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A GENERALIZED LIKELIHOOD RATIO TEST FOR A FAULT-TOLERANT CONTROL SYSTEM

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Abstract. *This paper deals with the problem of diagnosis of multiple sequential faults using statistical test GLR. Based on the work of Willsky and Jones [50], we propose a modified Generalized Likelihood Ratio (GLR) test, allowing detection, isolation and estimation of multiple sequential faults. Our contribution aims to maximise the rate of good decision of fault detection, using another updating strategy. This is based on a reference model updated on line after each detection and isolation of one jump. To reduce the computational requirement, the passive GLR test will be derived from a state estimator designed on a fixed reference model directly sensitive to system changes. We will show that the active and passive GLR tests give an interesting result compared with the GLR of Willsky and Jones [50], and can be easily integrated in a reconfigurable Fault-Tolerant Control System (FTCS) to asymptotically recover the nominal system performances of the jump-free system.*
Keywords: Generalized likelihood ratio, Sequential jumps detection, Augmented state Kalman filter, Two-stage Kalman filter, Nuisance parameter, Fault-tolerant control system

1. **Introduction.** The diagnosis of multiple faults in stochastic systems has been solved by many approaches: observers, parity space and fault detection filters. All these approaches are interested by residual generation, but missing appropriate test for decision. In this work, we will develop a method which takes into account the residual generation problem based Kalman filter, associated with GLR test decision for multiple faults.

The GLR test has been used in a wide variety of applications including the detection of sensor and actuator failures [15,49,50], electrocardiogram analysis [18], geophysical signal processing [4] and freeways supervision [51]. For sequential jumps detection in discrete-time stochastic linear systems, the GLR test is made of the following steps:

1) Detection and isolation of one possible jump by applying a GLR detector on the innovation sequence of the Kalman filter designed on the jump-free system.
2) Updating of the Kalman filter using the jump magnitude estimate given by the GLR detector.
3) Go to the Step 1 to detect, isolate and estimate another possible jump from measurements immediately available after the detection time of the last jump.

Following the notations used in [52] or in [5], the updating strategy of [50], based on the incrementation of the state estimate $\hat{x}_k$ and the state estimate error covariance matrix P_k of the Kalman filter, works as follows:

$$\hat{x}_k^{new} = \hat{x}_k^{old} + [\alpha_j(k,\hat{r}) - \beta_j(k,\hat{r})]\,\hat{\nu}(k,\hat{r}) \quad (1)$$

$$P_k^{new} = P_k^{old} + [\alpha_j(k,\hat{r}) - \beta_j(k,\hat{r})]\,P^{\nu}(k,\hat{r})\,[\alpha_j(k,\hat{r}) - \beta_j(k,\hat{r})]^T \quad (2)$$

where $(\hat{x}_k^{new}, P_k^{new})$ and $(\hat{x}_k^{old}, P_k^{old})$ represent the new and the old state estimation of the Kalman filter, $\alpha_j(k_j,\hat{r})$ and $\beta_j(k_j,\hat{r})$ the jump signatures on the state and the state estimate, $\hat{r}$ the jump occurrence time estimate and $(\hat{\nu}(k,\hat{r}), P^{\nu}(k,\hat{r}))$ the jump magnitude estimate given by the GLR detector. Immediately after the updating strategy (1), the innovation sequence of the resulting Kalman filter is given by

$$\gamma_k^{new} = \gamma_k^{old} - \rho_j(k,\hat{r})\hat{\nu}(k,\hat{r}) \quad (3)$$

$$H_k^{new} = H_k^{old} + \rho_j(k,\hat{r})P^{\nu}(k,\hat{r})\rho_j(k,\hat{r})^T \quad (4)$$

where $(\gamma_k^{new}, H_k^{new})$ and $(\gamma_k^{old}, H_k^{old})$ represent the new and the old innovation sequence of the Kalman filter and $\rho_j(k,\hat{r}) = C\,[\alpha_j(k,\hat{r}) - \beta^j(k,\hat{r})]$ the signature of jump. A GLR detector is then applied on γ_k^{new} to detect another possible jump. Some critics can be made about this updating strategy:

- What is the significant meaning of $(\hat{x}_k^{new}, P_k^{new})$ and $(\gamma_k^{new}, H_k^{new})$ consequently;
- What are the stability and convergence conditions of the resulting Kalman filter;
- The threshold level of the GLR detector must be chosen to solve a tradeoff between fast detection and accurate jump estimation;
- γ_k^{new} is not guaranteed to be minimum variance white innovation sequence, a necessary condition to minimize the rate of false alarms;
- The Kullback divergence is not guaranteed to be maximized with respect to the new possible jumps, a necessary condition to maximize the rate of good decisions [4].

The first part of the paper revisits the standard GLR test of Willsky and Jones [50] in relation with the fault detectability indexes [27] describing the time delay between the occurrence of a jump and its effect on measurements. In this part, we also derive a modified form of the standard GLR test avoiding the tradeoff between fast detection and accurate estimation. The necessary and sufficient conditions for multiple jumps detectability and distinguishability are established.

The second part presents the active GLR test. We will show that the updating strategy (1) and (2) have a significant meaning for the Kalman filter designed on a new reference model including the original state vector of the system and the states of jumps detected and isolated during the processing. The stability and convergence conditions of the augmented state Kalman filter designed on the new reference model will be established and linked with the multiple jumps detectability conditions given in part 1. Its innovation sequence will be guaranteed to be a minimum variance white innovation sequence and the Kullback divergence,

$$\delta_i^{new}(k,r) = \sum_{t=r}^{k} \left[\rho_t^{new}(t,r)^T H_t^{-1} \rho_t^{new}(t,r)\right] (\nu^{new})^2 \quad (5)$$

depending to the new jump signatures $\rho_i^{new}(k,\hat{r})$, will be maximized with respect to the new possible ν^{new}. By considering the old jumps (the jumps detected and isolated during the processing) as nuisance parameters, the equivalence between the active GLR test and the minmax test of basseville and Nikiforov [5] will be established. The active GLR test will be made of the following steps:

1) Detection and isolation of one jump by the GLR detector;

2) Updating the reference model of the system with the jump state initializing the augmented state Kalman filter using the jump magnitude estimate given by the GLR detector;
3) Go to the first step to treat another possible jump from measurements immediately available after the detection time of the last jump.

To reduce the computational requirement of the active GLR test, the third part of this paper presents the passive GLR test. Based on the augmented state Kalman filter designed on a fixed reference model including all the states of hypothetical jumps at the beginning of the processing, the updating strategy will be based on the incrementation of the state estimate and the state estimate error covariance matrix after each detection of one jump as in [50]. Less power than the active GLR test, we will show that it can be mixed with the active GLR test to derive a complete strategy allowing the treatment of the appearance and the disappearance of sequential jumps.

A Fault-Tolerant Control System (FTCS) is a control system that possesses the ability to accommodate system component failures automatically. The existing methods for reconfigurable controller design can be classified as linear quadratic regulator [31], eigenstructure assignment [24], multiple model [35], adaptive control [7], pseudo-inverse [8] and model following [19]. In general, a FTCS works as follows: a suitable Fault Detection and Isolation (FDI) strategy identifies the faults and their estimations are used to generate additional input signals which are superimposed to the nominal control inputs in such a way that the influence of the faults on the regulated variables is rejected. To integrate the standard GLR test in a FTCS, Willsky [49] has proposed a control law of the form $u_k = -L\hat{x}_k^{new}$. To do the same with our active GLR test, the part 5 proposes the design of a linear Quadratic Gaussian (LQG) regulator [2] of the form $u_k = -L\hat{X}_k$, where $\hat{X}_k$ will be the state prediction of the updated reference model, and thus reconfigurated on-line after each detection and isolation of one jump.

The last part of this paper proposes a comparative study between the modified GLR test presented in part 2 and the active GLR test for the treatment of sequential actuator jumps in dynamic systems. We will show that the active GLR test is very power when quick detections lead to bad jump estimations and thus very useful in a FTCS to maximize the rate of good decisions specially in regard to the appearance of big jumps which may greatly affect the nominal performance of the closed-loop system.

2. **The GLR Test of Willsky and Jones.**

This part revisits the GLR test of Willsky and Jones [50] in relation with the multi-hypotheses GLR test described in [52]. Assume that the model of the "no jump" hypothesis h_0 is given by

$$x_{k+1} = Ax_k + Bu_k + u_k \tag{6}$$

$$y_k = Cx_k + v_k \tag{7}$$

where $x_k \in \Re^n$ is the state vector, $y_k \in \Re^m$ the measurement vector, $u_k \in \Re^r$ the input vector. The zero mean white gaussian noises w_k and v_k satisfy

$$E\left\{\begin{bmatrix}\omega_k \\ \upsilon_k\end{bmatrix}\begin{bmatrix}\omega_j^T & \upsilon_j^T\end{bmatrix}\right\} = \begin{bmatrix}W & 0 \\ 0 & V\end{bmatrix}\delta_{k,j} \tag{8}$$

where $W \geq 0$ and $V > 0$ and the Gaussian initial state x_0, $\hat{\bar{x}}_0 = E(x_0)$ and $\bar{P}_0 = E\left((x_0 - E(x_0))(x_0 - E(x_0))^T\right)$. The jump hypotheses h_i for $i \in [1, \dots, N]$ are modelized by

$$x_{k+1} = Ax_k + Bu_k + f_i(k,r)\nu(k,r) + w_k \tag{9}$$

$$y_k = Cx_k + v_k \tag{10}$$

where $f_i(k,r)$ is the fault distribution vector, $\nu(k,r)$ the scalar fault magnitude, r the unknwon time of failure occurrence. Without lost of generality, the fault magnitude $\nu(k,r)$ is assumed to follows a constant bias model: $\nu(k,r) = \nu,\ (k \geq r)$ with step profile: $f_i(k,r) = f_i,\ (k \geq r)$ and $f_i(k,r) = 0,\ (k < r)$.

Let $\rho_i = \min(s : CA^{s-1}f_i \neq 0, s = 1, 2, \ldots)$ for $i \in [1, \ldots, N]$.

The jump detectability indexes [27] and assume that $\rho_i < \infty$ signifying that the first information about the jump h_i, appearing at time r, will be present on measurement $y_{r+\rho_i}$. In the case where the jumps may occur relatively infrequently, the most likelihood reference model at the beginning of the processing is the model of h_0. So, under the stability and convergence conditions

$$rang \begin{bmatrix} zI - A \\ C \end{bmatrix} = n,\ |z| \geq 1. \tag{11}$$

and

$$rang \begin{bmatrix} -e^{jw}I + A & W^{1/2} \end{bmatrix} = n,\ w \in [0, 2\pi] \tag{12}$$

The Kalman filter which gives the maximum likelihood state prediction of the jump-free system is given by

$$\hat{\bar{x}}_{k+1} = A\hat{\bar{x}}_k + Bu_k + \bar{K}_k(y_k - C\hat{\bar{x}}_k) \tag{13}$$

$$\bar{P}_{k+1} = (A - K_kC)\bar{P}_k(A - K_kC)^T + W + \bar{K}_kV\bar{K}_k^T \tag{14}$$

$$\bar{H}_k = C\bar{P}_kC^T + V \tag{15}$$

$$\bar{K}_k = A\bar{P}_kC^T\bar{H}_k^{-1} \tag{16}$$

The additive effect of jump h_i on the state prediction error $e_{k+1} = x_{k+1} - \hat{\bar{x}}_{k+1}$ and on the innovation sequence $\bar{\gamma} = y_k - C\hat{\bar{x}}_k$ can be expressed as

$$e_{k+1} = \tilde{e}_{k+1} + \zeta_i(k+1, r)\nu \tag{17}$$

$$\bar{\gamma}_k = \tilde{\gamma}_{k+1} + \varrho_i(k, r)\nu \tag{18}$$

where $\tilde{e}_{k+1}$ and $\tilde{\gamma}_k$ represent the state prediction error and the innovation sequence on the jump-free system and where $\zeta_i(k,r)$ and $\varrho_i(k,r)$ describes the additive effect of the jump satisfying

$$\zeta_i(k+1, r) = (A - \bar{K}_kC)\zeta_i(k, r) + f_i,\ \zeta_i(r, r) = 0 \tag{19}$$

$$\varrho_i(k, r) = C\zeta_i(k, r) \tag{20}$$

In [52] or [5], $\zeta_i(k,r) = \alpha_i(k,r) - \beta_i(k,r)$ where $\alpha_i(k,r)$ and $\beta_i(k,r)$ describe the additive effect of the jump on the state vector of the system and on the state vector of the Kalman filter, respectively. Our formulation shows that the jump signatures (19) only depend to the state prediction error of the Kalman filter and thus are decoupled from u_k (this property will be used in the design of the FTCS). The jump hypothesis h_i can be confronted to the "no-jump" hypothesis h_0 as

$$H_0 : E(\bar{\gamma}_t) = 0,\ t < r \tag{21}$$

$$H_i : E(\bar{\gamma}_t) = \varrho_i(t, r)\nu,\ k \geq t \geq r,\ i \in [1, \ldots, N] \tag{22}$$

