International Journal of Innovative Computing, Information and Control
Volume **8**, Number **5(A)**, May **2012**

ICIC International ©2012 ISSN 1349-4198
pp. 3215–3222

WAVELET ANALYSIS OF LINEAR OPTIMAL CONTROL SYSTEMS INCORPORATING OBSERVERS

Zahra Razavinasab[1], Mohammad Ali Vali[2] and Mahmoud Samavat[1]

[1]Department of Electrical Engineering
[2]Department of Mathematics
Shahid Bahonar University of Kerman
P.O.Box 76169, 133 Afzalipour Square, Kerman 76169-14111, Iran
z.razavinasab@gmail.com; { msamavat; mvali }@mail.uk.ac.ir

Received November 2010; revised May 2011

Abstract. *The extended Legendre wavelets and its operational matrix of integration are successfully applied for the analysis of linear optimal control systems incorporating observers using two approaches (the Kronecker method and the recursive method). The two methods simplify the system of state equations into the solution of a set of linear algebraic equations. Furthermore, the proposed algorithms can be easily implemented in a digital computer and the solutions can be obtained for any length of time. It seems, even with a relatively low number of terms, the proposed algorithms give very accurate results when compared with the results of the existing approaches. Further, owing to the simplicity of the recursive approach, the approach presents considerable computational advantages when compared with the Kronecker approach and the other existing approaches. Finally a numerical example is given to support these claims.*

Keywords: Extended Legendre wavelets, Operational matrix of integration, Optimal control, Observers

1. **Introduction.** In recent years, many researches have applied various methods of optimization on different types of systems (see, e.g., [1-4] and the references therein).

It appears that in many practical applications, only some of the states can always be measured. Since effective control of a process requires sufficient information on the states of the process, the research for this problem is important both in theoretical and practical applications. If the plant is observable, the problem can be solved by using an observer incorporated to estimate the unknown states.

Many orthogonal functions or polynomial series such as Walsh functions [5], block-pulse functions [6], Shifted Legendre series [7,8], Laguerre series [9], Shifted-Jacobi series [10], Taylor series [11] and Fourier series [12] were developed to help the analysis of linear optimal control systems incorporating observers. A distinguishing feature of the previously mentioned approaches is reducing the differential equations into a set of linear algebraic equations, which is very convenient for digital computation. The approaches given in [7,8] would lead to the same results but the recursive approach in [8] is faster than the non-recursive approach in [7].

On the other hand, wavelets permit the accurate representation of a variety of functions and operators. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis and many other areas. Special attention has been given to application of Haar wavelets [13], Legendre wavelets [14], general Legendre wavelets [15], Sine-cosine wavelets [16] and Chebyshev wavelets [17]. The main advantage of the wavelet analysis is its accuracy and ability to transform complex problems into a system of algebraic equations and thus making it computationally feasible. In [18], the

Legendre wavelet operational matrix of integration is defined on the interval [0,1). Using the translation property of the Legendre wavelets, extended Legendre wavelets defined on the interval $(-r, r)$ have been achieved in [19] where r is any rational constant. The extended Legendre wavelet operational matrix of integration (P_{ELW}) is sparse, equal to every subinterval and is low dimensional [20]. Thus using (P_{ELW}) instead of using the Legendre wavelet operational matrix of integration is more attractive computationally and simplifies the solution of system of algebraic equations. Also the extended Legendre wavelets can efficiently and accurately model both continuous and discontinuous problems [20].

In this paper, for the first time, the extended Legendre wavelets are used for the analysis of linear optimal control systems incorporating observers. A simple and powerful computational algorithm is proposed to obtain the optimal control signal from $t = 0$ to any length of time. By illustrating a numerical example, the effectiveness of the proposed methods (the Kronecker method and the recursive method) are demonstrated and a comparison is made between ELW approximations and the exact results. Also it is shown that the recursive approach has considerable computational advantages when compared with the other possible methods. Finally the results of [7,8] are modified.

