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INTRINSIC VALUE FUNCTION OF ASSETS DETERMINED BY THE YIELD EQUATION AND ITS PROPERTIES

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Abstract. *This paper tries to find, in view of function, some rules of intrinsic value of assets as time changes. In fact, the intrinsic value of assets at each moment is estimated by means of the classical principle of discounted cash flow model. A formula of the intrinsic value sequence of assets from the yield equation in discrete form and a formula of the intrinsic value function from the yield equation in continuous form are derived and some properties of the intrinsic value sequence and function for assets are given, such as linearity, non-arbitrage, and certain characteristics of the curve of intrinsic value function. In addition, we give the corresponding analytic formula of the intrinsic value functions for some commonly used cash earnings functions. Finally, several application modes to estimate the rationality prices of assets are discussed, making use of the concepts and the mathematical formula of the intrinsic value function proposed in this paper.*
Keywords: Discounted cash flow model, Intrinsic value sequence, Intrinsic value function, Cash earnings function

1. **Introduction.** Discounted cash flow model (DCF) is one of the most important mathematical models in estimating the value of assets when analyzing investment. We have the well-known dividend discount model (DDM) to estimate the value of stocks, and the free cash flow discount model to estimate the value of bonds and other financial assets. Cash earnings in the future, the price of assets at the last period, and the discount rate in the future are three important factors to determine the intrinsic value of current assets. Through the DCF, the quantitative relationship between the changes in interest rates and changes in the intrinsic value of assets can be easily figured out. The market prices of certain assets can also be used as its intrinsic value to solve the corresponding internal rate of return by DCF. Comparing the internal rate of return obtained above with the interest rates of commercial bank, bearing in mind the factor of risk premium which the investors required for the risk assets, whether the price of assets in the market is overvalued or undervalued can be assessed, so as to guide the investment activities for investors.

Many in-depth discussions are made in the respects of discounted cash flow model. The discounted cash flow model in discrete form has been well-known by financial experts, but the model in continuous form has not been systematically studied. It is necessary to define a cash earnings function in continuous form and to derive the cash flow discounting formula in continuous form. Another respect ignored by financial experts is that we can estimate the intrinsic value of assets at all the different moments by the DCF if the cash earnings at each time in the future has been known, and thus we can use the view and method of function to study the various properties of the intrinsic value function. The intrinsic value function is composed of the intrinsic value of the assets at all different moments. Usually people use the DCF to calculate the level of the intrinsic value of assets at the current time with the value of cash flow in the future and the discount rate, then to assess the value of the assets, but this process shows a lack of in-depth discussion about some rules of the intrinsic value of assets as the time changes. The studies on discounted cash flow model in continuous form, the intrinsic value sequence and the intrinsic value function have not been found, which are quite meaningful to enrich the evaluation methods for the risk assets.

Although the discounted cash flow model is established on the basis of the time value of cash, but the model is still a definition expression. If a formula for the discounted cash through more fundamental financial principles can be derived, that will be a meaningful attempt. Such attempts may be helpful to the more in-depth study on the discounted cash flow model. In this paper, we mainly focus on the study of the intrinsic value function and its various properties through the view and method of function. Instead of beginning with the existing discounted cash flow model and creating a new model by speculation, our approach is to carry on a deductive reasoning based on the definition of return on assets (ROA). By the definition of ROA in discrete form, we derive the yield equation on assets in discrete form, which is a difference equation. Then we can define the intrinsic value sequence of the assets by the solution of the equation. By the definition of ROA in continuous form, we derive the yield equation on assets in continuous form, which is a differential equation. And we can also define the intrinsic value function of the assets by the solution of the equation. Deriving the formulas of the intrinsic value sequence and the intrinsic value function of the assets, and further studying on the investment properties of these formulas constitute the main contents of this paper. The main innovation of this paper is that the intrinsic value sequence and the intrinsic value function are derived from the yield equations. This method is guaranteed by both the solid foundation of investments and the rigorous mathematical methods, so the results are reliable.

In Section 2, we present the yield equations on assets both in discrete form and in continuous form by the definition of ROA. In Section 3, we get the mathematical formula of the intrinsic value sequence by solving the yield equation in discrete form, including the case of the infinite period and the finite period. In Section 4, we get the mathematical formula of the intrinsic value function by solving the yield equation in continuous form, also including the case of the infinite period and the finite period. In Section 5, we prove several properties of the intrinsic value sequence and the intrinsic value function, including linearity, no-arbitrage, certain characteristics of the curve of intrinsic value function, the relationship between intrinsic value and the discount rate and so on. In Section 6, we give the corresponding intrinsic value functions for some cash earnings functions, and also the corresponding cash earnings functions for some intrinsic value functions, and explain the relationship between the two. In Section 7, we propose several application modes to estimate the rationality prices of assets with the concepts and the mathematical formula of the intrinsic value function. The last part is the conclusion.

2. **The Yield Equation in Discrete Form and in Continuous Form.** Suppose there is an assets A, which provide the cash earnings at fixed time intervals, then we have the following yield equation:

$$r_i = \frac{D_i + (P_{i+1} - P_i)}{P_i}$$

where D_i is the cash earnings of per unit of the assets in Phase i, P_i is the price of per unit of the assets in Phase i, r_i is the rate of return in Phase i, $P_{i+1} - P_i$ is the capital gain of per unit of the assets in Phase i.

We use r to represent the expected rate of return which the investors required for the investment in the assets A, r is also the discount rate that the investors required. In equilibrium financial market, as there is no risk-free arbitrage opportunity, so the expected rate of return of each phase should take the same value r, that is:

$$r = \frac{D_i + (P_{i+1} - P_i)}{P_i}$$

Rearrange the equation above, we have:

$$P_{i+1} = (1 + r) P_i - D_i \tag{1}$$

We call Equation (1) the yield equation on assets in discrete form, which is a difference equation. For a fixed discount rate r, if the cash earnings $D_i\,(i = 1, 2, 3, \cdots)$ of each phase in the future are known, we can solve the prices $P_i\,(i = 1, 2, 3, \cdots)$ of the assets in each phase by Equation (1). The price series $\{P_i\}$ which satisfies Equation (1) is fair and reasonable price and it is bound to be the intrinsic value sequence of the assets A. Similarly, for a fixed discount rate r, if the prices $P_i\,(i = 1, 2, 3, \cdots)$ of the assets in each phase are known, we can solve the cash earnings $D_i\,(i = 1, 2, 3, \cdots)$ of each phase in the future by Equation (1).