Since $E\left\{(\bar{\gamma}_{k-t} - E(\bar{\gamma}_{k-t}))(\bar{\gamma}_k - E(\bar{\gamma}_k))^T\right\} = 0\ \forall t < k$, the likelihood ratio between (21) and (22) gives

$$\lambda_i(k, r, \nu) = \frac{P(\bar{\gamma}_0/h_0)\ldots P(\bar{\gamma}_{r_i+\rho_i}/h_i)\ldots P(\bar{\gamma}_k/h_i)}{P(\bar{\gamma}_0/h_0)\ldots P(\bar{\gamma}_{r_i+\rho_i}/h_0)\ldots P(\bar{\gamma}_k/h_0)} \tag{23}$$

From Equation (20), we have $\varrho_i(r,r) = \varrho_i(r+1,r) = \ldots = \varrho_i(r+\rho_i-1,r) = 0$ leading to

$$\begin{aligned} P(\bar{\gamma}_{r_i}/h_i) &= P(\bar{\gamma}_{r_i}/h_0), \\ P(\bar{\gamma}_{r_i+1}/h_i) &= P(\bar{\gamma}_{r_i+1}/h_0), \\ &\vdots \\ P(\bar{\gamma}_{r_i+\rho_i-1}/h_i) &= P(\bar{\gamma}_{r_i}+\rho_i-1/h_0) \end{aligned} \tag{24}$$

and the likelihood ratio (23) can be rewritten

$$\lambda_i(k,r,\nu) = \frac{P(\bar{\gamma}_{r_i+\rho_i}/h_i)\ldots P(\bar{\gamma}_k/h_i)}{P(\bar{\gamma}_{r_i+\rho_i}/h_0)\ldots P(\bar{\gamma}_k/h_0)} = \frac{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|\bar{\gamma}_t - \varrho_i(t,r)\nu\|^2_{\bar{H}_t^{-1}}\right)}{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|\bar{\gamma}_t\|^2_{\bar{H}_t^{-1}}\right)} \tag{25}$$

Based on $\bar{\gamma}_0,\ldots,\bar{\gamma}_k$, the maximum likelihood prediction of ν conditioned on a particular assumed value of r is given by

$$\hat{\nu}(k+1,r) = \left[\sum_{t=r+\rho_i}^{k}\varrho_i^T(t,r)\bar{H}_t^{-1}\varrho_i(t,r)\right]^{-1}\sum_{t=r+\rho_i}^{k}\varrho_i^T(t,r)\bar{H}_t^{-1}\bar{\gamma}_t \tag{26}$$

and the log-likelihood ratio $T_i(k,r) = 2\log(\lambda_i(k,r,\hat{\nu}(k+1,r)))$ can be written

$$T_i(k,r) = b_i(k,r)^2 a_i(k,r)^{-1} \tag{27}$$

$$a_i(k,r) = \sum_{t=r+\rho_i}^{k}\varrho_i^T(t,r)\bar{H}_t^{-1}\varrho_i(t,r) \tag{28}$$

$$b_i(k,r) = \sum_{t=r+\rho_i}^{k}\varrho_i^T(t,r)\bar{H}_t^{-1}\bar{\gamma}_t \tag{29}$$

So, the decision rule of the GLR detector is as follows:

$$\max_{i\in[1,\ldots,N],\ \tilde{r}\in[0,\ldots,k]}\{T_i(k,\tilde{r}-\rho_i)\} > \varepsilon \tag{30}$$

where ε is the decision threshold. For a practical implementation of the GLR detector (30), the translated jump occurrence time $\tilde{r}$ must be constrained to belong to the sliding windows $W = [k-M \leq \tilde{r} \leq k]$ of size M. In this case, (30) can be rewritten

$$\max_{i\in[1,\ldots,N],\ \tilde{r}\in W}\{T_i(k,\tilde{r}-\rho_i)\} > \varepsilon \tag{31}$$

if $\max\{T_i(k,\tilde{r}-\rho_i\} > \varepsilon$ then one jump is detected at time k, isolated from $(j,\hat{\tilde{r}}) = \arg\max\{T_i(k,\tilde{r}-\rho_i)\}$ where $\hat{r} = \hat{\tilde{r}} - \rho_j$ is the estimation of the jump occurrence time and where

$$\hat{\nu}(k+1,\hat{r}) = a_j(k,\hat{r})^{-1}b_j(k,\hat{r}) \tag{32}$$

$$P^{\nu}(k+1,\hat{r}) = a_j(k,\hat{r})^{-1} \tag{33}$$

represent the maximum likelihood prediction of the jump magnitude based on measurements until time k under the assumption that ν has an infinite a priori covariance.

The second step of the standard GLR test, intuitively obtained from relation (17), consists to update the Kalman filter (13) by prediction and covariance incrementation of the Kalman filter (13) as follows:

$$\hat{x}_{k+1}^{new} = \hat{\bar{x}}_{k+1}^{old} + \zeta_j(k+1,\hat{r})\hat{\nu}(k+1,\hat{r}) \tag{34}$$

$$P_{k+1}^{new} = \bar{P}_{k+1}^{old} + \zeta_j(k+1,\hat{r})P\zeta_j(k+1,\hat{r})^T \tag{35}$$

where the new state prediction $(\hat{x}_{k+1}^{new}, P_{k+1}^{new})$ is substituted in the Kalman filter (13) (see Figure 1 in [50]. In the original version of the GLR test, the state variables of the matched filters $\zeta_i(k,r)$ for $i \in [1,\dots,N]$ are reinitialized to zero immediately following the filter incrementation leading to consider $(\hat{x}_{k+1}^{new}, P_{k+1}^{new})$ as a new initialization of the Kalman filter more appropriate than the initial value given at the beginning of the processing [5]. With this implementation, Equation (32) cannot be used to improve the jump estimation from measurements available after its detection leading to choose the threshold level ε by solving a compromize between fast detection and accurate estimation.

To avoid this treadeoff, we can modify this implementation in such a ways that $(\hat{x}_{k+1}^{new}, P_{k+1}^{new})$ is not substituted in the Kalman filter but used to generate the auxiliary innovation sequence $\gamma_k = y_k - C\hat{x}_k^{new}$ and its covariance matrix $H_k = CP_k^{new}C^T + V$ expressed

$$\gamma_k = \bar{\gamma}_k - \varrho_j(k,\hat{r})\hat{\nu}(k,\hat{r}) \tag{36}$$

$$H_k = \bar{H}_k + \varrho_j(k,\hat{r})P^{\nu}(k,\hat{r})\varrho_j(k,\hat{r})^T \tag{37}$$

where $\bar{\gamma}_k$ and $\bar{H}_k$ represent the innovation sequence of the Kalman filter (13) and its covariance matrix. Now decoupled to the updating strategy, the Kalman filter (13), (32) and (19) can be used to improve the jump estimation from measurements available after its detection time and another possible jump can then be treated by the following GLR detector

$$\max_{i\in[1,\dots,N],\ i\neq j\ \ \tilde{r}\in W}\{\tilde{T}_i(k-\rho_i,\tilde{r})\} > \varepsilon \tag{38}$$

with

$$\tilde{T}_i(k,r) = \tilde{b}_i(k,r)^2\tilde{a}_i(k,r)^{-1} \tag{39}$$

$$\tilde{a}_i(k,r) = \sum_{t=r+\rho_i}^{k} \varrho_i^T(t,r)H_t^{-1}\varrho_i(t,r) \tag{40}$$

$$\tilde{b}_i(k,r) = \sum_{t=r+\rho_i}^{k} \varrho_i^T(t,r)H_t^{-1}\gamma_t \tag{41}$$

where the jump signatures $\varrho_i(t,r)$ are computed from Equation (19) after having reinitialized $\zeta_i(k,r)$ for $i \neq j$ to zero immediately after the detection time of the first jump. The active GLR test will give a significant meaning of the auxiliary innovation sequence 36 and 37.

3. **Statistical and Geometrical Detectability and Distinguishability Conditions of Jumps.** The probability of false alarms is given by $P^F = \int_{\varepsilon}^{\infty} p(T = x/h_0)dx$ where $P(T = x/h_0)$ is the probability density of $T_i(k,r)$ conditioned on h_0 which follows a Chi-squared density with one degrees of freedom. The probabilities P_i^D of correct detection of the jump h_i are given by $P_i^D = \int_{\varepsilon}^{\infty} p(T_i = x/h_i)dx$ where $p(T_i = x/h_i)$ is the probability density of $T_i(k,r)$ conditioned on h_i which follows a noncentral Chi-squared density with one degrees of freedom with the noncentrality parameter $\delta_i(k,r) = a_i(k,r)\nu^2$. The probability of false isolations, i.e., the probability to detect a jump h_i when the jump h_j $(j \neq i)$ is appeared on the system, is given by $P_{ij}^D = \int_{\varepsilon}^{\infty} p(T_i = x/h_j)dx$ where $p(T_i = x/h_j)$ is the

probability density of $T_i(k,r)$ conditioned on h_j which follows a noncentral Chi-squared density with one degrees of freedom with the noncentrality parameter

$$\delta_{ij}(k,r) = \left[\frac{a_{ij}(k,r)^2\nu^2}{a_i(k,r)}\right] \tag{42}$$

with $a_{ij}(k,r) = \sum_{t=r+\rho_i}^{k} \varrho_i^T(t,r)\bar{H}_t^{-1}\varrho_j(t,r)$.

The probabilities P_i^D and P_{ij}^D depend to the size the sliding window and the choice of the decision threshold ε is the main drawback of the GLR detector [4]. In this paper, ε will be only chosen to fixe the false alarms rate P^F.

Theorem 3.1. *Under Equations (11) and (12), the jump hypothesis h_i is statistically detectable and distinguishable from each other if and only if*

$$rank\begin{bmatrix} CA^{\rho_1-1}f_1 & \dots & CA^{\rho_i-1}f_i & \dots & CA^{\rho_N-1}f_N \end{bmatrix} = N \tag{43}$$

$$rank\begin{bmatrix} I-A & f_1 \ \dots \ f_N \\ C & 0 \end{bmatrix} = n+N \tag{44}$$

Proof: The jump hypothesis h_i is statistically detectable and distinguishable from each other if and only if the Kullback divergence (Kullback, 1959) defined by $\delta(k,r) = \nu^T a(k,r)\nu$ satisfies $\delta(k,r) > 0\ \forall k \geq r$ with $\nu \in \Re^N \neq 0$ where

$$a(k,r) = \begin{bmatrix} a_1(k,r-\rho_1) & \dots & a_{1i}(k,r-\rho_1) & \dots & a_{1N}(k,r-\rho_1) \\ \vdots & \vdots & \vdots & \vdots & \\ a_{i1}(k,r-\rho_i) & \dots & a_i(k,r-\rho_i) & \dots & a_{iN}(k,r-\rho_i) \\ \vdots & \vdots & \vdots & \vdots & \\ a_{N1}(k,r-\rho_N) & \dots & a_{Ni}(k,r-\rho_N) & \dots & a_N(k,r-\rho_N) \end{bmatrix} \tag{45}$$

Let

$$\zeta(k,r) = \begin{bmatrix} \zeta_1(k,r-\rho_1) & \dots & \zeta_i(k,r-\rho_i) & \dots & \zeta_N(k,r-\rho_N) \end{bmatrix} \tag{46}$$

$$\varrho(k,r) = \begin{bmatrix} \varrho_1(k,r-\rho_1) & \dots & \varrho_i(k,r-\rho_i) & \dots & \varrho_N(k,r-\rho_N) \end{bmatrix} \tag{47}$$

satisfying

$$\zeta(k+1,r) = (A-\bar{K}_kC)\zeta(k,r) + F \tag{48}$$

$$\varrho(k,r) = C\zeta(k,r) \tag{49}$$

with $F = \begin{bmatrix} f_1 & \dots & f_i & \dots & f_N \end{bmatrix}$.