2. **Properties of the Extended Legendre Wavelets (ELW).** The ELW are constructed through a translation operator transformation on the Legendre wavelets. They are defined on the interval $\left(-\frac{i}{a^n}, \frac{i}{a^n}\right)$ as [20]

$$\psi_{nm}^k(t) = \begin{cases} (2k+1)^{1/2} a^{\frac{n}{2}} L_k(2a^n t - 2m - 1) & \text{for } \frac{m}{a^n} \le t < \frac{(m+1)}{a^n} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $(a \in N, a \geq 2)$, $m = -i, \ldots, 0, 1, \ldots, i-1$, $(i \in N)$ and $n = 0, 1, \ldots$ denotes decomposition level. The functions $L_k(t)$ are the k degree Legendre polynomials and satisfy the following recursive formula:

$$L_0(t) = 1, \quad L_1(t) = t,$$

$$L_{k+1}(t) = \left(\frac{2k+1}{k+1}\right) t L_k(t) - \left(\frac{k}{k+1}\right) L_{k-1}(t), \quad k = 1, 2, 3, \ldots$$

where k is the degree of the Legendre polynomials.

For any rational constant r, there exist two positive integers a and i such that $r = \frac{i}{a^n}$ or $r = -\frac{i}{a^n}$ and the ELW on the interval $(-r, r)$ can be obtained.

The ELW, defined on the interval $\left(-\frac{i}{a^n}, \frac{i}{a^n}\right)$, is an orthogonal set [20].

The function $f(t)$ defined on the interval $\left(-\frac{i}{a^n}, \frac{i}{a^n}\right)$ may be approximated as

$$f(t) \cong \sum_{m=-i}^{i-1} \sum_{k=0}^{K} c_{nm}^k \psi_{nm}^k(t) = C^T \Psi(t) \tag{2}$$

where n is decomposition level, C and Ψ are $2i(K+1) \times 1$ matrices and given by

$$C = \left[c_{n,-i}^0, c_{n,-i}^1, \ldots, c_{n,-i}^K, \ldots, c_{n0}^0, \ldots, c_{n0}^K, \ldots, c_{n,i-1}^0, \ldots, c_{n,i-1}^K\right]^T \tag{3}$$

$$\Psi(t) = \left[\psi_{n,-i}^0(t), \psi_{n,-i}^1(t), \ldots, \psi_{n,-i}^K(t), \ldots, \psi_{n0}^0(t), \ldots, \psi_{n0}^K(t), \ldots, \psi_{n,i-1}^0(t), \ldots, \psi_{n,i-1}^K(t)\right]^T \tag{4}$$

and

$$c_{nm}^k = \langle f(t), \psi_{nm}^k(t) \rangle \tag{5}$$

in Equation (5), $\langle \cdot, \cdot \rangle$ denotes the inner product.

For any t belongs to the interval $[\frac{m}{a^n}, \frac{m+1}{a^n})$, let

$$\Psi_{nm}(t) = \left(\psi_{nm}{}^0(t), \psi_{nm}{}^1(t), \ldots, \psi_{nm}{}^K(t)\right)^T, \tag{6}$$

then the integration of the vector $\Psi_{nm}(t)$ can be approximated by

$$\int_{\frac{m}{a^n}}^{t} \Psi_{nm}(t) d\tau \cong P_{ELW} \Psi_{nm}(t), \tag{7}$$

where $\Psi_{nm}(t)$ is given in Equation (6) and P_{ELW} is the operational matrix of integration of order $(K+1) \times (K+1)$ and is given by

$$P_{ELW} = \frac{1}{2a^n} \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{5}\times\sqrt{3}}{5\times 3} & 0 & \cdots & 0 \\ 0 & -\frac{\sqrt{5}\times\sqrt{3}}{5\times 3} & 0 & \frac{\sqrt{7}\times\sqrt{5}}{7\times 5} & \cdots & 0 \\ 0 & 0 & -\frac{\sqrt{7}\times\sqrt{5}}{7\times 5} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\sqrt{2K+1}\times\sqrt{2K-1}}{(2K+1)\times(2K-1)} \\ 0 & 0 & 0 & \cdots & \frac{-\sqrt{2K+1}\times\sqrt{2K-1}}{(2K+1)\times(2K-1)} & 0 \end{bmatrix} \tag{8}$$

3. **Problem Statements.** A linear time-invariant completely observable and completely controllable system is considered as follows

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{9}$$
$$y(t) = Cx(t), \quad x(0) = X_0, \tag{10}$$

where $x(t)$ is the n-state vector, $u(t)$ is the m-control vector, $y(t)$ is the p-output vector, and A, B and C are constant matrices of appropriate dimensions.