Suppose there is an assets B, the cash earnings of which is continuous. We use function $D(t)$ to express the cash earnings in one unit of time at time t, we call $D(t)$ the cash earnings function. Consider the time period $[t, t + \Delta t]$, we have the following yield equation:

$$r(t)\,\Delta t = \frac{D(t)\,\Delta t + [P(t + \Delta t) - P(t)]}{P(t)}$$

where $P(t)$ is the price of per unit of the assets at time t, $D(t)\,\Delta t$ is the cash earnings in the period $[t, t + \Delta t]$, $r(t)$ is the instantaneous rate of return in time t, $r(t)\,\Delta t$ is the return in the period $[t, t + \Delta t]$, $P(t + \Delta t) - P(t)$ is the capital gain in the period $[t, t + \Delta t]$.

As discussed previously, we take a fixed discount rate (or expected rate of return) r. Then we have:

$$r\Delta t = \frac{D(t)\,\Delta t + [P(t + \Delta t) - P(t)]}{P(t)}$$

Rearrange the equation above, and let $\Delta t \to 0$. We have:

$$\frac{dP(t)}{dt} = rP(t) - D(t) \tag{2}$$

We call Equation (2) the yield equation on assets in continuous form, which is also a differential equation about $P(t)$. For a fixed discount rate r, if the cash earnings function $D(t)$ for some time in the future are known, we can solve the price function $P(t)$ of the assets by Equation (2). The price function $P(t)$ which satisfies Equation (2) is the fair and reasonable price and it is bound to be the intrinsic value function of the assets B. Similarly, for a fixed discount rate r, if the price function $P(t)$ of the assets is known, we can obtain the cash earnings function $D(t)$ of the assets B by Equation (2).

3. **Derive the Intrinsic Value Sequence of the Assets from the Yield Equation in Discrete Form.** In this section, we try to derive the mathematical formulas of the intrinsic value sequence $\{P_i\}$ by Equation (1). First, we consider a situation where the cash earnings sequence $\{D_i\}$ has an infinite period.

We first introduce a few assumptions. Because D_i is the cash earnings of per unit of the assets A achieved in Phase i, so $D_i \geq 0$ is reasonable. If $D_i < 0$, it means that the investors who hold the assets A are required to add additional investment in period i, this requirement is unreasonable. In addition, because P_i is the fair and reasonable price of the assets in period i, so there should be $P_i \geq 0$ for all i. What is more, r is the expected rate of return, of course, there should be $r > 0$.

According to Equation (1), we have:

$$\begin{aligned}
&P_2 = (1+r)\,P_1 - D_1 \\
&P_3 = (1+r)^2\,P_1 - (1+r)\,D_1 - D_2 \\
&P_4 = (1+r)^3\,P_1 - (1+r)^2\,D_1 - (1+r)\,D_2 - D_3 \\
&\quad\vdots \\
&P_n = (1+r)^{n-1}\,P_1 - (1+r)^{n-2}\,D_1 - (1+r)^{n-3}\,D_2 - \cdots - D_{n-1}
\end{aligned}$$

Divide both ends of the expression of P_n by $(1+r)^{n-1}$, we have:

$$\frac{P_n}{(1+r)^{n-1}} = P_1 - \sum_{i=1}^{n-1} \frac{D_i}{(1+r)^i}$$

Rearrange the equation, we have:

$$P_n = (1+r)^{n-1}\left(P_1 - \sum_{i=1}^{n-1} \frac{D_i}{(1+r)^i}\right)$$

Since $0 < r < 1, D_i \geq 0$ is the premise, $P_n \geq 0$ is reasonable.

Under a wider premise, $\sum\limits_{i=1}^{+\infty} \frac{D_i}{(1+r)^i}$ is convergent, thus

$$\lim_{n\to\infty} \sum_{i=n}^{+\infty} \frac{D_i}{(1+r)^i} = 0$$

If $P_1 < \sum\limits_{i=1}^{+\infty} \frac{D_i}{(1+r)^i}$, then $P_n < (1+r)^{n-1} \sum\limits_{i=n}^{+\infty} \frac{D_i}{(1+r)^i}$. When n is large enough, $P_n < 0$, which, however, is unreasonable. Therefore, we set $P_1 = \sum\limits_{i=1}^{\infty} \frac{D_i}{(1+r)^i} + C$, where $C \geq 0$.

Substitute the expression of P_1 to the above expression, we have:

$$P_n = (1+r)^{n-1}\left(\sum_{i=n}^{+\infty} \frac{D_i}{(1+r)^i} + C\right)$$

According to the investment meaning, if $D_i = 0$, $i = 1, 2, 3, \cdots$, then there should be $P_i = 0$, $i = 1, 2, 3, \cdots$. That means $C = 0$. So we get the solution of the difference Equation (1) which is meaningful in the investment. The intrinsic value P_n of per unit of the assets A in Phase n is given by the following formula:

$$P_n = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{D_i}{(1+r)^i} \tag{3}$$

In particular, when $n = 1$, we have

$$P_1 = \sum_{i=1}^{+\infty} \frac{D_i}{(1+r)^i}$$

This is the discounted cash flow model of the intrinsic value that we have known.