We have

$$a(k,r) > 0 \Leftrightarrow \sum_{t=r+1}^{k} \varrho^T(t,r)\bar{H}_t^{-1}\varrho(t,r) + \varrho(r,r)^T\bar{H}_r^{-1}\varrho(r,r) > 0 \tag{50}$$

where $\varrho(r,r) = \begin{bmatrix} CA^{\rho_1-1}f_1 & \dots & CA^{\rho_i-1}f_i & \dots & CA^{\rho_N-1}f_N \end{bmatrix}$ and $a(k,r) > 0$ is satisfied under Equation (43).

Another jump detectability condition has been proposed by [60]: The jump hypothesis h_i is statisticaly detectable and distinguishable from each other if the Kullback divergence $a(k,r)$ strictly increases with time, in other words if

$$\tilde{a}(k,r) > 0 \quad \forall k > r \text{ where } \tilde{a}(k,r) = a(k,r) - a(k-1,r) \tag{51}$$

The Kullback increment is given by $\tilde{a}(k,r) = \varrho(k,r)^T\bar{H}_k^{-1}\varrho(k,r)$. Under Equations (11) and (12), $A-\bar{K}C$ has all its eigenvalues inside the unit circle (i.e., $\bar{P}$ is a stabilizing solution of (14)) and Equation (48) gives

$$\lim_{|k|\to\infty}\varrho(k,r) = C\left[I-(A-\bar{K}C)\right]^{-1}F = \left[I+C(I-A)^{-1}\bar{K}\right]^{-1}C(I-A)^{-1}F \tag{52}$$

If $I-A$ is nonsingular, the relation $C(I-A)^{-1}F=0$ cannot be satisfied under $rank\,(\varrho(r,r)) = N$ and Equation (43), Equation (44) ensures that F does not belong to the null space of $C(I-A)^{-1}$. So, $\lim\limits_{|k|\to\infty}\tilde{a}(k,r)>0$ and the Kullback divergence $a(k,r)$ is not satured.

What is the condition to have $\tilde{a}(k,r)>0$, $\forall k>r$ with $k<\infty$ signifying that any new observation bring new information about possible jump: Let $P^{\nu}(k,r) = a(k-1,r)^{-1}$ satisfy the following Riccati difference equation:

$$\begin{bmatrix}\bar{P}_{k+1} & 0\\ 0 & P^{\nu}(k+1,r)\end{bmatrix} = \begin{bmatrix}A\bar{P}A^T+W-\bar{K}_kC\bar{P}A^T & 0\\ 0 & \left[I-K^{\nu}(k,r)\varrho(k,r)\right]P^{\nu}(k,r)\end{bmatrix} \tag{53}$$

where $K^{\nu}(k,r) = P^{\nu}(k,r)\varrho(k,r)^T\left[C\bar{P}_kC^T+V+\varrho(k,r)P^{\nu}(k,r)\varrho^T(k,r)\right]^{-1}$ or equivalently

$$\Omega(k+1,r) = \bar{A}\Omega(k,r)\bar{A}^T+\bar{\Gamma}W\bar{\Gamma}^T-\bar{A}\Omega(k,r)\bar{C}^T(\bar{C}\Omega(k,r)\bar{C}^T+V)^{-1}\bar{C}\Omega(k,r)\bar{A}^T \tag{54}$$

where

$$\begin{aligned}\Omega(r,r) &= \begin{bmatrix}\bar{P}_r+\zeta(r,r)P^{\nu}(r,r)\zeta(r,r)^T & \zeta(r,r)P^{\nu}(r,r)\\ P^{\nu}(r,r)\zeta(r,r)^T & P^{\nu}(r,r)\end{bmatrix},\\ \bar{A} &= \begin{bmatrix}A & F\\ 0 & I\end{bmatrix},\ \bar{C} = \begin{bmatrix}C & 0\end{bmatrix}\end{aligned} \tag{55}$$

and $\bar{\Gamma} = \begin{bmatrix}I\\ 0\end{bmatrix}$ (the equivalence between (53) and (54) will be more clearly explained in the design of the active GLR test). Under Equation (12), the pair $\left(\bar{A},\bar{\Gamma}W^{1/2}\right)$ has N unreachable modes on the unit circle and the convergence condition of (53) to a strong solution is given by

$$rank\begin{bmatrix}Iz-\bar{A}\\ \bar{C}\end{bmatrix} = n+N,\ \forall z\in C,\ |z|\geq 1 \tag{56}$$

The detectability of the pair $(\bar{A},\ \bar{C})$ corresponds to the geometrical detectability condition of jumps given by Caglayan [8]. Under Equation (11), Equation (56) is reduced to Equation (44) and ensures the asymptotical convergence of $P^{\nu}(k,r)$ to zero with the initial condition $P^{\nu}(r,r)>0$. So, $P^{\nu}(k+1,r)<P^{\nu}(k,r)$ and $\tilde{a}(k,r)>0$, $\forall k>r$ with $k<\infty$. Under Equation (43), we conclude that the results of [60] coincide with the results of [8].

4. **The Active GLR Test.** The first part of this chapter gives a significant meaning of the auxiliary innovation sequence (36) and (37) used in the modified GLR test by showing that Equations (1) and (2) are include in the augmented state Kalman filter

$$\begin{bmatrix}\hat{x}_{k+1}\\ \hat{\nu}_{k+1}\end{bmatrix} = \hat{X}_{k+1} = \bar{A}\hat{X}_k+\bar{B}u_k+K_k\gamma_k \tag{57}$$

$$\begin{bmatrix}P^{x}_{k+1} & P^{x\nu}_{k+1}\\ P^{\nu x}_{k+1} & P^{\nu}_{k+1}\end{bmatrix} = \Omega_{k+1} = \bar{A}\Omega_k\bar{A}^T+\bar{\Gamma}W\bar{\Gamma}^T-\bar{A}\Omega_k\bar{C}^T\left(\bar{C}\Omega_k\bar{C}^T+V\right)^{-1}\bar{C}\Omega_k\bar{A}^T \tag{58}$$

$$K_k = \begin{bmatrix}K^x_k\\ K^{\nu^j}_k\end{bmatrix} = \bar{A}\Omega_k\bar{C}^TH_k^{-1},\ H_k = \bar{C}\Omega\bar{C}^T+V \tag{59}$$

designed on the new reference model h_j rewritten

$$X_{k+1} = \bar{A}X_k + \bar{B}u_k + \bar{\Gamma}w_k \tag{60}$$
$$y_k = \bar{C}X_k + v_k \tag{61}$$

with $X_k = \begin{bmatrix} x_k \\ \nu_k \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & f_j \\ 0 & 1 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$ and $\bar{\Gamma} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ where $\hat{x}_{k+1} = \hat{x}^{new}_{k+1}$ and $P^x_{k+1} = P^{new}_{k+1}$ represents the maximum likelihood prediction of the original state x_k but the belonging now to augmented state X_k.

Theorem 4.1. *In the two-stage Kalman filter of Friedland [16] which optimaly implement (5), the updating strategy (1) rewritten*

$$\hat{x}_{k+1} = \hat{\bar{x}}_{k+1} + \zeta_j(k+1,\hat{r})\hat{\nu}(k+1,\hat{r}) \tag{62}$$
$$P_{k+1} = \bar{P}_{k+1} + \zeta_j(k+1,\hat{r})P^{\nu}(k+1,\hat{r})\zeta_j(k+1,\hat{r})^T \tag{63}$$

have the following significant meaning: $\left(\hat{\bar{x}}_{k+1}, \bar{P}_{k+1}\right)$ *is the state prediction of the jump-free system;* $(\hat{x}_{k+1}, P_{k+1})$ *is the reconfigurated state prediction of the faulty system* $(\hat{\nu}(k+1,\hat{r})$*;* $P^{\nu}(k+1,\hat{r}))$ *is the prediction of the jump magnitude, and is optimal under* $\hat{r}$ *extremely well estimated if the augmented sate Kalman filter (5) is correctly initialized at the detection time of the jump with*

$$\begin{aligned} \hat{X}_k &= \begin{bmatrix} I & \zeta_j(k,\hat{r}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_k \\ \hat{\nu}(k,\hat{r}) \end{bmatrix} \\ \Omega_k &= \begin{bmatrix} I & \zeta_j(k,\hat{r}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{P}_k & 0 \\ 0 & P^{\nu}(k,\hat{r}) \end{bmatrix} \begin{bmatrix} I & \zeta_j(k,\hat{r}) \\ 0 & 1 \end{bmatrix}^T \end{aligned} \tag{64}$$

from the quantities $\left(\hat{\bar{x}}_k, \bar{P}_k\right)$, $(\hat{\nu}(k,\hat{r}), P^{\nu}(k,\hat{r}))$ *and* $\zeta_j(k,\hat{r})$ *given by the GLR detector.*

Proof: At time $t_j = \hat{r} + \rho_j$, (32) represents the minimum-time prediction of ν given by

$$\hat{\nu}(t_j+1,\hat{r}) = \left[\left(CA^{\rho_j-1}f_j\right)^T \bar{H}_{tj}^{-1}\left(CA^{\rho_j-1}f_j\right)\right]^{-1}\left(CA^{\rho_j-1}f_j\right)^T \bar{H}_{tj}^{-1}\bar{\gamma}_{tj} \tag{65}$$
$$P^{\nu}(t_j+1,\hat{r}) = \left[\left(CA^{\rho_j-1}f_j\right)^T \bar{H}_{tj}^{-1}\left(CA^{\rho_j-1}f_j\right)\right]^{-1} \tag{66}$$

under the assumption that ν has an infinite a priori covariance since $\varrho_j(k,\hat{r}) = 0$ for $k < t_j$ and $\varrho_j(t_j,\hat{r}) = CA^{\rho_j-1}f_j$. So, the updating strategy (1) and (2) applied at time t_j gives by

$$\hat{x}_{t_j+1} = \hat{\bar{x}}_{t_j+1} + \zeta_j(t_j+1,\hat{r}_j)\hat{\nu}(t_j+1,\hat{r}) \tag{67}$$
$$P_{t_j+1} = \bar{P}_{t_j+1} + \zeta_j(t_j+1,\hat{r})P^{\nu}(t_j+1,\hat{r})\zeta_j(t_j+1,\hat{r})^T \tag{68}$$

and can be used to define the Gaussian state prediction of the initial state X_{t_j+1} as

$$\begin{aligned} \hat{X}_{t_j+1} &= \begin{bmatrix} \hat{x}_{t_j+1} \\ \hat{\nu}(t_j+1,\hat{r}) \end{bmatrix} \\ \Omega_{t_j+1} &= \begin{bmatrix} P_{t_j+1} & \zeta_j(t_j+1,\hat{r})P^{\nu}(t_j+1,\hat{r}) \\ P^{\nu}(t_j+1,\hat{r})\zeta_j(t_j+1,\hat{r})^T & P^{\nu}(t_j+1,\hat{r}) \end{bmatrix} \end{aligned} \tag{69}$$

The augmented state Kalman filter (5) can be implemented from the two-stage Kalman filter of Friedland [16] described by

$$\hat{x}_{k+1} = \hat{\bar{x}}_{k+1} + \zeta_{k+1}\hat{\nu}_{k+1} \tag{70}$$
$$P_{k+1} = \bar{P}_{k+1} + \zeta_{k+1}P^{\nu}_{k+1}\zeta^T_{k+1} \tag{71}$$

where $\left(\hat{\bar{x}}_{k+1}, \bar{P}_{k+1}\right)$ are given by the Kalman filter designed under h_0

$$\begin{aligned}
\hat{\bar{x}}_{k+1} &= A\hat{\bar{x}}_k + Bu_k + \bar{K}_k(y_k - C\hat{\bar{x}}_k) && (72)\\
\bar{P}_{k+1} &= A\bar{P}_kA^T + W - A\bar{P}_kC^T(C\bar{P}_kC^T + V)^{-1}C\bar{P}_kA^T && (73)\\
\bar{K}_k &= A\bar{P}_kC^T\bar{H}_k^{-1} && (74)\\
\bar{H}_k &= C\bar{P}_kC^T + V && (75)
\end{aligned}$$

where $(\hat{\nu}_{k+1}, P^{\nu}_{k+1})$ are given the jump filter

$$\begin{aligned}
\hat{\nu}_{k+1} &= \hat{\nu}_k + K^{\nu}_k\gamma_k && (76)\\
P^{\nu}_{k+1} &= P^{\nu}_k - P^{\nu}_k\varrho_k^T H_k^{-1}\varrho_k P^{\nu}_k && (77)\\
K^{\nu}_k &= P^{\nu}_k\varrho_k^T H_k^{-1} && (78)\\
\gamma_k &= \bar{\gamma}_k - \varrho_k\hat{\nu}_k && (79)\\
H_k &= \bar{H}_k + \varrho_k P^{\nu}_k\varrho_k^T && (80)
\end{aligned}$$

from the coupling equations

$$\begin{aligned}
\zeta_{k+1} &= (A - \bar{K}_kC)\zeta_k + f_j && (81)\\
\varrho_k &= C\zeta_k && (82)
\end{aligned}$$