When an observer is incorporated to generate an estimate $\hat{x}(t)$ of the plant state vector, we need to choose the matrix L in the feedback law

$$u^*(t) = L\hat{x}(t), \tag{11}$$

so that the cost function

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt, \quad Q \geq 0, \quad R > 0 \tag{12}$$

is minimized. $u^*(t)$ in Equation (11) is the optimal control signal. In Equation (12), Q is a real symmetric positive semidefinite matrix of order $n \times n$ and the $m \times m$ matrix R is a real symmetric positive definite. L in Equation (11) is given by

$$L = -R^{-1}B^T P, \tag{13}$$

where P is the positive-definite solution of the following Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{14}$$

It has been shown that an $(n-p)$-dimensional state observer for the system of Equations (9) and (10) can be constructed as [7]

$$\dot{z}(t) = Dz(t) + Gy(t) + Hu(t), \tag{15}$$
$$\hat{x}(t) = My(t) + Nz(t), \tag{16}$$

where $z(t)$ is the $(n-p)$-state vector, and D, G, H, M and N are $(n-p) \times (n-p)$, $(n-p) \times p$, $(n-p) \times m$, $n \times p$ and $n \times (n-p)$ matrices, respectively. The observer given in

Equations (15) and (16) can produce the estimate $\hat{x}(t)$ when the following relationships are satisfied:

$$z(t) = Ux(t) + e(t), \tag{17}$$
$$\dot{e}(t) = De(t), \tag{18}$$

where

$$UA - DU = GC, \tag{19}$$
$$H - UB = 0, \tag{20}$$
$$MC + NU = I_n \tag{21}$$

Substituting Equations (16), (17) and (21) into Equation (11), we obtain

$$u^*(t) = Lx(t) + LNe(t) \tag{22}$$

Inserting Equation (22) into Equation (9) gives

$$\begin{aligned} \dot{x}(t) &= (A + BL)x(t) + BLNe(t), \\ &\triangleq \hat{A}x(t) + \hat{B}e(t). \end{aligned} \tag{23}$$

Equation (18) and Equation (23) are used for computing the optimal control signal $u^*(t)$.

4. **ELW Analysis of Linear Optimal Control Systems Incorporating Observers.** In this section we propose an effective computational algorithm in order to compute the numerical solutions of $u^*(t)$ over an arbitrary interval $[0, T]$, using the ELW and the matrix of integration defined in Section 2.

First, an arbitrary time interval $\frac{1}{a}$ $(a \geq 2, a \in N)$ is chosen for the independent variable t. The state vector $x(t)$ and error vector $e(t)$ over the subinterval $\left[\frac{m}{a}, \frac{m+1}{a}\right]$ can be approximated in terms of ELW with $n = 1$ as follows:

$$x(t) \approx X^{(m)}\Psi(\tau), \tag{24}$$
$$e(t) \approx E^{(m)}\Psi(\tau), \tag{25}$$

where $0 \leq \tau \leq \frac{1}{a}$, and

$$t = \frac{m}{a} + \tau, \quad m = 0, 1, 2, \ldots, (T \times a) - 1 \tag{26}$$
$$X^{(m)} = \left[{x_{1m}}^0\, {x_{1m}}^1 \ldots {x_{1m}}^K\right], \tag{27}$$
$$E^{(m)} = \left[{e_{1m}}^0\, {e_{1m}}^1 \ldots {e_{1m}}^K\right], \tag{28}$$

and m denotes those variables calculated within the time interval $\frac{m}{a} \leq t \leq \frac{m+1}{a}$.

$\Psi(\tau)$ in Equations (24) and (25) is defined as

$$\Psi(\tau) = \left({\psi_{10}}^0(\tau), {\psi_{10}}^1(\tau), \ldots, {\psi_{10}}^K(\tau)\right)^T. \tag{29}$$

Integrating $\dot{x}(t)$ and $\dot{e}(t)$ once with respect to t, over the interval $\frac{m}{a} \leq t \leq \frac{m+1}{a}$, we have

$$\int_{\frac{m}{a}}^{t} \dot{x}(t)dt = x(t) - x_{\frac{m}{a}}, \tag{30}$$

$$\int_{\frac{m}{a}}^{t} \dot{e}(t)dt = e(t) - e_{\frac{m}{a}}. \tag{31}$$

Integrating Equations (18) and (23) over the interval $\frac{m}{a} \leq t \leq \frac{m+1}{a}$ and using the approximated values of $x(t)$, $e(t)$, $x_{\frac{m}{a}}$ and $e_{\frac{m}{a}}$ and using the matrix of integration P_{ELW}, yield

$$\left(E^{(m)} - E_{\frac{m}{a}}\right) = DE^{(m)}P_{ELW}, \tag{32}$$

$$\left(X^{(m)} - X_{\frac{m}{a}}\right) = \hat{A}X^{(m)}P_{ELW} + \hat{B}E^{(m)}P_{ELW}, \tag{33}$$

where

$$X_{\frac{m}{a}} = \left[\frac{x_{\frac{m}{a}}}{a^{1/2}}, 0, 0, \dots, 0\right], \tag{34}$$

$$E_{\frac{m}{a}} = \left[\frac{e_{\frac{m}{a}}}{a^{1/2}}, 0, 0, \dots, 0\right]. \tag{35}$$