Let us consider the cash earnings sequence $\{D_i\}$ which has a finite period, that is $\{D_i\}$, $i = 1, 2, \cdots, m$. Let $P_1 = \sum\limits_{i=1}^{m} \frac{D_i}{(1+r)^i} + C$, for $n \leq m$, according to the derivation above, we have:

$$\frac{P_n}{(1+r)^{n-1}} = P_1 - \sum_{i=1}^{n-1} \frac{D_i}{(1+r)^i}$$

Substitute the expression of P_1 to the above expression, we have:

$$P_n = (1+r)^{n-1} \left(\sum_{i=n}^{m} \frac{D_i}{(1+r)^i} + C \right)$$

Similarly, if $D_i = 0$, $i = 1, 2, 3, \cdots$, then there also should be $P_i = 0$, $i = 1, 2, 3, \cdots$. That means $C = 0$. So we get the solution of the difference Equation (1), which is meaningful in the investment. The intrinsic value P_n $(1 \le n \le m)$ of per unit of the assets A in Phase n is given by the following formula:

$$P_n = (1+r)^{n-1} \sum_{i=n}^{m} \frac{D_i}{(1+r)^i} \tag{4}$$

It is worth noting that if an assets is valid for a finite period, then the cash earnings sequence in Formula (4) should include the interest income, and the cash flow of capital redemption or that we call the cash flow of realization of assets at the end of the phase. For bonds, if the investors receive the one-time capital redemption at the end of the phase (Phase m), then D_m will be much larger than $D_i\,(i < m)$. For the investor that plan to hold on the stock until Phase m, D_m should include not only the dividend income of Phase m, but the income they sell the stock at the price P_m at the end of Phase m as well. Usually people are accustomed to use $\{D_i\}$ to represent the interest or the dividend income, not including the cash flow of capital redemption or the cash flow of realization of assets. In this case, Formula (4) should be modified. Let P_m to be the price of per unit of the assets at the end of the phase, or the price of the one-time capital redemption at the end of the phase, and then we have the following formula of the intrinsic value of the assets A:

$$P_n = (1+r)^{n-1} \left(\sum_{i=n}^{m} \frac{D_i}{(1+r)^i} + \frac{P_m}{(1+r)^m} \right) \tag{5}$$

4. **Derive the Intrinsic Value Function of the Assets from the Yield Equation in Continuous Form.** In this section, we try to derive the mathematical formulas of the intrinsic value function $P(t)$ by Equation (2). First, we consider the cash earnings function $D(t)$, $t \in [0, +\infty)$, which has an infinite period. With the same situation as Section 3, we assume that $D(t) \ge 0$, $P(t) \ge 0$, and $r > 0$ for $t \in [0, +\infty)$. Equation (2) is a first order differential equation. For a common first order differential equation $y' + p(x)\,y = q(x)$, the general solution is:

$$y = e^{-\int p(x)dx} \left[\int q(x)\, e^{\int p(x)dx} dx + c \right]$$

The independent variable in Equation (2) is the time parameter t. Accordingly, $p(t) = -r$ and $q(t) = -D(t)$. Substitute the expression into the general solution, we have

$$P(t) = e^{rt} \left[\int_0^t -D(\xi)\, e^{-r\xi} d\xi + C_0 \right] \tag{6}$$

where C_0 is a constant. Let $t = 0$, by the formula above, $C_0 = P(0)$. That is:

$$P(t) = e^{rt} \left[\int_0^t -D(\xi)\, e^{-r\xi} d\xi + P(0) \right]$$

Since $0 < r < 1$, $D(t) \geq 0$ is a premise, $P(t) \geq 0$ is reasonable. Under a wider premise, the integral $\int_0^\infty D(\xi) e^{-r\xi} d\xi$ is convergent, so

$$\lim_{t\to\infty} \int_t^\infty D(\xi) e^{-r\xi} d\xi = 0$$

If $P(0) < \int_0^\infty D(\xi) e^{-r\xi} d\xi$, then $P(t) < e^{rt} \int_t^\infty D(\xi) e^{-r\xi} d\xi$. When t is large enough, $P(t) < 0$, which, however, is unreasonable. Therefore, set $P(0) = \int_0^\infty D(\xi) e^{-r\xi} d\xi + C$, where $C \geq 0$.

Substitute the expression of $P(0)$ to the above expression, we have:

$$P(t) = e^{rt} \left[\int_t^{+\infty} D(\xi) e^{-r\xi} d\xi + C \right]$$

If $D(t) = 0$, there must be $P(t) = 0$. We have $P(t) = e^{rt} \cdot C = 0$, which means $C = 0$. So we get the solution $P(t)$ of the differential Equation (2) which is meaningful in the investment. The intrinsic value function $P(t)$ of the assets B at time t is given by the following formula:

$$P(t) = e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi \tag{7}$$

In particular, when $t = 0$, we have

$$P(0) = \int_0^{+\infty} D(\xi) e^{-r\xi} d\xi$$

This is the discounted cash flow model of the intrinsic value in continuous form.

Let us consider the case of the finite time period, that is $t \in [0, T]$. Let the C_0 in general solution of expression (6) be $C_0 = \int_0^T D(\xi) e^{-r\xi} d\xi + C$. Substitute into the general solution of expression (6) and rearrange:

$$P(t) = e^{rt} \left[\int_t^T D(\xi) e^{-r\xi} d\xi + C \right]$$

Assume that $D(t)$ is a continuous bounded function in $[0, T]$. It certainly has:

$$\lim_{t\to T} e^{rt} \int_t^T D(\xi) e^{-r\xi} d\xi = 0$$

So $P(T) = Ce^{rT}$, meaning $C = e^{-rT} P(T)$. $P(T)$ is the price for per unit of the assets B at time T. C is the price for the assets B discounted to time 0. So we have the following formula:

$$P(t) = e^{rt} \left[\int_t^T D(\xi) e^{-r\xi} d\xi + e^{-rT} P(T) \right] \tag{8}$$

This is the formula of the intrinsic value function for assets B in a finite period. In particular, when $t = 0$, we have the following discount formula of the intrinsic value in continuous form:

$$P(0) = \int_0^T D(\xi) e^{-r\xi} d\xi + e^{-rT} P(T)$$

5. **Several Properties of the Intrinsic Value Sequence and the Intrinsic Value Function.** In this section we give several properties of the intrinsic value sequence $\{P_i\}$ and the intrinsic value function $P(t)$, including linearity, no-arbitrage, certain characteristics of the curve of intrinsic value function, the relationship between intrinsic value and the discount rate and so on.