From Equation (19), the initial values of the two-stage Kalman filter are

$$\begin{bmatrix} \hat{\bar{x}}_{t_j+1} \\ \hat{\nu}_{t_j+1} \end{bmatrix} = \begin{bmatrix} I & -\zeta_{t_j+1} \\ 0 & 1 \end{bmatrix} \hat{X}_{t_j+1} = \begin{bmatrix} \hat{\bar{x}}_{t_j+1} \\ \hat{\nu}(t_j+1, \hat{r}) \end{bmatrix} \quad (83)$$

$$\begin{bmatrix} \bar{P}_{t_j+1} & 0 \\ 0 & P^{\nu}_{t_j+1} \end{bmatrix} = \begin{bmatrix} I & -\zeta_{t_j+1} \\ 0 & 1 \end{bmatrix} \Omega_{t_j+1} \begin{bmatrix} I & -\zeta_{t_j+1} \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \bar{P}_{t_j+1} & 0 \\ 0 & P^{\nu}(t_j+1, \hat{r}) \end{bmatrix} \quad (84)$$

with $\zeta_{t_j+1} = \zeta_j(t_j+1, \hat{r})$.

After some manipulations, $\hat{\nu}$ can be rewritten in the form of a recursive filter

$$\begin{aligned}
\hat{\nu}(k+1, \hat{r}) &= \hat{\nu}(k, \hat{r}) + K^{\nu}_k\gamma_k && (85)\\
P^{\nu}(k+1, \hat{r}) &= P^{\nu}(k, \hat{r}) - P^{\nu}(k, \hat{r})\varrho_j^T(k, \hat{r})H_k^{-1}\varrho_j(k, \hat{r})P^{\nu}(k, \hat{r}) && (86)\\
K^{\nu}_k &= P^{\nu}(k, \hat{r})\varrho_j^T(k, \hat{r})H_k^{-1} && (87)\\
\gamma_k &= \bar{\gamma}_k - \varrho_j(k, \hat{r})\hat{\nu}_k && (88)\\
H_k &= \bar{H}_k + \varrho_j(k, \hat{r})P^{\nu}(k, \hat{r})\varrho_j^T(k, \hat{r}) && (89)
\end{aligned}$$

We can verify that Equations (76) and (81) optimally implement Equation (85) closing the proof of Theorem 4.1.

To avoid the tradeoff between fast detection and accurate estimation, we conclude that the innovation sequence which must be used to detect, isolate and estimate a new jump is the innovation sequence of the jump filter (85) and (89) equals to the auxiliary innovation sequence (36) and (37) used in modified GLR test. This innovation sequence is also the innovation sequence of the augmented state Kalman filter guaranted to be a minimum variance white innovation sequence allowing the design of a GLR detector.

Theorem 4.2. *After having initialized the augmented state Kalman filter (57) at the detection time of the first jump with the help of Theorem 4.1, another possible jump can be detected, isolated and estimated by the following GLR detector*

$$\max_{i\in[1,\ldots,N],\ i\neq j\ \ \tilde{r}\in W}\{T_i(k, \tilde{r} - \rho_i)\} > \varepsilon \quad (90)$$

with

$$T_i^{new}(k,r) = b_i^{new}(k,r)^2 a_i^{new}(k,r)^{-1} \tag{91}$$

$$a_i^{new}(k,r) = \sum_{t=r+\rho_i}^{k} [\varrho_i^{new}(t,r)]^T H_t^{-1} \varrho_i^{new}(t,r) \tag{92}$$

$$b_i^{new}(k,r) = \sum_{t=r+\rho_i}^{k} [\varrho_i^{new}(t,r)]^T H_t^{-1} \gamma_t \tag{93}$$

where the new jump signatures $\varrho_i^{new}(t,r)$ *are recursively computed as*

$$\zeta_i^{new}(k+1,r) = (\bar{A} - \bar{K}_k \bar{C})\zeta_i^{new}(k,r) + \begin{bmatrix} f_i \\ 0 \end{bmatrix}, \ \zeta_i^{new}(r,r) = 0 \tag{94}$$

$$\varrho_i^{new}(k,r) = \bar{C}\zeta_i^{new}(k,r) \tag{95}$$

where $\zeta_i^{new}(t,r)$ *represents the additive effect of a new jump on the augmented state prediction error of the Kalman filter (5).*

Proof: The jump hypotheses (9) and (10), noted h_i^{new} for $i \in [1,\ldots,N]$ and $i \neq j$, can be modelized in relation with the new reference model (60) as

$$X_{k+1} = \bar{A}X_k + \bar{B}u_k + \begin{bmatrix} f_i(k,r) \\ 0 \end{bmatrix} \nu^{new}(k,r) + \bar{\Gamma} w_k \tag{96}$$

$$y_k = \bar{C}X_k + v_k \tag{97}$$

and can be confronted from the augmented state Kalman filter (57) as

$$h_j : E(\gamma_t) = 0, \ t < r \tag{98}$$

$$h_i^{new} : E(\gamma_t) = \varrho_i^{new}(t,r)\nu^{new}, \ k \geq t \geq r, \ i \in [1,\ldots,N] \text{ and } i \neq j \tag{99}$$

where the additive effect of h_i^{new} on its state prediction error $\begin{bmatrix} e_k^x \\ e_k^\nu \end{bmatrix} = X_k - \hat{X}_k$ and on its innovation sequence $\gamma_k = y_k - \bar{C}\hat{X}_k$ is described by Equations (94) and (95). We have

$$E\{(\gamma_{k-t} - E(\gamma_{k-t}))(\gamma_k - E(\gamma_k))^T\} = 0, \ \forall t < k \tag{100}$$

and $\varrho_i^{new}(r,r) = 0$ and so until $\varrho_i^{new}(r+\rho_i-1,r) = 0$ where $\varrho_i^{new}(r+\rho_i,r) = CA^{\rho_i-1}f_i$ (the detectability indexes ρ_i have not lost their significant meaning). So, the likelihood ratio between h_i^{new} $(i \neq j)$ and h_j is given by

$$\lambda_i^{new}(k,r,\nu^{new}) = \frac{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k} \|\gamma_t - \varrho_i^{new}(t,r)\nu^{new}\|^2_{\bar{H}_t^{-1}}\right)}{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k} \|\gamma_t\|^2_{\bar{H}_t^{-1}}\right)} \tag{101}$$

Based on measurements until time k, the maximum likelihood prediction of ν^{new} conditioned on r is given by

$$\hat{\nu}^{new}(k+1,r) = \left[\sum_{t=r+\rho_i}^{k} \varrho_i^{new}(t,r)^T \bar{H}_t^{-1} \varrho_i^{new}(t,r)\right]^{-1} \sum_{t=r+\rho_i}^{k} \varrho_i^T(t,r)\bar{H}_t^{-1}\gamma_t \tag{102}$$

Substituting Equation (102) in (101), we obtain the log-likelihood ratio

$$T_i^{new}(k,r) = 2\log(\lambda_i^{new}(k,r,\hat{\nu}^{new}(k+1,r))) \tag{103}$$

So, if $\max_{i\in[1,\dots,N],\ \tilde{r}\in[0,\dots,k]}\{T_i^{new}(k,\tilde{r}-\rho_i)\} > \varepsilon$, then a new jump is detected and isolated from $(j,\hat{\tilde{r}}) = \arg\max\{T_i^{new}(k,\tilde{r}-\rho_i)\}$ and its estimation is given by

$$\hat{\nu}(k+1,\hat{r}) = a_j^{new}(k,\hat{r})^{-1}b_j^{new}(k,\hat{r}) \tag{104}$$

$$P^{\nu}(k+1,\hat{r}) = a_j^{new}(k,\hat{r})^{-1} \tag{105}$$

with $\hat{r} = \hat{\tilde{r}} - \rho_j$ closing the proof.

Theorem 4.3. *The first step of the active GLR test described by Theorems 4.1 and 4.2 follows the minmax strategy developed by Basseville and Nikiforov [5]. The Kullback divergence between h_i^{new} and h_j given by*

$$\delta_i^{new}(k,r) = \sum_{t=r}^{k}\left[\varrho_i^{new}(t,r)^T H_t^{-1}\varrho_i^{new}(t,r)\right](\nu^{new})^2 \tag{106}$$

is maximized with respect to ν^{new} and satisfies $\delta_i^{new}(k,r) \geq \tilde{\delta}_i(k,r)$ where

$$\tilde{\delta}_i(k,r) = \sum_{t=r}^{k}\left[\varrho_i(t,r)^T H_t^{-1}\varrho_i(t,r)\right](\nu^{new})^2 \tag{107}$$

is the Kullback divergence derived from the modified GLR test presented in part 2. The rate of good decisions will then be always superior to those obtained by the modified GLR test.

Proof: From the Kalman filter (13), the jump hypotheses h_i^{new} can be confronted as

$$h_j : \bar{\gamma}_t = \varrho_t\nu^{old},\ t<r \tag{108}$$

$$h_i^{new} : \bar{\gamma}_t = \begin{bmatrix} \varrho_i(t,r) & \varrho_t \end{bmatrix}\begin{bmatrix} \nu^{new} \\ \nu^{old} \end{bmatrix},\quad k \geq t \geq r+\rho_i,\ i\in[1,\dots,N] \text{ for } i\neq j \tag{109}$$

where ν^{old} can be viewed as a nuisance parameter. Using the optimal prediction of ν^{old} under h_j and h_i^{new} given by $\hat{\nu}_{t+1}$ and $\hat{\nu}_{t+1}+\zeta_i^{\nu}(t+1,r)\nu^{new}$ respectively where $\zeta_i^{\nu}(t+1,r)$ describes the additive effect of the new jump on the bias filter (76) given by

$$\zeta_i^{\nu}(t+1,r) = (I-K_t^{\nu}\varrho_t)\zeta_i^{\nu}(t,r) - K_t^{\nu}\varrho_i(t,r) \text{ with } \zeta_i^{\nu}(r,r)=0 \tag{110}$$

the jump hypotheses (108) and (109) can be equivalently confronted as

$$\begin{aligned} h_j &: \bar{\gamma}_t = \varrho_t\hat{\nu}_{t+1},\ t<r \\ h_i^{new} &: \bar{\gamma}_t = \begin{bmatrix} \varrho_i(t,r) & \varrho_t \end{bmatrix}\begin{bmatrix} \nu^{new} \\ \hat{\nu}_{t+1}+\zeta_i^{\nu}(t+1,r)\nu^{new} \end{bmatrix}, \\ & \quad t\geq r+\rho_i,\ i\in[1,\dots,N] \text{ for } i\neq j \end{aligned} \tag{111}$$

or

$$h_j :\ E(\bar{\gamma}_t - \varrho_t\hat{\nu}_{t+1}) = 0,\ t<r \tag{112}$$

$$h_i^{new} :\ E(\bar{\gamma}_t - \varrho_t\hat{\nu}_{t+1}) = \left[\varrho_i(t,r) + \varrho_t\zeta_i^{\nu}(t+1,r)\right]\nu^{new},\quad t\geq r+\rho_i,\ i\in[1,\dots,N], i\neq j \tag{113}$$