4.1. **The Kronecker product method.** Equations (32) and (33) define a set of algebraic equations. Applying the operation of Kronecker product ($\otimes$) [21] to Equations (32) and (33), we obtain

$$vec\left(E^{(m)}\right) = \left({I_q}^T \otimes I_{(n-p)\times(n-p)} - {P_{ELW}}^T \otimes D\right)^{-1} vec\left(E_{\frac{m}{a}}\right), \tag{36}$$

$$vec\left(X^{(m)}\right) = \left({I_q}^T \otimes I_{n\times n} - {P_{ELW}}^T \otimes \hat{A}\right)^{-1} vec\left(\hat{Z}\right), \tag{37}$$

where

$$\hat{Z} \triangleq X_{\frac{m}{a}} + \hat{B}E^{(m)}P_{ELW}. \tag{38}$$

In Equations (36) and (37), $q = (K+1) \times (K+1)$ and the operation of vec, stacks the columns of an appropriate matrix into a single column vector [21]. We can solve Equations (36) and (37) for $X^{(m)}$ and $E^{(m)}$. Finally, the optimal control signal $u^*(t)$ can be obtained within any time interval $\frac{m}{a} \leq t \leq \frac{m+1}{a}$ as follows

$$u^*(t) = \left(LX^{(m)} + LNE^{(m)}\right)\Psi(\tau), \tag{39}$$

where $\Psi(\tau)$ given by Equation (29).

Solving Equations (36) and (37) involves inversion of a matrix of size $(K+1) \times n$ or $(K+1) \times (n-p)$ which becomes large as the value of K increases. On the other hand, more accurate results can be obtained with increasing the value of K. As can be seen from Equation (8), P_{ELW} is sparse, equal to every subinterval and is similar to the integration operational matrix of shifted Legendre polynomials in [8]. So, a recursive algorithm will be developed (in order to overcome this difficulty) in the same manner of [8] in the next section.

4.2. **The recursive ELW method.** Substituting matrix P_{ELW} into Equation (32) and rearranging the terms, gives

$$\begin{bmatrix} Z_{00} & Z_{01} & 0 & \cdots & 0 & 0 \\ Z_{10} & Z_{11} & Z_{12} & \cdots & 0 & 0 \\ 0 & Z_{21} & Z_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z_{(K-1)(K-1)} & Z_{(K-1)K} \\ 0 & 0 & 0 & \cdots & Z_{K(K-1)} & Z_{KK} \end{bmatrix} \begin{bmatrix} S_m^0 \\ S_m^1 \\ S_m^2 \\ \vdots \\ S_m^{K-1} \\ S_m^K \end{bmatrix} = \begin{bmatrix} V_m^0 \\ V_m^1 \\ V_m^2 \\ \vdots \\ V_m^{K-1} \\ V_m^K \end{bmatrix} \tag{40}$$

where

$$Z_{ij} = \begin{cases} I_q, & \text{if} \quad i = j = 1, 2, \ldots, K \\ \frac{D}{2a}\gamma(i+1), & \text{if} \quad i = 0, 1, 2, \ldots, K-1 \text{ and } j = i+1 \\ -\frac{D}{2a}\gamma(i), & \text{if} \quad i = 1, 2, \ldots, K \text{ and } j = i-1 \\ I_q - \frac{D}{2a}, & \text{if} \quad i = j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{41}$$

$$V_m^i = \begin{cases} \frac{e(\frac{m}{a})}{a^{1/2}}, & \text{if} \quad i = 0 \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

and

$$S_m^i = e_{1m}{}^i, \quad \text{for all } i \tag{43}$$

Similarly, substituting matrix P_{ELW} into Equation (33) and rearranging the terms, leads to Equation (40) where