Theorem 5.1. *The corresponding intrinsic value sequences for the cash earnings sequences determined by Equations (3)-(5) are linear. That means if* $\{P_i\}$ *is determined by* $\{D_i\}$*, then for arbitrary* $\alpha \geq 0$, $\{\alpha P_i\}$ *is determined by* $\{\alpha D_i\}$ *and if* $\{P_i^1\}$ *is determined by* $\{D_i^1\}$, $\{P_i^2\}$ *is determined by* $\{D_i^2\}$*, then for arbitrary* $\alpha \geq 0$, $\beta \geq 0$, $\{\alpha P_i^1 + \beta P_i^2\}$ *is determined by* $\{\alpha D_i^1 + \beta D_i^2\}$.

Proof: From Equation (3), we have:

$$P_n = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{D_i}{(1+r)^i}$$

For cash earnings sequence $\{\alpha D_i\}$, if the intrinsic value sequence is represented by $\{P_i^\alpha\}$, then we have:

$$P_n^\alpha = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{\alpha D_i}{(1+r)^i} = \alpha P_n$$

It means $\{\alpha P_i\}$ is determined by $\{\alpha D_i\}$. As regards the case of Equation (4), the proof is the same as above. For Equation (5), since the cash earnings sequence $\{\alpha D_i\}$ must be produced by α unit of the assets A, so the value P_m^α of the assets at the termination time must be αP_m. By Equation (5), we get

$$P_n = (1+r)^{n-1} \left(\sum_{i=n}^{m} \frac{D_i}{(1+r)^i} + \frac{P_m}{(1+r)^m} \right)$$

For cash earnings sequence $\{\alpha D_i\}$, if the intrinsic value sequence is represented by $\{P_i^\alpha\}$, then we have:

$$P_n^\alpha = (1+r)^{n-1} \left(\sum_{i=n}^{m} \frac{\alpha D_i}{(1+r)^i} + \frac{P_m^\alpha}{(1+r)^m} \right)$$

According to the above discussion, substitute $P_m^\alpha = \alpha P_m$ into the above equation, we have $P_n^\alpha = \alpha P_n$, which means that $\{\alpha P_i\}$ is determined by $\{\alpha D_i\}$.

Assuming the cash earnings sequence $\{P_i^1\}$ is determined by $\{D_i^1\}$, $\{P_i^2\}$ is determined by $\{D_i^2\}$, then for the sequence in the infinite time period, there are:

$$P_n^1 = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{D_i^1}{(1+r)^i} \qquad P_n^2 = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{D_i^2}{(1+r)^i}$$

For arbitrary $\alpha \geq 0$, $\beta \geq 0$, the following formula is easily perceived:

$$\alpha P_n^1 + \beta P_n^2 = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{\alpha D_i^1 + \beta D_i^2}{(1+r)^i}$$

It means $\{\alpha P_i^1 + \beta P_i^2\}$ is determined by the cash earnings sequence $\{\alpha D_i^1 + \beta D_i^2\}$. For the case of the finite time period, we have exactly the same proof, so the theorem is proved.

Theorem 5.1 shows that there is a linear relationship between the intrinsic value sequence of the assets and the cash earnings sequence.

Theorem 5.2. *The corresponding intrinsic value functions* $P(t)$ *for the cash earnings functions* $D(t)$ *determined by Equations (7) and (8) are linear. That means: If* $P(t)$ *is determined by* $D(t)$*, then for arbitrary* $\alpha \geq 0$, $\alpha P(t)$ *is determined by* $\alpha D(t)$*. If* $P_1(t)$ *is determined by* $D_1(t)$, $P_2(t)$ *is determined by* $D_2(t)$*, then for arbitrary* $\alpha \geq 0$, $\beta \geq 0$*, the intrinsic value function* $\alpha P_1(t) + \beta P_2(t)$ *is determined by the cash earnings function* $\alpha D_1(t) + \beta D_2(t)$.

Proof: From Equation (7), we have:

$$P(t) = e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi$$

For cash earnings function $\alpha D(t)$, if the intrinsic value function is represented by $P_\alpha(t)$, then we have:

$$P_\alpha(t) = e^{rt} \int_t^{+\infty} \alpha D(\xi) e^{-r\xi} d\xi = \alpha P(t)$$

It means that $\alpha P(t)$ is determined by $\alpha D(t)$. As regards the case of Formula (8), since the cash earnings function $\alpha D(t)$ must be produced by α unit of the assets B, the value of the assets at time T must be $\alpha P(T)$. Through Formula (8), we have:

$$P(t) = e^{rt} \left[\int_t^T D(\xi) e^{-r\xi} d\xi + e^{-rT} P(T) \right]$$

For cash earnings function $\alpha D(t)$, we have:

$$P_\alpha(t) = e^{rt} \left[\int_t^T \alpha D(\xi) e^{-r\xi} d\xi + e^{-rT} \alpha P(T) \right] = \alpha P(t)$$

It also means that $\alpha P(t)$ is determined by $\alpha D(t)$.

Assuming the intrinsic value function $P_1(t)$ is determined by $D_1(t)$, $P_2(t)$ is determined by $D_2(t)$, then for the function in the infinite time period, there are:

$$P_1(t) = e^{rt} \int_t^{+\infty} D_1(\xi) e^{-r\xi} d\xi \quad P_2(t) = e^{rt} \int_t^{+\infty} D_2(\xi) e^{-r\xi} d\xi$$

For arbitrary $\alpha \geq 0$, $\beta \geq 0$, it is easy to see the establishment of the following formula:

$$\alpha P_1(t) + \beta P_2(t) = e^{rt} \int_t^{+\infty} (\alpha D_1(\xi) + \beta D_2(\xi)) e^{-r\xi} d\xi$$

So the intrinsic value function $\alpha P_1(t) + \beta P_2(t)$ is determined by the cash earnings function $\alpha D_1(t) + \beta D_2(t)$. For the case of the finite time period, we have exactly the same proof, so the theorem is proved.

Theorem 5.2 shows that there is a linear relationship between the intrinsic value function of the assets and the cash earnings function.

