70,80,90)	(30,40,50)
A_3	(50,60,70)	(50,60,70)	(40,50,60)	(30,40,50)	(40,50,60)
A_4	(60,70,80)	(20,30,40)	(30,40,50)	(30,40,50)	(60,70,80)
A_5	(50,60,70)	(70,80,90)	(30,40,50)	(20,30,40)	(40,50,60)

After careful analysis of available information, the experts derive some probabilistic information about which state of nature will most likely occur in the future. This information is represented by the following belief structure about the states of nature.

Focal element

$$m(B_1) = \{S_1, S_2, S_3\} = 0.3.$$
$$m(B_2) = \{S_2, S_3, S_4\} = 0.3.$$
$$m(B_3) = \{S_3, S_4, S_5\} = 0.4.$$

The attitudinal character of the enterprise is very complex because it involves the opinion of different members of the board of directors. After careful evaluation, the experts establish the following weighting vectors for both the UWA and UOWA operators. For simplicity, we assume that the weights are represented with exact numbers. Note also that for the induced aggregation operators, we use the order-inducing variables $U = (10, 22, 24, 34, 17)$.

Weighting vector

$$W_3 = \omega_3 = (0.3, 0.3, 0.4).$$
$$W_4 = \omega_4 = (0.2, 0.2, 0.3, 0.3).$$
$$W_5 = \omega_5 = (0.1, 0.2, 0.2, 0.2, 0.3).$$

With this information, we can generate the uncertain aggregated results. They are shown in Table 2.

Table 2. Aggregated results

	UA	UWA	UOWA	UHA	UIOWA
V_{11}	(33.3,43.3,53.3)	(33,43,53)	(32,42,52)	(28,36,44)	(32,42,52)
V_{12}	(46.6,56.6,66.6)	(48,58,68)	(45,55,65)	(45,55,65)	(47,57,67)
V_{13}	(50,60,70)	(51,61,71)	(48,58,68)	(57,68.5,80)	(51,61,71)
V_{21}	(43.3,53.3,63.3)	(42,52,62)	(42,52,62)	(33,41,49)	(45,55,65)
V_{22}	(46.6,56.6,66.6)	(49,59,69)	(45,55,65)	(45,55,65)	(46,56,66)
V_{23}	(43.3,53.3,63.3)	(42,52,62)	(42,52,62)	(46.5,58,69.5)	(42,52,62)
V_{31}	(46.6,56.6,66.6)	(46,56,66)	(46,56,66)	(37,45,53)	(47,57,67)
V_{32}	(40,50,60)	(39,49,59)	(39,49,59)	(39,49,59)	(41,51,61)
V_{33}	(36.6,46.6,56.6)	(37,47,57)	(36,46,56)	(46.5,58,69.5)	(37,47,57)
V_{41}	(36.6,46.6,56.6)	(36,46,56)	(35,45,55)	(26,36,46)	(39,49,59)
V_{42}	(26.6,36.6,46.6)	(27,37,47)	(26,36,46)	(26,36,46)	(26,36,46)
V_{43}	(40,50,60)	(42,52,62)	(39,49,59)	(48,58,68)	(42,52,62)
V_{51}	(50,60,70)	(48,58,68)	(48,58,68)	(40,50,60)	(50,60,70)
V_{52}	(40,50,60)	(38,48,58)	(38,48,58)	(38,48,58)	(43,53,63)
V_{53}	(30,40,50)	(31,41,51)	(29,39,49)	(35,45,55)	(31,41,51)

Once we have generated the aggregated results, we can calculate the uncertain generalized expected value. The results are shown in Table 3.

As we can see, depending on the uncertain aggregation operator used, the results and decisions may differ. Note that in this example, our optimal choice is the same for all aggregation operators, but in other situations, we may find that different decisions are derived from different aggregation operators.

TABLE 3. Uncertain generalized expected value

	UA	UWA	UOWA	UHA	UIOWA
A_1	(44,54,64)	(40.7,50.7,60.7)	(42.3,52.3,62.3)	(44.7,54.7,64.7)	(44.1,54.1,64.1)
A_2	(44.3,54.3,64.3)	(44.1,54.1,64.1)	(42.9,52.9,62.9)	(42,52,62)	(44.1,54.1,64.1)
A_3	(40.6,50.6,60.6)	(40.3,50.3,60.3)	(39.9,49.9,59.9)	(41.4,51.4,61.4)	(41.2,51.2,61.2)
A_4	(35,45,55)	(35.7,45.7,55.7)	(33.9,43.9,53.9)	(34.6,44.6,54.6)	(36.3,46.3,56.3)
A_5	(39,49,59)	(38.2,48.2,58.2)	(37.4,47.4,57.4)	(37.4,47.4,57.4)	(40.3,50.3,60.3)

Another interesting issue involves ordering the strategies. The ordering is very useful when a decision maker wants to consider more than one alternative. The results are shown in Table 4.

TABLE 4. Ordering of the strategies

	Ordering		Ordering
UA	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4$	UHA	$A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$
UWA	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4$	UIOWA	$A_1 = A_2 \succ A_3 \succ A_5 \succ A_4$
UOWA	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4$		

As we can see, depending on the aggregation operator used, the ordering of the strategies may be different. Note that in this example, the optimal choice is A_2 for the UA, UWA and UOWA operators, but the optimal choice is A_1 for the UHA operator. For the UIOWA operator, both A_1 and A_2 are optimal.

8. **Conclusions.** We have studied the use of Dempster-Shafer belief structures in decision-making under uncertain environments assessed with interval numbers. By using interval numbers, we can analyze information in a more complete way because we can consider the maximum and minimum values that may occur as well as the most possible results in certain cases. We developed this approach by using a wide range of uncertain aggregation operators, such as the UWA, the UOWA, the UHA, the UGWA, the Quasi-UWA, the UGOWA, the Quasi-UOWA, the UGHA, the Quasi-UHA, the UIGOWA and the Quasi-UIOWA operators. We also considered many particular cases of these aggregation operators. Finally, we also presented a new method for addressing interval weights.

By using these aggregations into a D-S framework, we obtained a wide range of new aggregation operators, such as the BS-UWA, the BS-UOWA, the BS-UHA, the BS-UGWA, the BS-Quasi-UWA, the BS-UGOWA, the BS-Quasi-UOWA, the BS-UGHA, the BS-Quasi-UHA, the BS-UIGOWA and the BS-Quasi-UIOWA operators. Thus, we can represent in a more complete way the D-S approach because we can consider uncertain information that can be assessed with interval numbers.

We also developed an application of the new approach to a decision-making problem regarding strategy selection, thereby, demonstrating, usefulness of using probabilistic information, the WA operator and the OWA operator in the same formulation. Moreover, we showed that depending on the particular type of aggregation operator used, the results may indicate different decisions. Note that this approach can be implemented in a wide range of decision making problems including political decision making and financial decision making.

In future research, we expect to further develop this approach by using other sources of information such as fuzzy numbers that will permit us to assess environments where we have more knowledge to represent the information and including more complex formulations to fully integrate the OWA operator, the WA operator and probabilistic information in the same formulation.

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