$$Z_{ij} = \begin{cases} I_n, & \text{if} \quad i = j = 1, 2, \ldots, K \\ \frac{\hat{A}}{2a}\gamma(i+1), & \text{if} \quad i = 0, 1, 2, \ldots, K-1 \text{ and } j = i+1 \\ -\frac{\hat{A}}{2a}\gamma(i), & \text{if} \quad i = 1, 2, \ldots, K \text{ and } j = i-1 \\ I_n - \frac{\hat{A}}{2a}, & \text{if} \quad i = j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

$$V_m^i = \begin{cases} \frac{x(\frac{m}{a})}{a^{1/2}} + \frac{\hat{B}}{2a}\left[e_{1m}^0 - \frac{\sqrt{3}}{3}e_{1m}^1\right], & \text{if} \quad i = 0 \\ \frac{\hat{B}}{2a}\left[\gamma(i)e_{1m}^{i-1} - \gamma(i+1)e_{1m}^{i+1}\right], & \text{if} \quad i = 1, 2, \ldots, K-1 \\ \frac{\hat{B}}{2a}\gamma(K)e_{1m}^{K-1}, & \text{if} \quad i = K \end{cases} \tag{45}$$

$$S_m^i = x_{1m}{}^i, \quad \text{for all } i \tag{46}$$

In the above equations we have the following formula for γ:

$$\gamma(i) = \frac{\sqrt{2i+1} \times \sqrt{2i-1}}{(2i+1)(2i-1)} \tag{47}$$

${S_{1m}}^i$ can be obtained by using the following recursive equations:

$$M_{ij} = \begin{cases} {Z_{ii}}^{-1} & \text{if} \quad i = K, \\ (Z_{ii} + Z_{i,i+1}R_{i+1,i})^{-1} & \text{if} \quad i = K-1, K-2, \ldots, 2, 1, 0. \end{cases} \tag{48}$$

$$d_m^i = \begin{cases} M_{ii}V_m^i & \text{if} \quad i = K, \\ M_{ii}(V_m^i - Z_{i,i+1}d_m^{i+1}), & \text{if} \quad i = K-1, K-2, \ldots, 2, 1, 0. \end{cases} \tag{49}$$

$$R_{i,i-1} = -M_{ii}Z_{i,i-1}, \quad i = K, K-1, \ldots, 2 \tag{50}$$

$${S_{1m}}^i = R_{i,i-1}{S_{1m}}^{i-1} + d_m^i, \quad \text{for } i = 1, 2, \ldots, K-1. \tag{51}$$

$${S_{1m}}^0 = d_m^0. \tag{52}$$

In Equation (48), the size of the matrix to be inverted can be kept to n or $n - p$ instead of $(K+1) \times n$ or $(K+1) \times (n-p)$ as in the case of using Kronecker product method. Therefore, in the case of recursive method, the size of the matrix becomes much smaller and we will have considerable computational advantages when compared with the Kronecker method.

Table 1. The shifted Legendre, ELW approximation and exact values of $u^*(t)$

t	Shifted Legendre ($m=6$) [7,8]	ELW. ($a=2$, $K=3$)	ELW. ($a=2$, $K=5$)	Exact solution
0.0	−1.292516640278664	−1.299904510932026	−1.299999866277128	−1.300000000000000
0.5	−0.405565387685947	−0.403114835887336	−0.403210190164076	−0.403210323886947
1.0	0.094265048432012	0.092486850114900	0.092392049038985	0.092391925874173
1.5	0.343861011331337	0.342700507906432	0.342625091370763	0.342625007255783
2.0	0.440089433996108	0.441560518009806	0.441502152249011	0.441502101774488
2.5	0.447566419692893	0.449309018011448	0.449262392340787	0.449262364071693
3.0	0.406425822525273	0.405911166379653	0.405872596999942	0.405872581476822
3.5	0.340087827988379	0.338015782678808	0.337983480920393	0.337983471945097
4.0	0.263027533523418	0.262957303971464	0.262930631469972	0.262930625494146
4.5	0.188543529072207	0.191373832323613	0.191352594098633	0.191352589331543
5.0	0.136526477631701	0.129108823872968	0.129092832692696	0.129092828379141

5. **Conclusion.** The extended Legendre wavelets and its operational matrix of integration are applied for the analysis of linear optimal control systems incorporating observers using two different approaches. In both cases, by using these approximations and the operational matrix of integration, the differential equations are reduced into a set of linear algebraic equations, which are very convenient for digital computation. The proposed algorithms compute the numerical solutions of the optimal control signal $u^*(t)$ over an arbitrary interval $[0, T]$. A numerical example is given to show that by using ELW, even with a relatively low number of terms, we have excellent results. Furthermore, a recursive approach in the same manner of [8] is developed to speed up the proposed algorithm.