According to the yield Equations (1) and (2), the determination of the intrinsic value of the assets is based on the condition that the expected rate of return is fixed as time changes. If the price of the assets is determined by its intrinsic value from time to time, then the rate of return by holding the assets at any time should always be r. In other words, holding the assets priced by its intrinsic value do not exist any arbitrage opportunities. If the initial amount of investment is W_0, for the discrete case, the value by holding certain assets until the last phase must be $W_0(1+r)^n$, and for the continuous case, the value by holding certain assets until the last time T must be $W_0 e^{rT}$. From the intrinsic value sequence $\{P_i\}$ and the intrinsic value function $P(t)$ given by Section 3 and Section 4, we cannot easily find that they are no-arbitrage as mentioned above. So we have the following theorem:

Theorem 5.3. *Suppose there is an assets A, its price is given by (3). The initial amount of investment is W_0. Holding the assets for n period, during which the cash earnings are all used to buy the assets A, then the total value at the end of the period must be $W_0(1+r)^n$.*

Proof: We only need to prove that the rate of return in any phase is always r. Consider the Phase i. The rate of return r_i in Phase i will be

$$r_i = \frac{\frac{W_i}{P_i} \cdot D_i + (P_{i+1} - P_i) \cdot \frac{W_i}{P_i}}{W_i} = \frac{D_i + (P_{i+1} - P_i)}{P_i}$$

where W_i is the total value of assets at the beginning of this phase, P_i is the price of the assets in Phase i, P_{i+1} is the price of the assets in Phase $i+1$, D_i is the cash earnings of per unite of assets A in Phase i, $\frac{W_i}{P_i}$ is the amount of the assets that we can purchase, $\frac{W_i}{P_i} \cdot D_i$ is the cash earnings in Phase i. Substitute Equation (3) into the above expression, we have:

$$r_i = \frac{D_i + \left[(1+r)^i \sum\limits_{j=i+1}^{+\infty} \frac{D_j}{(1+r)^j} - (1+r)^{i-1} \sum\limits_{j=i}^{+\infty} \frac{D_j}{(1+r)^j}\right]}{(1+r)^{i-1} \sum\limits_{j=i}^{+\infty} \frac{D_j}{(1+r)^j}}$$

Let $\sum\limits_{j=i}^{+\infty} \frac{D_j}{(1+r)^j} = \delta_i$, then we have:

$$\begin{aligned} r_i &= \frac{D_i + \left[(1+r)^i \left(\delta_i - \frac{D_i}{(1+r)^i}\right) - (1+r)^{i-1} \delta_i\right]}{(1+r)^{i-1} \delta_i} \\ &= \frac{D_i + \left[(1+r)^i \delta_i - D_i - (1+r)^{i-1} \delta_i\right]}{(1+r)^{i-1} \delta_i} \\ &= \frac{(1+r)^{i-1} \delta_i (1+r-1)}{(1+r)^{i-1} \delta_i} = r \end{aligned}$$

The theorem is proved.

Theorem 5.4. *Suppose there is an assets B, its price is given by (7). The initial amount of investment is W_0. Holding the assets in the period $[0, T]$, during which the cash earnings are all used to buy the assets B continuously, then the total value at the end of the period must be $W_0 e^{rT}$.*

Proof: We only need to prove that the rate of return in any sufficiently small time interval $[t, t+\Delta t]$ is always r. Let's considering the time t, the rate of return $r(t)$ in the period $[t, t+\Delta t]$ will be

$$r(t)\Delta t = \frac{\frac{W(t)}{P(t)} \cdot D(t)\Delta t + [P(t+\Delta t) - P(t)] \cdot \frac{W(t)}{P(t)}}{W(t)} = \frac{D(t)\Delta t + [P(t+\Delta t) - P(t)]}{P(t)}$$

where $W(t)$ is the total value of assets at time t, $P(t)$ is the price of the assets at time t, $\frac{W(t)}{P(t)}$ is the amount of the assets that we can purchase, $P(t+\Delta t)$ is the price of the assets at time $t+\Delta t$, $D(t)\Delta t$ is the cash earnings in the period $[t, t+\Delta t]$.

Substitute Equation (7) to the above expression, we have:

$$r(t)\Delta t = \frac{D(t)\Delta t + \left[e^{r(t+\Delta t)} \int_{t+\Delta t}^{+\infty} D(\xi) e^{-r\xi} d\xi - e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi\right]}{e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi}$$

Let $\int_t^{+\infty} D(\xi) e^{-r\xi} d\xi = \delta(t)$, then we have:

$$r(t)\Delta t = \frac{D(t)\Delta t + \left[e^{r(t+\Delta t)}\left(\delta(t) - \int_t^{t+\Delta t} D(\xi) e^{-r\xi} d\xi\right) - e^{rt}\delta(t)\right]}{e^{rt}\delta(t)}$$
$$= \frac{D(t)\Delta t + \delta(t)\left[e^{r(t+\Delta t)} - e^{rt}\right] - e^{r(t+\Delta t)} \int_t^{t+\Delta t} D(\xi) e^{-r\xi} d\xi}{e^{rt}\delta(t)}$$

That is:

$$r(t) = \lim_{\Delta t \to 0} \frac{D(t) + \delta(t)\left[\dfrac{e^{r(t+\Delta t)} - e^{rt}}{\Delta t}\right] - e^{r(t+\Delta t)} \cdot \dfrac{1}{\Delta t} \int_t^{t+\Delta t} D(\xi) e^{-r\xi} d\xi}{e^{rt}\delta(t)}$$
$$= \frac{D(t) + re^{rt}\delta(t) - e^{rt}D(t)e^{-rt}}{e^{rt}\delta(t)} = r$$

The theorem is proved.

The conclusions of Theorem 5.3 and Theorem 5.4 also show that the mathematical formulas of the intrinsic value sequence and the intrinsic value function we have derived are correct.