The likelihood ratio between (113) and (114) gives

$$\lambda_i^{new}(k,r,\nu^{new}) = \frac{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|(I-\varrho_t K_t^\nu)\left(\bar{\gamma}_t - \varrho_t\hat{\nu} - [\varrho_i(t,r)+\varrho_t\zeta_i^\nu(t,r)]\,\nu^{new}\right)\|^2_{Q_t^{-1}}\right)}{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|(I-\varrho_t K_t^\nu)\left(\bar{\gamma}_t - \varrho_t\hat{\nu}_t\right)\|^2_{Q_t^{-1}}\right)} \tag{114}$$

where

$$Q_t = (I-\varrho_t K_t^\nu)H_t(I-\varrho_t K_t^\nu)^T \tag{115}$$

since

$$\bar{\gamma}_t - \varrho_t\hat{\nu}_{t+1} = (I-\varrho_t K_t^\nu)\left(\bar{\gamma}_t - \varrho_t\hat{\nu}_t\right) \tag{116}$$

$$\varrho_i(t,r)+\varrho_t\zeta_i^\nu(t+1,r) = (I-\varrho_t K_t^\nu)\left(\varrho_i(t,r)+\varrho_t\zeta_i^\nu(t,r)\right) \tag{117}$$

or

$$\lambda_i(k,r,\nu^{new}) = \frac{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|\left(\bar{\gamma}_t - \varrho_t\hat{\nu}_t - [\varrho_i(t,r)+\varrho_t\zeta_i^\nu(t,r)]\,\nu^{new}\right)\|^2_{H_t^{-1}}\right)}{\exp\left(-\frac{1}{2}\sum_{t=r+\rho_i}^{k}\|\left(\bar{\gamma}_t - \varrho_t\hat{\nu}_t\right)\|^2_{H_t^{-1}}\right)} \tag{118}$$

From the transformation $T_k = \begin{bmatrix} I & -\zeta_k \\ 0 & I \end{bmatrix}$ used in appendix 1, let $T_k\zeta_i^{new}(k,r) = \begin{bmatrix} \zeta_i(k,r) \\ \zeta_i^\nu(k,r) \end{bmatrix}$. So, the new fault signatures (94) can then be equivalently computed as

$$\begin{aligned} T_{k+1}\zeta_i^{new}(k+1,r) &= T_{k+1}(\bar{A}-K_k\bar{C})T_k^{-1}T_k\zeta_i^{new}(k,r) + T_{k+1}\begin{bmatrix} f_i \\ 0 \end{bmatrix} \\ \varrho_i^{new}(k,r) &= \bar{C}T_k^{-1}T_k\zeta_i^{new}(k,r) \end{aligned} \tag{119}$$

leading to

$$\begin{aligned} \begin{bmatrix} \zeta_i(k+1,r) \\ \zeta_i^\nu(k+1,r) \end{bmatrix} &= \begin{bmatrix} A-\bar{K}_kC & 0 \\ -K_k^\nu C & I-K_k^\nu\varrho_k \end{bmatrix}\begin{bmatrix} \zeta_i(k,r) \\ \zeta_i^\nu(k,r) \end{bmatrix} + \begin{bmatrix} f_i \\ 0 \end{bmatrix}, \\ \begin{bmatrix} \zeta_i(r,r) \\ \zeta_i^\nu(r,r) \end{bmatrix} &= 0, \\ \varrho_i^{new}(k,r) &= C\begin{bmatrix} I & \zeta_k \end{bmatrix}\begin{bmatrix} \zeta_i(r,r) \\ \zeta_i^\nu(r,r) \end{bmatrix} \end{aligned} \tag{120}$$

where Equation (120) gives $\varrho_i^{new}(k,r) = \varrho_i(k,r) + \varrho_k\zeta_i^\nu(k,r)$. We conclude that Equation (118) is equivalent to Equation (101). From the two-stage Kalman filter results, we can verify that $P_i^\nu(k+1,r) = a_i^{new}(k,r)^{-1}$ satisfying the following Riccati Difference equation

$$\begin{bmatrix} \Omega & 0 \\ 0 & P_i^\nu(k+1,r) \end{bmatrix} = \begin{bmatrix} \bar{A}\Omega\bar{A}^T + \bar{W} - K_k\bar{C}\Omega_k\bar{A}^T & 0 \\ 0 & [I-K_i^\nu(k,r)\varrho_i^{new}(k,r)]\,P_i^\nu(k,r) \end{bmatrix} \tag{121}$$

where $K_i^\nu(k,r) = P_i^\nu(k,r)\varrho_i^{new}(k,r)^T\left[\bar{C}\Omega_k\bar{C}^T + V + \varrho_i^{new}(k,r)P_i^\nu(k,r)\varrho_i^{new}(k,r)^T\right]^{-1}$ minimizes the trace of $P_i^\nu(k+1,r)$ (and where the gain of the augmented state Kalman filter K_k minimize the trace of Ω_{k+1}) and the Kullback divergence $\delta_i^{new}(k,r) = [P_i^\nu(k,r)]^{-1}(\nu^{new})^2$ is then maximized with respect to ν^{new}. We have $\begin{bmatrix} \Omega_{k+1} & 0 \\ 0 & P_i^\nu(k+1,r) \end{bmatrix}$ equivalent to

$$\begin{bmatrix} \bar{P}_{k+1} & 0 & 0 \\ 0 & P^{\nu}_{k+1} + \zeta^{\nu}_i(k+1,r)P^{\nu}_i(k+1,r)\zeta^{\nu T}_i(k+1,r) & \zeta^{\nu}_i(k+1,r)P^{\nu}_i(k+1,r)^{-1} \\ 0 & P^{\nu}_i(k+1,r)^{-1}\zeta^{\nu T}_i(k+1,r) & P^{\nu}_i(k+1,r) \end{bmatrix} \quad (122)$$

From Equation (122), the Kullback divergence between h_i^{new} and h_0 can be expressed as

$$\begin{bmatrix} \nu^{old} \\ \nu^{new} \end{bmatrix}^T \begin{bmatrix} P^{\nu}_{k+1} + \zeta^{\nu}_i(k+1,r)P^{\nu}_i(k+1,r)\zeta^{\nu T}_i(k+1,r) & \zeta^{\nu}_i(k+1,r)P^{\nu}_i(k+1,r)^{-1} \\ P^{\nu}_i(k+1,r)^{-1}\zeta^{\nu T}_i(k+1,r) & P^{\nu}_i(k+1,r) \end{bmatrix}^{-1} \begin{bmatrix} \nu^{old} \\ \nu^{new} \end{bmatrix}$$
$$= \begin{bmatrix} \nu^{old} - \zeta^{\nu}_i(k+1,r)\nu^{new} \\ \nu^{new} \end{bmatrix}^T \begin{bmatrix} P^{\nu}_{k+1} & 0 \\ 0 & P^{\nu}_i(k+1,r) \end{bmatrix}^{-1} \begin{bmatrix} \nu^{old} - \zeta^{\nu}_i(k+1,r)\nu^{new} \\ \nu^{new} \end{bmatrix} \quad (123)$$

The Kullback divergence (123) gives its minimal value $\delta_i^{new}(k,r) = [P_i^{\nu}(k+1,r)]^{-1} (\nu^{new})^2$ for $\nu^{old} = \zeta_i^{\nu}(k+1,r)\nu^{new}$. So, we conclude that the first step of the active GLR test follows a minmax strategy (see appendix) closing the proof of Theorem 4.3. Based on an inductive reasonning with the help of Theorems 4.1 and 4.2, the proposed active GLR test is then derived leading to a GLR detector of the form

$$\max_{i\in[1,\dots,N],\, i\neq[jumps\ already\ treated]\ \tilde{r}\in W} \{T_i^{new}(k, \tilde{r} - \rho_i)\} > \varepsilon \quad (124)$$

where the state vector $X_k = \begin{bmatrix} x_k^T & (\nu^{old})^T \end{bmatrix}^T$ of the reference model (96) includes the q states of jumps $\nu_k^{old} = \begin{bmatrix} \nu_k^1 & \dots & \nu_k^q \end{bmatrix}^T$ detected and isolated during the recursive processing. In [5], the off-line statistical decoupling of nuisance parameters is reduced to a static decoupling problem in a regression model. Our active GLR test solves on-line a dynamic statistical decoupling problem by rejecting the nuisance parameters which are statistically significant (see also appendix).

Under the multiple jumps detectability and distinguishability conditions of Theorem 3.1, the augmented state Kalman filter is guaranted to be stable at each step of the recursive treatment. With this implementation, the estimation of jumps detected and isolated during the processing will be improved from measurements available after their detection. In the case where the old jumps are extremely well estimated (the jump prediction errors does not converge exponentially to zero as the state prediction of the jump-free system but only asymptotically), then $\delta_i^{new}(k,r) = \tilde{\delta}_i(k,r)$ and the GLR test coincides with the modified GLR test.

5. **The Passive GLR Test.** The passive GLR is based on the assumption that the jumps occur frequently. So assume that the fixed reference model noted H_N is described

$$X_{k+1} = \bar{A}X_k + \bar{B}u_k + \bar{\Gamma}w_k \quad (125)$$
$$y_k = \bar{C}X_k + v_k \quad (126)$$

with $\bar{A} = \begin{bmatrix} A & F \\ 0 & 1 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{\Gamma} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$ and $X_k = \begin{bmatrix} x_k \\ \nu_k \end{bmatrix}$, the augmented state model including all jump's states $\nu_k = \begin{bmatrix} \nu_k^1 & \dots & \nu_k^j & \dots & \nu_k^N \end{bmatrix}$ that we wish to detect and isolate. From Equations (125) and (126), the jump hypothesis h_j described by Equations (9) and (10) can be viewed as an impulsive abrupt change on the j^{th} hypothetical jump's state, modelized as

$$\nu_{k+1}^j = \nu_k^j + \Delta\nu\delta_{kr},\ \forall j \in [1,\dots,N] \quad (127)$$

where r is the unknown occurrence time of impulsive abrupt change, $\Delta\nu$ the jump's state increment and δ_{kr} is the Kronecker operator.

Substituting Equation (127) in (125), we obtain the impulsive jump hypotheses, noted h_j^Δ, described by

$$\begin{aligned} X_{k+1} &= \bar{A}X_k + \bar{B}u_k + f_j^\Delta(k,r)\Delta\nu^j(k,r) + \bar{\Gamma}w_k \\ y_k &= \bar{C}X_k + v_k \end{aligned} \tag{128}$$

with $\Delta\nu^j(k,r) = \Delta\nu^j\delta_{kr_i}$, $f_j^\Delta(k,r) = f_j^\Delta\delta_{kr_i}$ with $f_j^\Delta = \begin{bmatrix} 0 \\ I_j \end{bmatrix}$ and $I_j^T = [0 \ldots 1 \ldots 0]$ has one at the j^{th} position and zero elsewhere. Based on an approach very similar to the modified GLR test of part 4, let

$$\begin{aligned} \hat{X}_{k+1} &= \bar{A}\hat{X}_k + \bar{B}u_k + K_k(y_k - \bar{C}\hat{X}_k) \\ \Omega_{k+1} &= \bar{A}\Omega_k\bar{A}^T + \bar{\Gamma}W\bar{\Gamma}^T - \bar{A}_k\Omega_k\bar{C}^T\left(\bar{C}\Omega_k\bar{C}^T + V\right)^{-1}\bar{C}\Omega_k\bar{A}^T \\ K_k &= \bar{A}\Omega_k\bar{C}^T H_k^{-1} \\ H_k &= \bar{C}\Omega_k\bar{C}^T + V \end{aligned} \tag{129}$$

the augmented state Kalman filter designed on the reference model Equations (125) and (126). The additive effect of the impulsive jump hypothese h_j^Δ on the state prediction error and on the innovation sequence of the augmented state Kalman filter propagates as

$$e_{k+1} = \tilde{e}_{k+1} + \zeta_j^\Delta(k+1,r)\Delta\nu \tag{130}$$

$$\gamma_k = \tilde{\gamma}_{k+1} + \varrho_j^\Delta(k,r)\Delta\nu \tag{131}$$

where $\tilde{e}_{k+1}$ and $\tilde{\gamma}_k$ represent the state prediction error and the innovation sequence on the jump-free system and where $\zeta_j^\Delta(k+1,r)$ and $\varrho_j^\Delta(k,r)$ propagate as

$$\begin{aligned} \zeta_j^\Delta(k+1,r) &= (\bar{A} - \bar{K}_k\bar{C})\zeta_j^\Delta(k,r),\ \zeta_j^\Delta(r,r) = f_j^\Delta \\ \varrho_j^\Delta(k,r) &= \bar{C}\zeta_j^\Delta(k,r) \end{aligned} \tag{132}$$