Acknowledgement. The authors wish to express their sincere thanks to the referees for valuable suggestions which improved the final manuscript.

REFERENCES

[1] X. T. Wang, Numerical solution of optimal control for scaled systems by hybrid functions, *International Journal of Innovative Computing, Information and Control*, vol.4, no.4, pp.849-855, 2008.
[2] H. A. Tehrani and S. M. Karbassi, Minimum norm time-optimal control of linear discrete-time periodic systems by parameterization of state feedback, *International Journal of Innovative Computing, Information and Control*, vol.5, no.8, pp.2151-2158, 2009.
[3] S. M. S. Modarres and S. M. Karbassi, Time-optimal control of discrete-time linear systems with state and input time-delays, *International Journal of Innovative Computing, Information and Control*, vol.5, no.9, pp.2619-2625, 2009.
[4] L. Tang, L. Zhao and J. Guo, Research on pricing policies for seasonal goods based on optimal control theory, *ICIC Express Letters*, vol.3, no.4(B), pp.1333-1338, 2009.
[5] S. Kawaji and R. Tada, Walsh series analysis in optimal control systems incorporating observers, *Int. J. Control*, vol.37, pp.455-462, 1983.
[6] S. Kawaji, Block-pulse series analysis of linear systems incorporating observers, *Int. J. Control*, vol.37, pp.1113-1120, 1983.
[7] J. Chou and I. Horng, Shifted Legendre series analysis of linear optimal control systems incorporating observers, *Int. J. Systems Science*, vol.16, pp.863-867, 1985.
[8] B. M. Mohan and S. K. Kar, Shifted Legendre polynomial approach to analysis of linear optimal control systems incorporating observers, *Proc. of INDICON 2008*, pp.383-387, 2008.

[9] M. Perng, Laguerre polynomial analysis in optimal control systems incorporating observers, *Int. J. Control*, vol.44, pp.43-48, 1986.

[10] T. Lee, S. Tsay and I. Horng, Shifted-Jacobi series analysis of linear optimal control systems incorporating observers, *J. Franklin Inst.*, vol.321, pp.289-298, 1986.

[11] I. R. Horng, J. H. Chou and R. Y. Tsai, Taylor series analysis of linear optimal control systems incorporating observers, *Int. J. Control*, vol.44, pp.1265-1272, 1986.

[12] H. Y. Chung and Y. Y. Sun, Fourier series analysis of linear optimal control systems incorporating observers, *Int. J. Systems Science*, vol.18, pp.213-220, 1987.

[13] C. H. Hsiao and S. P. Wu, Numerical solution of time-varying functional differential equations via Haar wavelets, *Appl. Math. Comp.*, vol.188, pp.1049-1058, 2007.

[14] R. Ebrahimi, M. A. Vali, M. Samavat and A. A. Gharavisi, A computational method for solving optimal control of singular systems using the Legendre wavelets, *ICGST. ACSE. J.*, vol.9, pp.1-6, 2009.

[15] X. T. Wang, Numerical solution of time-varying generalized delay systems via general Legendre wavelets, *International Journal of Innovative Computing, Information and Control*, vol.2, no.6, pp.1355-1363, 2006.

[16] S. H. Nasehi, M. Samavat and M. A. Vali, Analysis of time-delay systems via sine-cosine wavelets, *Proc. of the 39th Annual Iranian Mathematics Conference*, 2008.

[17] E. Babolian and F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, *Appl. Math. Comp.*, vol.188, pp.417-426, 2007.

[18] M. Razzaghi and S. Yousefi, The Legendre wavelets operational matrix of integration, *Int. J. Systems Science*, vol.32, pp.500-502, 2001.

[19] X. Y. Zheng, X. F. Yang and Y. Wu, Properties of extended Legendre wavelets, *Proc. of ICWAPR*, vol.2, pp.629-633, 2008.

[20] X. Y. Zheng, X. F. Yang and Y. Wu, Technique for solving differential equation by extended Legendre wavelets, *Proc. of ICWAPR*, vol.2, pp.667-671, 2008.

[21] J. W. Brewer, Kronecker products and matrix calculus in system theory, *IEEE Trans. on Circuits and Systems*, vol.25, no.9, pp.772-781, 1978.