Theorem 5.5. *Suppose that $D(t)$ is the cash earnings function, then the intrinsic value function $P(t)$ determined by Equation (2) composes a price curve. The maximum and minimum point t of the price curve satisfies to the following equation:*

$$rP(t) - D(t) = 0$$

If $rP(t) > D(t)$, then $\frac{dP}{dt} > 0$, the price curve increases; If $rP(t) < D(t)$ then $\frac{dP}{dt} < 0$, the price curve decreases. The inflection point t of the price curve satisfies to the following equation:

$$r^2P(t) - rD(t) - D'(t) = 0$$

Proof: The conclusions on extreme-value problem in Theorem 5.5 are the clear results of Equation (2). So we just need to prove the inflection point problem. As the inflection point t of the price curve should satisfy to $\frac{d^2P}{dt^2} = 0$, and from Equation (2), we have:

$$\frac{dP(t)}{dt} = rP(t) - D(t)$$

Derivative of both sides of the above expression, and let $\frac{d^2P}{dt^2} = 0$, we have:

$$\frac{d^2P}{dt^2} = r \cdot \frac{dP(t)}{dt} - D'(t) = 0$$

After substituting Formula (2) into the formula above, we have

$$r^2P(t) - rD(t) - D'(t) = 0$$

The theorem is proved.

Equation (7) shows that, when the discount rate is a constant, the intrinsic value $P(t)$ of the assets B at time t is:

$$P(t) = e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi$$

Obviously, when the value of r is different, the corresponding intrinsic value $P(t)$ will be different too. Then the expression should be written as $P(t, r)$, that is:

$$P(t, r) = e^{rt} \int_t^{+\infty} D(\xi) e^{-r\xi} d\xi$$

Similarly, Equation (3) should also be written in the following form:

$$P_n(r) = (1+r)^{n-1} \sum_{i=n}^{+\infty} \frac{D_i}{(1+r)^i}$$

Thus we can examine the sensitivity of the intrinsic value function $P(t, r)$ and the intrinsic value sequence $P_n(r)$ as r changes. We have the following theorem:

Theorem 5.6. *The intrinsic value P_n determined by Equation (3) is negatively correlated with the discount rate r. The intrinsic value function $P(t)$ determined by Equation (7) is negatively correlated with the discount rate r.*

Proof: We only need to prove $\frac{dP_n(r)}{dr} \leq 0$ and $\frac{\partial P(t,r)}{\partial r} \leq 0$. According to the expression of $P_n(r)$, we have:

$$\frac{dP_n(r)}{dr} = -\sum_{i=n}^{+\infty} \frac{D_i(i-n+1)}{(1+r)^{i-n+2}}$$

As each term in the sign of summation is non-negative, so $\frac{dP_n(r)}{dr} \leq 0$.

According to the expressions of $P(t, r)$, we have:

$$\frac{\partial P(t,r)}{\partial r} = -\int_t^{+\infty} (\xi - t) D(\xi) e^{r(t-\xi)} d\xi$$

Since $\xi \geq t$, $D(\xi) \geq 0$, $e^{r(t-\xi)} > 0$, the integrand in the sign of integration is non-negative, so $\frac{\partial P(t,r)}{\partial r} \leq 0$. The theorem is proved.

6. **Several Cash Earnings Functions and the Corresponding Intrinsic Value Functions.** From the perspective of function transformation, Equation (7) defines a transformation from the cash earnings function $D(t)$ to the intrinsic value function $P(t)$, while the yield Equation (2) defines a transformation from the intrinsic value function $P(t)$ to the cash earnings function $D(t)$. In this section, we give the corresponding intrinsic value functions for some commonly used cash earnings functions and the corresponding cash earnings functions for some intrinsic value functions, and indicate their properties. Some intrinsic value functions determined by the cash earnings functions are listed in Table 1.

Each case in Table 1 has a corresponding investment background. The cash earnings function $D(t)$ in case (1) describes the characteristics of a kind of enterprise whose future earnings remains the same; The $D(t)$ in case (2) describes the characteristics of another kind of enterprise whose future earnings grow steadily; The cash earnings function $D(t)$ in case (3) describes the characteristics of a kind of enterprise whose future earnings remain the same at first, but grow steadily after a period of time. And its intrinsic value function is initially an exponential growth, then a linear growth; The $D(t)$ in case (6) reflects the characteristics of another kind of enterprise which use the profits they get in a period of time in the future to reinvest (the dividend income keeps 0 during the period). And its intrinsic value function is initially an exponential growth, and then maintains the same after a period of time; The $D(t)$ in case (7) reflects the characteristics of a kind of enterprise whose future earnings fluctuate at a fixed rule. And its intrinsic value function is also volatile, and has the same cycle. The explanation for the background of case (4), case (5), and case (8) are omitted here for brevity.

Some cash earnings functions determined by the intrinsic value functions are listed in Table 2.

TABLE 1. Several cash earnings functions and the corresponding intrinsic value functions