So, we can apply the following GLR detector

$$\max_{j\in[1,\ldots,N],\ \tilde{r}\in W}\{T_j^\Delta(k-\rho_j,\tilde{r})\} > \varepsilon \tag{133}$$

with

$$T_j^\Delta(k,r) = b_j^\Delta(k,r)^2 a_j^\Delta(k,r)^{-1} \tag{134}$$

$$a_j^\Delta(k,r) = \sum_{t=r+\rho_j}^{k} \varrho_j^{\Delta T}(t,r)H_t^{-1}\varrho_j^\Delta(t,r) \tag{135}$$

$$b_j^\Delta(k,r) = \sum_{t=r+\rho_j}^{k} \varrho_j^{\Delta T}(t,r)H_t^{-1}\gamma_t \tag{136}$$

if $\max\limits_{j,\ \tilde{r}}\{T_j^\Delta(k-\rho_j,\tilde{r}\} > \varepsilon$, then $(i,\hat{\tilde{r}}) = \arg\max\limits_{j,\tilde{r}}\{T_j^\Delta(k,\tilde{r}-\rho_j)\}$ and the implusive jump h_i^Δ is declared to be occurred at time where $\hat{r} = \hat{\tilde{r}} - \rho_i$, where

$$\Delta\hat{\nu}(k+1,\hat{r}) = a_i^\Delta(k,\hat{r})^{-1}b_i^\Delta(k,\hat{r}) \tag{137}$$

$$P^{\Delta\nu}(k+1,\hat{r}) = a_i^\Delta(k,\hat{r})^{-1} \tag{138}$$

represents the maximum likelihood prediction of the jump increment $\Delta\nu$ (under the assumption that $\Delta\nu$ has an infinite a priori covariance). At the detection time of the first jump, the tracking ability of the augmented state Kalman filter (129) can be improved from the updating strategy as

$$\begin{aligned} \hat{X}_{k+1}^{new} &= \hat{X}_{k+1}^{old} + \zeta_i^\Delta(k+1,\hat{r})\Delta\hat{\nu}(k+1,\hat{r}) \\ \Omega_{k+1}^{new} &= \Omega_{k+1}^{old} + \zeta_i^\Delta(k+1,\hat{r})P^{\Delta\nu}(k+1,\hat{r})\zeta_i^\Delta(k+1,\hat{r})^T \end{aligned} \tag{139}$$

In our case, the state of the matched filters given by $\zeta_j^{\Delta}(k,r)$ are spanned in the trajectory space of the augmented state Kalman filter's prediction errors. So, Equation (139) substituted in the augmented state Kalman filter (129) improves its tracking ability without to produce a possible instability on the resulting filter (under the stability and convergence conditions of the augmented state Kalman filter given by Jamouli (2007)). The treatment of another impulsive jump is then obtained by applying GLR detector (133) on the resulting filter after having reinitialized $\zeta_j^{\Delta}(k,r) = 0$, $\forall j \in [1,\ldots,N]$ immediately after the filter incrementation. The new initialization (139) allows $E(\hat{X}_k^{new})$ to reach the true state of the system very quickly (and $E(\gamma_t)$ to reach zero for jump compensation, consequently) avoiding the detection of the same jump several times. From an inductive reasonning, the passive GLR test is then derived and consists of the following steps:

1) Detection, isolation and estimation of one impulsive jump with the GLR detector (133),
2) Updating of the augmented state Kalman filter (129) with (139) to improve its tracking,
3) Go to Step 1.

The sequential multiple decision theory is not complete and the choise of the threshold level ε is not studied in this paper. However, some simulation results not presented in this paper show that only statistical tuning parameter ε can be fixed at the beginning of the processing (this is not the threshold level which is adaptive but the augmented Kalman filter).

If the updated reference model (60) is substituted to the fixed reference model (125), the jump hypothesis h_i^{Δ} can modelize another jump on the old changes or also the disappearance of the old jumps. In this case, a mixed active/passive GLR test can be derived for a complete strategy allowing the treatment of the appearance and the disappearance of sequential jumps.

6. **Results.** To illustrate the proposed approach we considered the following system described by the matrix

$$A = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{140}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{141}$$

$$W = 0.01 * Id(4,4) \text{ and } V = 0.5 * Id(3,3) \tag{142}$$

The faults isolability is satisfied with $rank\,[CAf_1\ CAf_2]$ and $F = [f_1\ f_2]$. The statistical variables describing the performances of the reconfigurable FTCS coincide with the statistical variables describing the performances of the statistical test. So, to simplify the Monte Carlo simulation, the proposed example will be realized in open loop. In the field od dynamic systems, the signal-to-noise ratio $\delta_i(k,r)$ are generally more greater than the signal to noise ratio treated in the fields of electrocardiogram analysis or geophysical signal processing and the size M of the sliding window $W = [k - M \leq \tilde{r} \leq k]$ can be generally chosen small. In our example, we take $M = 0$ (we do not optimize $\tilde{r}$ at all and we fixe the following rate of false alamrs: $P^F = 0.005$ from a table of Chi-Squared distribution.

In first, we suppose one fault occurred at 350 with magnitude 2, we obtain the following results.

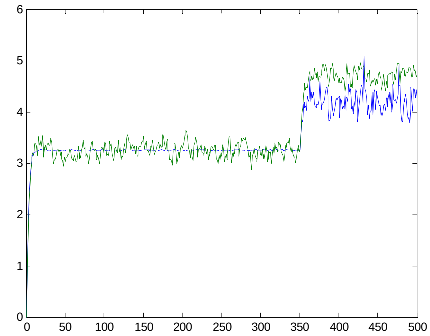

FIGURE 1. First state component and its estimation given by Willsky algorithm

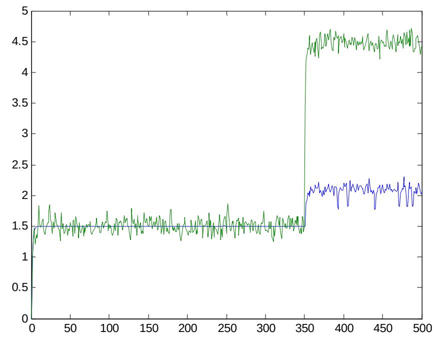

FIGURE 2. Second state component and its estimation given by Willsky algorithm

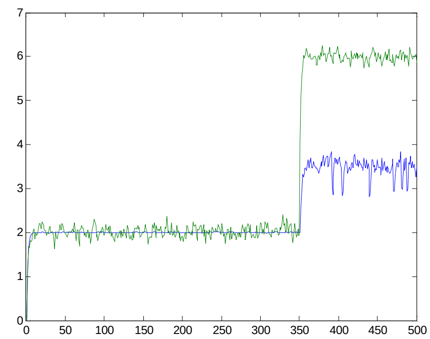

FIGURE 3. Third state component and its estimation given by Willsky algorithm

In the first case, the results shows that the proposed state estimation giving by our filter is more adaptive to the fault occurence than the Willsky algorithm.

In second case, we suppose two sequentiel faults occurred at 350 and 400 with magnitudes 2, we obtain the following results.

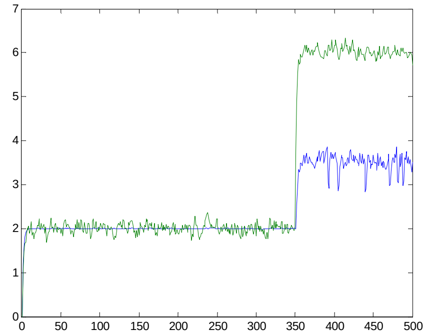

FIGURE 4. Fourth state component and its estimation given by Willsky algorithm

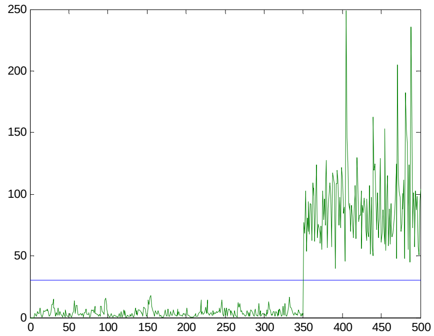

FIGURE 5. GLR test applied on Willsky algorithm

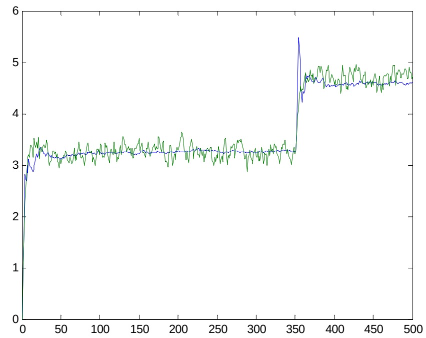

FIGURE 6. First state component and its estimation given by our adaptive filter algorithm

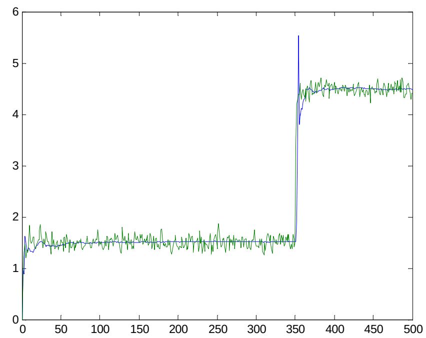

FIGURE 7. Second state component and its estimation given by our adaptive filter algorithm

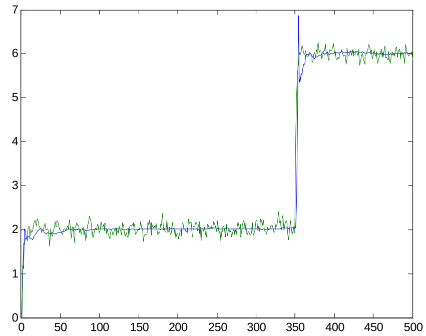

FIGURE 8. Third state component and its estimation given by our adaptive filter algorithm

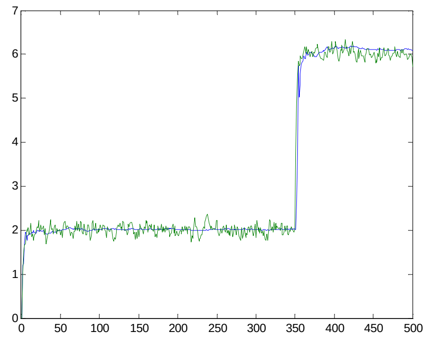

FIGURE 9. Fourth state component and its estimation given by our adaptive filter algorithm

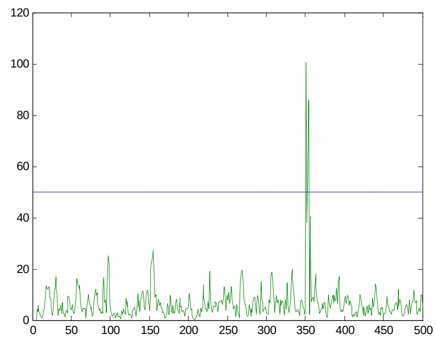

FIGURE 10. GLR test applied on our adaptive filter

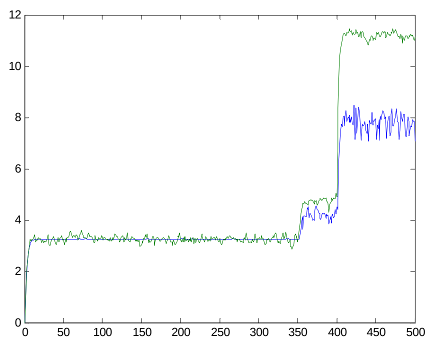

FIGURE 11. First state component and its estimation given by Willsky algorithm with presence of two sequential faults

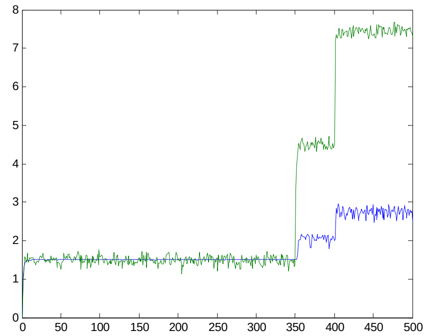

FIGURE 12. Second state component and its estimation given by Willsky algorithm with presence of two sequential faults