	Cash Earnings Function $D(t)$	Intrinsic Value Function $P(t)$	Intrinsic Value when $t=0$	Properties
(1)	$D(t)=D_0$ $D_0>0$	$\frac{D_0}{r}$	$\frac{D_0}{r}$	$P(t)$ is also a constant
(2)	$D(t)=a+bt$ $a>0,\ b>0$	$\frac{a+bt}{r}+\frac{b}{r^2}$	$\frac{1}{r}\left(a+\frac{b}{r}\right)$	$P(t)$ is a linear function of t
(3)	$D(t)=\begin{cases} a & 0\le t\le t_0 \\ a+b(t-t_0) & t>t_0 \end{cases}$ $a>0,\ b>0$	$\begin{cases} \frac{1}{r}\left[a+\frac{b}{r}e^{r(t-t_0)}\right] & 0\le t\le t_0 \\ \frac{1}{r}\left[a+b(t-t_0)+\frac{b}{r}\right] & t>t_0 \end{cases}$	$\frac{1}{r}\left(a+\frac{b}{r}e^{-rt_0}\right)$	$P(t)$ is an exponential function at $[0,t_0]$, a linear function at $[t_0,+\infty)$
(4)	$D(t)=a+bt+ct^2$ $c>0$	$\frac{a+bt+ct^2}{r}+\frac{b+2ct}{r^2}+\frac{2c}{r^3}$	$\frac{a}{r}+\frac{b}{r^2}+\frac{2c}{r^3}$	$P(t)$ is also a quadratic function of t
(5)	$D(t)=Ae^{\lambda(t-t_0)}$ $\lambda<r$	$\frac{Ae^{\lambda(t-t_0)}}{r-\lambda}$	$\frac{Ae^{-\lambda t_0}}{r-\lambda}$	$P(t)$ is different from $D(t)$ by a constant factor $\frac{1}{r-\lambda}$
(6)	$D(t)=\begin{cases} 0 & 0\le t\le t_0 \\ A & t>t_0 \end{cases}$	$\begin{cases} \frac{A}{r}e^{r(t-t_0)} & 0\le t\le t_0 \\ \frac{A}{r} & t>t_0 \end{cases}$	$\frac{A}{r}e^{-rt_0}$	$P(t)$ is an exponential function at $[0,t_0]$, a constant at $[t_0,+\infty]$
(7)	$D(t)=A_0+A\sin\omega t$ $A_0>A>0$	$\frac{A_0}{r}+\frac{A}{r^2+\omega^2}(r\sin\omega t+\omega\cos\omega t)$	$\frac{A_0}{r}+\frac{A\omega}{r^2+\omega^2}$	$P(t)$ is still a fluctuant function with the same period
(8)	$D(t)=D_0+A\delta(t-t_0)$ $\delta(t-t_0)=\begin{cases} 0 & t\ne t_0 \\ +\infty & t=t_0 \end{cases}$ and $\int_{-\infty}^{+\infty}\delta(t-t_0)dt=1$	$\begin{cases} \frac{D_0}{r}+Ae^{r(t-t_0)} & 0\le t\le t_0 \\ \frac{D_0}{r} & t>t_0 \end{cases}$	$\frac{D_0}{r}+Ae^{-rt_0}$	$P(t)$ is discontinuous at the point t_0

TABLE 2. Several intrinsic value functions and the corresponding cash earnings functions

	Intrinsic Value Function $P(t)$	Cash Earnings Function $D(t)$	Properties
(1)	$P(t)=P_0$ $P_0>0$	rP_0	$D(t)$ is also a constant
(2)	$P(t)=a+bt$ $a>0,\ b>0$	$r(a+bt)-b$	$D(t)$ is a linear function of t
(3)	$P(t)=a+bt+ct^2$ $c>0,-\frac{b}{2c}\ge 0$	$(ra+b)+(rb+2c)t+rct^2$	$D(t)$ is also a quadratic function of t
(4)	$P(t)=A_0+A\sin\omega t$ $A_0>A>0$	$rA_0+A(r\sin\omega t-\omega\cos\omega t)$	$D(t)$ is still a fluctuant function with the same period
(5)	$P(t)=Ae^{\lambda(t-t_0)}$ $\lambda<r$	$A(r-\lambda)e^{\lambda(t-t_0)}$	$D(t)$ is different from $P(t)$ by a constant factor $r-\lambda$
(6)	$P(t)=A\ln(t+\delta)$ $A>0,\delta>1$	$A\left[r\ln(t+\delta)-\frac{1}{t+\delta}\right]$	When $t\to+\infty$, the main part of $D(t)$ is the form of logarithmic function

7. **The Practical Application of the Intrinsic Value Function.** In this section we discuss the practical application of the intrinsic value function. We estimate the assets with the intrinsic value function presented in this paper in order to find profitable investment opportunities, and to discover and escape the over-expected risk. In the existing finance theory, people use the discounted cash flow model to determine the intrinsic value of the assets at the current time, then they compare the intrinsic value with the transaction price on the market to determine whether the value of the assets is overvalued or undervalued. Our approach is to examine the intrinsic value function in a period of time (rather than one point of time), comparing which with the market price in the corresponding period of time. As the prices of securities in the financial market are generally fluctuant, so it is more reasonable and more reliable to examine in a period of time than one point of time. We only discuss the situation on the continuous form, and give the following three application modes:

Application mode I: Let r be the discount rate which is also the average investment rate of return that investors require. $D(t)$ is the cash earnings function of the assets B in the future. By Equation (7), we get the intrinsic value function $P(t)$ of the assets B:

$$P(t) = e^{rt} \int_t^{+\infty} D(\xi)\, e^{-r\xi} d\xi$$

Select a period of time $[t_1, t_2]$, and study the intrinsic value function $P(t)$ and the market price $P_M(t)$. If $P_M(t) < P(t)$, $t \in [t_1, t_2]$, it means that the value of assets B in the market is undervalued. We should hold a long position of the assets B at time t_2. If $P_M(t) > P(t)$, $t \in [t_1, t_2]$, it means that the value of assets B in the market is overvalued. We should hold a short position of the assets B at time t_2. If the curve $P_M(t)$ crosses and twists $P(t)$ in the period of time $[t_1, t_2]$, and there is no significant deviation from each other, then the market prices are generally reasonable.

Application mode II: Let r be the discount rate, and the study period is $[t_1, t_2]$. Take the market price $P_M(t)$ of the assets B in this period as the intrinsic value of the assets. By Equation (2), we find that the cash earnings function $D_M(t)$ in the period of time $[t_1, t_2]$ is:

$$D_M(t) = rP_M(t) - \frac{dP_M(t)}{dt}$$

Compare the cash earnings function $D_M(t)$ obtained from $P_M(t)$ with the actual cash earnings function $D_{real}(t)$ of the assets B in the period of time $[t_1, t_2]$. Actual cash earnings function $D_{real}(t)$ will be a flat and smooth curve, while the cash earnings function $D_M(t)$ derived from the market price $P_M(t)$ of the assets B is a volatile curve. If $D_{real}(t)$ is significantly less than that $D_M(t)$, it means that the assets B is priced too low, since the inverse relationship between the price and the earnings of the assets. And If $D_{real}(t)$ is significantly larger than that $D_M(t)$, it means that the assets B is priced too high. If the curve $D_{real}(t)$ crosses and twists $D_M(t)$ in the period of time $[t_1, t_2]$, and there is no significant deviation from each other, then the market prices are roughly reasonable.

Since we have got the two cash earnings curves $D_{real}(t)$ and $D_M(t)$ in $[t_1, t_2]$, we can use a more accurate and more reasonable way to estimate the investment value of the two curves corresponding to different situations. Let us derive the cash flow value of one unit of the assets B in $[t_1, t_2]$. For the general cash earnings function $D(t)$, its value at time t_2 can be calculated by continuous compounding principle, and the formula is as follows:

$$V(t_1, t_2) = \int_{t_1}^{t_2} D(t)\, e^{r(t_2 - t)} dt$$

Let $V_{real}(t_1,t_2)$ indicate the cash flow value corresponding to the actual cash earnings function $D_{real}(t)$, and $V_M(t_1,t_2)$ indicate the cash flow value corresponding to the cash earnings function $D_M(t)$ calculated by $P_M(t)$, then we have:

$$V_{real}(t_1,t_2) = \int_{t_1}^{t_2} D_{real}(t)\, e^{r(t_2-t)}dt$$
$$V_M(t_1,t_2) = \int_{t_1}^{t_2} D_M(t)\, e^{r(t_2-t)}dt$$

We have the following decision rules:

(1) If $V_{real}(t_1,t_2)$ is significantly less than $V_M(t_1,t_2)$, then the assets B is priced too low.

(2) If $V_{real}(t_1,t_2)$ is significantly larger than $V_M(t_1,t_2)$, then the assets B is priced too high.

(3) If $V_{real}(t_1,t_2)$ is equal or roughly equal to $V_M(t_1,t_2)$, then the price of the assets B is reasonable.

Application mode III: Let $D(t)$ indicate the cash earnings function of the assets B. Take the market price $P_M(t)$ of the assets B as the intrinsic value of the assets. By Equation (2), we find the instantaneous rate of return $r_M(t)$ is:

$$r_M(t) = \frac{D(t) + \frac{dP_M(t)}{dt}}{P_M(t)}$$

Compare $r_M(t)$ with the discount rate r in the period of time $[t_1,t_2]$. If $r_M(t) < r$, it means that the assets B is priced too high, since the inverse relationship between the price and the rate of return of the assets. If $r_M(t) > r$, it means that the assets B is priced too low. If $r_M(t)$ fluctuates around the value of r slightly, then the price of the assets B is reasonable.

Since we have got two curves $y = r_M(t)$ and $y = r$ in $[t_1,t_2]$, we can use a more accurate and more reasonable way to estimate the investment value of the two curves corresponding to different situations. Let us calculate the value of one unit of the assets at time t_2, which is invested at time t_1. For general rate of return function $r(t)$, its value at time t_2 can be calculated by continuous compounding principle, and the formula is as follows:

$$E = e^{\int_{t_1}^{t_2} r(t)dt}$$

To illustrate that the formula above is correct, we divide $[t_1,t_2]$ into N small intervals, and the length of each interval is Δt. The separation points are t_i, $i = 1,2,3,\cdots,N$, then the approximate value of one unit of the assets at time t_2 is given by:

$$E_N = (1 + r(t_1)\Delta t)(1 + r(t_2)\Delta t)\cdots(1 + r(t_N)\Delta t)$$

Take natural logarithm on both sides, then we have:

$$\ln E_N = \sum_{i=1}^{N} \ln(1 + r(t_i)\Delta t)$$

Since $\ln(1 + r(t_i)\Delta t) = r(t_i)\Delta t + o(\Delta t)$, substitute it into the equation above and take the limit, we have:

$$\begin{aligned}
\lim_{N\to\infty} \ln E_N &= \lim_{N\to\infty} \sum_{i=1}^{N} r(t_i)\Delta t + \lim_{N\to\infty} \sum_{i=1}^{N} o(\Delta t) \\
&= \int_{t_1}^{t_2} r(t)dt + \lim_{N\to\infty} \sum_{i=1}^{N} 1\cdot\Delta t\cdot(o(\Delta t)/\Delta t) \\
&= \int_{t_1}^{t_2} r(t)dt + (t_2 - t_1) \lim_{N\to\infty}(o(\Delta t)/\Delta t) \\
&= \int_{t_1}^{t_2} r(t)dt
\end{aligned}$$

That is $E = e^{\int_{t_1}^{t_2} r(t)dt}$. Then we have the following decision rules:

(1) If $e^{\int_{t_1}^{t_2} r_M(t)dt}$ is significantly larger than $e^{r(t_2-t_1)}$, the rate of return of the assets sold at the market price $P_M(t)$ in $[t_1, t_2]$ is higher than r, thus the assets B is priced too low.

(2) If $e^{\int_{t_1}^{t_2} r_M(t)dt}$ is significantly less than $e^{r(t_2-t_1)}$, the rate of return of the assets sold at the market price $P_M(t)$ in $[t_1, t_2]$ is lower than r, thus the assets B is priced too high.

(3) If $e^{\int_{t_1}^{t_2} r_M(t)dt}$ is equal or roughly equal to $e^{r(t_2-t_1)}$, then the price of the assets B is reasonable.

8. **Conclusion.** In this paper, the yield equations on assets both in discrete form and in continuous form are presented based on the definition of rate of return on assets. The yield equation on assets in discrete form is a difference equation. The mathematical formula of the intrinsic value sequence is obtained by solving this difference equation. The yield equation on assets in continuous form is a differential equation. The mathematical formula of the intrinsic value function is obtained as well by solving this differential equation. In general, people use the discounted cash flow model in discrete form, and they usually only calculate the intrinsic value at present time ($t = 0$). This paper mainly focuses on the intrinsic value sequence and intrinsic value function which are composed of the intrinsic value of the assets at all the different moments and on its various properties, such as linearity, no-arbitrage, certain characteristics of the curve of intrinsic value function, the negative relationship between intrinsic value and the discount rate and so on, by means of the view and method of function. Then, the corresponding intrinsic value functions for some cash earnings functions and the corresponding cash earnings functions for some intrinsic value functions are given. Finally, several application modes to estimate the rationality of asset prices are proposed with the concepts and the mathematical formula of the intrinsic value function.

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