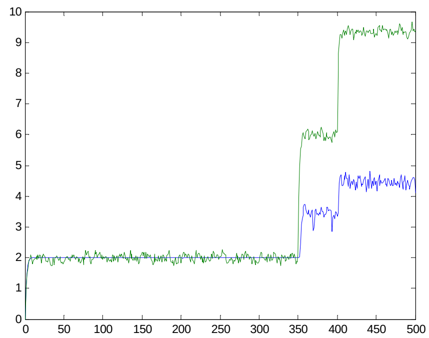

FIGURE 13. Third state component and its estimation given by Willsky algorithm with presence of two sequential faults

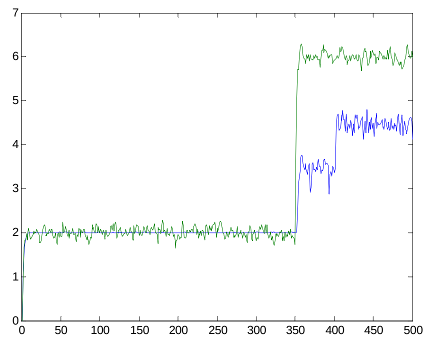

FIGURE 14. Fourth state component and its estimation given by Willsky algorithm with presence of two sequential faults

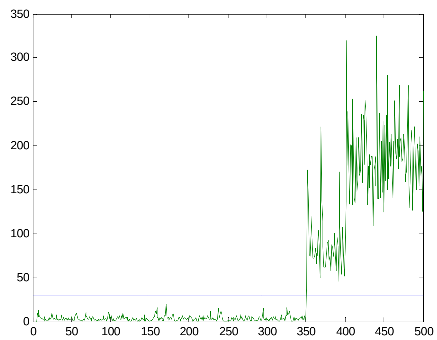

FIGURE 15. GLR test applied on Willsky algorithm with presence of two sequential faults

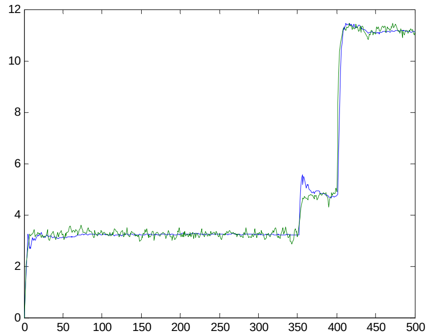

FIGURE 16. First state component and its estimation given by our adaptive algorithm with presence of two sequential faults

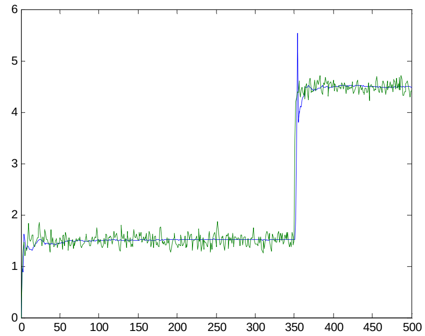

FIGURE 17. Second state component and its estimation given by our adaptive algorithm with presence of two sequential faults

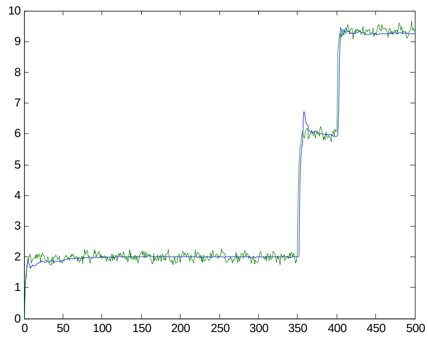

FIGURE 18. Third state component and its estimation given by our adaptive algorithm with presence of two sequential faults

In case of two sequential faults, the GLR detector applied on the passive augmented model allows to detect the first at 350s and the second faults at 400s. The Willsky GLR detect just the first fault at 350, and cannot detect the second.

Remarks and Discussion

- We have also computed the rate of false alarms and the rate of good detections with 10^5 Monte Carlo trials. We have obtained $\hat{P}^F \simeq 0,01$; $\hat{P}_D \simeq 0,85$ for the modified GLR test and $\hat{P}^F \simeq 0,0055$, $\hat{P}^D \simeq 0,91$ for the passive GLR test clearly much power. We conclude that the passive GLR test is very power when quick detections lead to bad jump estimations and thus very usefull for FTCS to maximize the rate of good decisions specially in reguard to the occurrence of a big jump which may greatly affect nominal performance of the system.

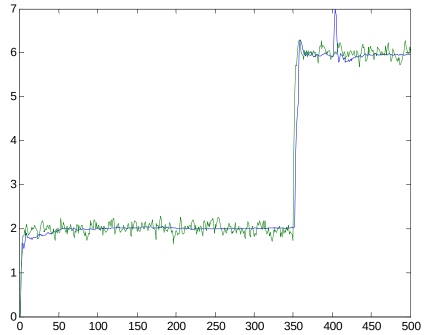

FIGURE 19. Fourth state component and its estimation given by our adaptive algorithm with presence of two sequential faults

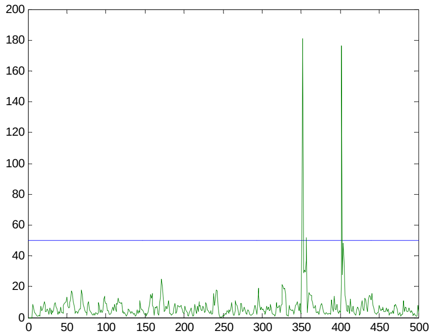

FIGURE 20. GLR test applied on adaptive algorithm with presence of two sequential faults

- We can improve this approach of detection and isolation with an active GLR based on free model which it will be augmented after each detection and isolation. The faults already detected and isolated will considered as perturbation and the we will update the new GLR in order to detect another fault.

The sequential multiple decision theory is not complete and the choise of the threshold level ε is not studied in this paper. However, some simulation results not presented in this paper show that only statistical tuning parameter ε can be fixed at the beginning of the processing (this is not the threshold level which is adaptive but the augmented Kalman filter).

7. **A Reconfigurable Fault-Tolerant Control System.** The purpose of the chapter is to show how the active and passive GLR test can be used in a FTCS. Our FTCS is only designed to reach the unique goal:

$$\underset{k\to\infty}{E(y_k)} = 0 \text{ under } r < \infty \tag{143}$$

in order words to asymptotically reject the effect of jumps on the output of the system (the reference input will be maintained equals to zero avoiding the need of a reconfigurable feedforward control law). The proposed FTCS is based on the active GLR test integrated via a reconfigurable control law designed on the model

$$X_{k+1} = \bar{A}X_k + \bar{B}u_k + \bar{\Gamma}w_k \tag{144}$$

$$y_k = \bar{C}X_k + v_k \tag{145}$$

where the main problem to reach our goal is that the pair $(\bar{A}, \bar{B})$ has N uncontrollable modes (under (A, B) controllable). The reconfigurable control law of the form $u_k =$

$u_k^n - G\hat{\nu}_k$ will be designed in such a way that the nominal control $u_k^n = -\bar{L}\hat{\bar{x}}_k$ of the jump-free system (6;7) (obtained by a LQG approach on an infinite horizon) is reconfigured on-line after each detection and isolation of one impulsive jump by the additive term $G\hat{\nu}_k$. In order to design G in relation with the available nominal control law, we assume that the implementation of the active GLR test is based on the two-stage Kalman filter, the only optimal filter which gives the state prediction of the jump-free system $\hat{\bar{x}}_k$. So, let

$$u_k = u_k^n - G\nu_k \tag{146}$$

the control law that we wish to design for a physical rejection of jumps ν_k. Under the state transformation

$$\begin{bmatrix} \bar{x}_k \\ \nu_k \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ \nu_k \end{bmatrix} \tag{147}$$

the system (141) with (144) can be expressed as

$$\begin{bmatrix} \bar{x}_{k+1} \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} A & (I-A)T+F \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \nu_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} (u_k^n - G\nu_k) \tag{148}$$

$$y_k = \begin{bmatrix} C & -CT \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \nu_k \end{bmatrix} \tag{149}$$

and the physical rejection of jumps will be obtained if and only if T let G satisfy the algebraic equations

$$(I-A)T + F = -BG \tag{150}$$

$$CT = 0 \tag{151}$$

Under the existence condition of a solution of (148) given by

$$rang \begin{bmatrix} A-I & B & -F \\ C & 0 & 0 \end{bmatrix} = rang \begin{bmatrix} I-A & B \\ C & 0 \end{bmatrix} \tag{152}$$

the gain G of the control law (143), solution of (148) gives

$$G \;=\; [C(I-A)^{-1}B]^{-1}C(I-A)^{-1}F \tag{153}$$

and $T = (I-A)^{-1}(BG - F)$. Under Equation (150), Equation (145) gives

$$\begin{bmatrix} \bar{x}_{k+1} \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \nu_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k^n + \begin{bmatrix} I \\ 0 \end{bmatrix} w_k \tag{154}$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \nu_k \end{bmatrix} + v_k \tag{155}$$

where $\bar{x}_k$ represent the state of the jump-free system. So, under (A, B) controllable, the LQG regulator $u_k^n = -\bar{L}\hat{\bar{x}}_k$ can be designed on the jump-free system h_0 (from the separation principle) to obtained the nominal system performances (not defined here). The reconfigurated control law, which reject the q jump's uncontrollable modes is given by

$$u_k = -\bar{L}\hat{\bar{x}}_k - G\hat{\nu}_k \tag{156}$$

from the two-stage Kalman filter or equivalently

$$u_k = -\begin{bmatrix} \bar{L} & G \end{bmatrix} \begin{bmatrix} I & -\zeta_k \\ 0 & I \end{bmatrix} \begin{bmatrix} I & \zeta_k \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_k \\ \hat{\nu}_k \end{bmatrix} = -\begin{bmatrix} \bar{L} & G - \bar{L}\zeta \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{\nu}_k \end{bmatrix} \tag{157}$$

from the augmented state Kalman filter. Note that after each detection and isolation of one jump, the nominal control law $u_k^n = -\bar{L}\hat{\bar{x}}_k$ is not affected by the active GLR test but only corrected by the additive term $G\hat{\nu}_k$ depending to the old jumps estimation (improved from measurement available after their detection). The active GLR test depends only to the state prediction errors of the Kalman filter decoupled from u_k and we can propose the following reconfigurable FTCS scheme:

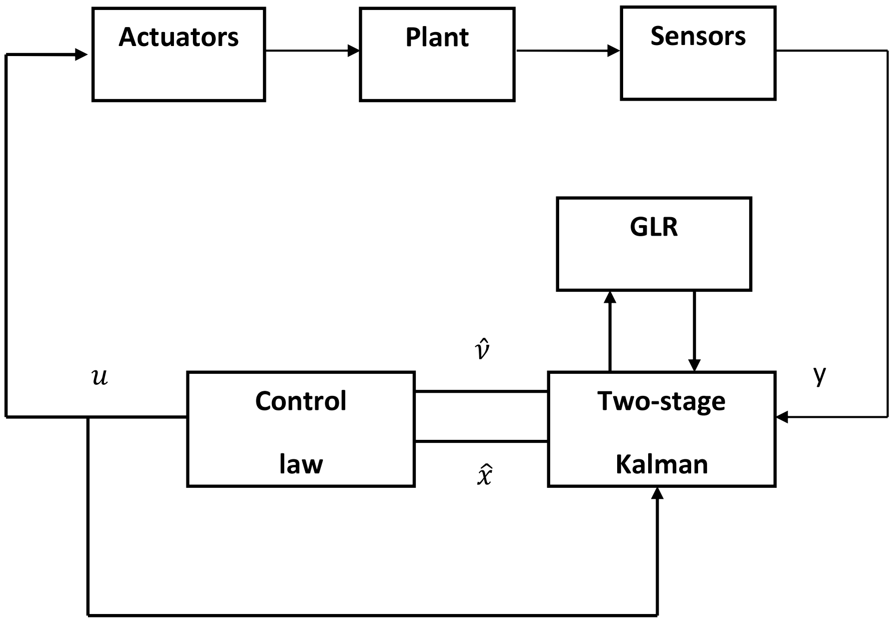

Figure 21. The reconfigurable FTCS scheme based on the active GLR test

The reference model used for the design of the control law GLR test coincides with the reference model used by the GLR Test. After each detection and isolation of one jump, the reference model is updated with the new state of jump and the three parts of the FTCS, i.e., the GLR detector, the Kalman filter, the control law, can be reconfigurated in harmony by the reconfiguration mechanism. To reduce the the computational requirement, the passive GLR test working on a fixed reference model can be used but the statistical performances of the reconfigurable FTCS will be closely related by the rates of false alarms and good decisions of the used statistical test.

8. **Numerical Example.** Consider the following discrete-time stochastic system

$$A = \begin{bmatrix} 0.5 & 2 & 0.2 \\ 0 & 0.4 & 1 \\ 0 & 0 & 0.1 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{158}$$

$V = 2I$ and $W = I$. Subject to two possible abrupt changes h_1 and h_2 on actuators modelized by $f_1 = B_1$, $f_2 = B_2$ with $\rho_1 = 2$ and $\rho_2 = 1$. From Theorem 3.1, the two jumps are statistically detectable and distinguishable, i.e., (11) and (12) hold with

$$rank \begin{bmatrix} CAf_1 & Cf_2 \end{bmatrix} \tag{159}$$

and

$$rank \begin{bmatrix} I - A & f_1 \; f_2 \\ C & 0 \end{bmatrix} = 3 + 2 = 5(I). \tag{160}$$

We can remark that the rank condition (I) guaranties the existence condition of the reconfigurable FTCS given by (149) for any jump's scenario. The statistical variables describing the performances of the reconfigurable FTCS coincide with the statistical variables describing the performances of the statistical test. So, to simplify the Monte Carlo simulation, the proposed example will be realized in open loop.

In the field of dynamic systems, the signal-to-noise ratio $\delta_i(k, r)$ are generally more greater than the signal to noise ratio treated in the fields of electrocardiogram analysis or geophysical signal processing and the size M of the sliding window $W = [k - M \leq \tilde{r} \leq k]$ can be generally chosen small. In our example, we take $M = 0$ (we do not optimize $\tilde{r}$ at all) and we fixe the following rate of false alarms: $P^F = 0.005$ from a table of Chi-Squared distribution. The first jump $\nu = 10$ appearing at time instant $k = 50$ has been chosen sufficiently important to ensure that the rate of correct decisions is very closed to one. The statistical comparaison will be made on the second jump $\nu = 3$ occurring at time instant $r = 60$.

We have computed the rate of false alarms and the rate of good detections with 10^5 Monte Carlo trials. We have obtained $\hat{P}^F \approx 0.01$, $\hat{P}_D \approx 0.85$ for the modified GLR test and $\hat{P}_2 \approx 0.0055$, $\hat{P}_D \approx 0.91$ for the active GLR test clearly much power. Many other simulations not presented in this paper, realized in the case where the second jump appears at times $r = 70, 80, \ldots$ have shown that the active GLR test always gives better results but with less significant results. In the limite case where the second jump appears after the first jump with a very long time delay, the two GLR tests have given the same results. We conclude that the active GLR test is very power when quick detections lead to bad jump estimations and thus very usefull for FTCS to maximize the rate of good decisions specially in reguard to the occurrence of a big jump which may greatly affect nominal performance of the system.

9. **Conclusion.** Derived from the works of Willsky and Jones [50], this paper has presented the active GLR test for sequential jumps detection in stochastic discrete-time linear systems. From a new updating strategy based on the statistical rejection of the jumps detected and isolated during the recursive treatment, the rate of false alarm has been minimized and the rate of good decisions maximized. The active GLR test has been integrated in a reconfigurable Fault-Tolerant Control System by using an LQG regulator designed on the jump-free system where the nominal control law is corrected on line to asymptotically reject the influence of jumps.

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Appendix 1. The predictive form of the Friendland's two-stage Kalman Filter which optimaly implement the following augmented state Kalman filter

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\nu}_{k+1} \end{bmatrix} = \hat{X}_{k+1} = \bar{A}\hat{X}_k + \bar{B}u_k + K_k\gamma_k \tag{161}$$

$$\begin{bmatrix} P^x_{k+1} & P^{x\nu}_{k+1} \\ P^{\nu x}_{k+1} & P^{\nu}_{k+1} \end{bmatrix} = \Omega_{k+1} = \bar{A}\Omega_k\bar{A}^T + \bar{\Gamma}W\bar{\Gamma}^T - \bar{A}\Omega_k\bar{C}^T\left(\bar{C}\Omega_k\bar{C}^T + V\right)^{-1}\bar{C}\Omega_k\bar{A}^T \tag{162}$$

$$K_k = \begin{bmatrix} K^x_k \\ K^{\nu^j}_k \end{bmatrix} = \bar{A}\Omega_k\bar{C}^T H_k^{-1},\ H_k = \bar{C}\Omega\bar{C}^T + V \tag{163}$$

with $\hat{X}_0 = \begin{bmatrix} \hat{x}_0 \\ \hat{\nu}_0 \end{bmatrix}$ and $\Omega_0 = \begin{bmatrix} P^x_0 & P^{x\nu}_0 \\ P^{\nu x}_0 & P^{\nu}_0 \end{bmatrix}$, where $\bar{A} = \begin{bmatrix} A & F \\ 0 & I \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$ and $\bar{\Gamma} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ is given by

$$\hat{x}_{k+1} = \hat{\bar{x}}_{k+1} + \zeta_{k+1}\hat{\nu}_{k+1} \tag{164}$$

$$P_{k+1} = \bar{P}_{k+1} + \zeta_{k+1}P^{\nu}_{k+1}\zeta^T_{k+1} \tag{165}$$

where $\left(\hat{\bar{x}}_{k+1}, \bar{P}_{k+1}\right)$ are given by the bias-free filter

$$\hat{\bar{x}}_{k+1} = A\hat{\bar{x}}_k + Bu_k + \bar{K}_k\bar{\gamma}_k \tag{166}$$

$$\bar{P}_{k+1} = A\bar{P}_kA^T + W - A\bar{P}_kC^T(C\bar{P}_kC^T + V)^{-1}C\bar{P}_kA^T \tag{167}$$

$$\bar{K}_k = A\bar{P}_kC^T\bar{H}_k^{-1} \tag{168}$$

$$\bar{\gamma}_k = y_k - C\hat{\bar{x}}_k \tag{169}$$

$$\bar{H}_k = C\bar{P}_kC^T + V \tag{170}$$

where $(\hat{\nu}_{k+1}, P^{\nu}_{k+1})$ are given the bias filter

$$\hat{\nu}_{k+1} = \hat{\nu}_k + K^{\nu}_k(\bar{\gamma}_k - \rho_k\hat{\nu}_k) \tag{171}$$

$$K^{\nu}_k = P^{\nu}_k\rho^T_k(\bar{H}_k + \rho_kP^{\nu}_k\rho^T_k)^{-1} \tag{172}$$

$$P^{\nu}_{k+1} = P^{\nu}_k - P^{\nu}_k\rho^T_k(\rho_kP^{\nu}_k\rho^T_k + \bar{H}_k)^{-1}\rho_kP^{\nu}_k \tag{173}$$

with the coupling equation

$$\zeta_{k+1} = (A - \bar{K}_kC)\xi_k + F \tag{174}$$

$$\rho_k = C\zeta_k \tag{175}$$

The initial conditions of the two-stage Kalman filter are given by

$$\begin{bmatrix} \hat{\bar{x}}_0 \\ \hat{\nu}_0 \end{bmatrix} = \begin{bmatrix} I & -\zeta_0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{x}_0 \\ \hat{\nu}_0 \end{bmatrix} \tag{176}$$

and

$$\begin{bmatrix} \bar{P}_0 & 0 \\ 0 & P^{\nu}_0 \end{bmatrix} = \begin{bmatrix} I & -\zeta_0 \\ 0 & I \end{bmatrix}\begin{bmatrix} P^x_0 & P^{x\nu}_0 \\ P^{\nu x}_0 & P^{\nu}_0 \end{bmatrix}\begin{bmatrix} I & -\zeta_0 \\ 0 & I \end{bmatrix}^T \tag{177}$$

with $\zeta_0 = P^{x\nu}_0(P^{\nu}_0)^{-1}$. The zero mean white innovation sequence of the bias-filter equals to the innovation sequence of the augmented state Kalman filter since $\gamma_k = y_k - \bar{C}\hat{X}_k = \bar{\gamma}_k - \rho_k\hat{\nu}_k$ where $H_k = \bar{H}_k + \rho_kP^{\nu}_k\rho^T_k$ is its covariance matrix. The optimality of two-stage Kalman filter can be proved under the transformation $T_k = \begin{bmatrix} I & -\zeta_k \\ 0 & I \end{bmatrix}$ with $\zeta_k =$

$P_k^{x\nu}[P_k^\nu]^{-1}$ applied on the augmented state filter

$$\hat{X}_{k+1} = \bar{A}\hat{X}_k + \bar{B}u_k + \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} (y_k - \bar{C}\hat{X}_k) \tag{178}$$

$$\Omega_{k+1} = \left(\bar{A} - \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} \bar{C}\right) \Omega_k \left(\bar{A} - \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} \bar{C}\right)^T + \bar{\Gamma} W \bar{\Gamma}^T + \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} V \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix}^T \tag{179}$$

as

$$T_{k+1}\hat{X}_{k+1} = T_{k+1}\bar{A}T_k^{-1}T_k\hat{X}_k + \bar{B}u_k + T_{k+1}\begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} (y_k - \bar{C}T_k^{-1}T_k\hat{X}_k) \tag{180}$$

$$\begin{aligned} T_{k+1}\Omega_{k+1}T_{k+1}^T = {} & T_{k+1}\left(\bar{A} - \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} \bar{C}\right) T_k^{-1}T_k\Omega_k T_k^T (T_k^T)^{-1} \left(\bar{A} - \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} \bar{C}\right)^T \\ & + \bar{\Gamma} W \bar{\Gamma}^T + T_{k+1}\begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix} V \begin{bmatrix} K_k^x \\ K_k^\nu \end{bmatrix}^T T_{k+1}^T \end{aligned} \tag{181}$$

From $\begin{bmatrix} \hat{\bar{x}}_k \\ \hat{\nu}_k \end{bmatrix} = T_k\hat{X}_k$, $\begin{bmatrix} \bar{P}_k & 0 \\ 0 & P_k^\nu \end{bmatrix} = T_k\Omega T_k^T$ and under the assumption that $K_k^\nu = \bar{K}_k + \zeta_{k+1}K_k^\nu$ with $\zeta_{k+1} = P_{k+1}^{x\nu}[P_{k+1}^\nu]^{-1}$, we obtain

$$\begin{bmatrix} \hat{\bar{x}}_{k+1} \\ \hat{\nu}_{k+1} \end{bmatrix} = \begin{bmatrix} A - \bar{K}_kC & 0 \\ -K_k^\nu C & (I - K_k^\nu\rho_k) \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_k \\ \hat{\nu}_k \end{bmatrix} + \bar{B}u_k + \begin{bmatrix} \bar{K}_k \\ K_k^\nu \end{bmatrix} y_k \tag{182}$$

$$\begin{aligned} \begin{bmatrix} \bar{P}_{k+1} & \times \\ \times & P_{k+1}^\nu \end{bmatrix} = {} & \begin{bmatrix} A - \bar{K}_kC & 0 \\ -K_k^\nu C & (I - K_k^\nu\rho_k) \end{bmatrix} \begin{bmatrix} \bar{P}_k & 0 \\ 0 & P_k^\nu \end{bmatrix} \begin{bmatrix} A - \bar{K}_kC & 0 \\ -K_k^\nu C & (I - K_k^\nu\rho_k) \end{bmatrix}^T \\ & + \begin{bmatrix} \bar{K}_k \\ K_k^\nu \end{bmatrix} V \begin{bmatrix} \bar{K}_k \\ K_k^\nu \end{bmatrix}^T + \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{183}$$

and with $\bar{K}_k = A\bar{P}_kC^T\bar{H}_k^{-1}$ minimizing the trace of $\bar{P}_{k+1}$.

The trace of P_{k+1}^ν is minimized by $K_k^\nu = P_k^\nu\rho_k^T(\bar{H}_k + \rho_kP_k^\nu\rho_k^T)^{-1}$ and the proof that $K_k^x = \bar{K}_k + \zeta_{k+1}K_k^\nu$ where $\zeta_{k+1} = P_{k+1}^{x\nu}[P_{k+1}^\nu]^{-1}$ is given in Keller and Darouach (1997). Note that the existence condition of $\zeta_k = P_k^{x\nu}[P_k^\nu]^{-1}$ given by $P_k^\nu > 0$ is always satisfied under (21.a).