International Journal of Innovative Computing, Information and Control
Volume **8**, Number **5(A)**, May **2012**

ICIC International ©2012 ISSN 1349-4198
pp. **2941–2952**

ROBUST CONTROL OF A NONLINEAR ELECTRICAL OSCILLATOR MODELED BY DUFFING EQUATION

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Received November 2010; revised March 2011

Abstract. *This paper studies the optimal control problem for a nonlinear electrical circuit exemplified by a Duffing equation. Two cases are considered: first, if the cost function to be optimized is a quadratic one and secondly, if the same quadratic cost function is optimized for a circuit that contains parameter uncertainties in the right-hand side of this nonlinear differential equation. Even though this type of circuit has been object of a variety of control strategies in the past, very few papers have been devoted to the design of optimal control laws with quadratic performance indexes and subject to uncertainties. For that propose, two main techniques are developed in this paper: the so-called state dependent Riccati equation (SDRE) and the Robust Maximum Principle (RMP). Simulation examples are presented to demonstrate that the optimal control strategy works well for a circuit without uncertainties, and that an optimal control of the mini-max type can also be implemented if there are uncertainties in the circuit.*
Keywords: Nonlinear electrical circuit, Optimal control, Mini-max control, Uncertain parameters

1. Introduction.

1.1. Antecedents and motivation.
An interesting example of a nonlinear electric oscillator is described by the *so-called* Duffing equation. Such a circuit has attracted the attention of the control community because it represents a complex chaotic nonlinear system, which has a variety of applications ranging from physics to engineering [17, 20, 21, 24]. Given the complex behavior of the circuit, the control task becomes really challenging. Most results found in the literature, regarding the control design, are based on nonlinear control theory [17, 21, 25]. For instance, the Lyapunov design method has been widely used in controlling this circuit. As usual, the control objective in the mentioned papers is to stabilize the circuit variables around an equilibrium point or follow a given trajectory (tracking problem, see [21]). We can also find control laws of the state feedback and observer-based design (for more methods of controlling the Duffing equation, see [11]). To the best of the authors' knowledge, neither the previously mentioned papers, nor other works, attempted to solve the control problem of the nonlinear circuit in an optimal fashion. The main advantage of looking for an optimal control law is that the control objective can be achieved at the same time as the control signal is minimized, which is of

critical importance in many applications. Evidently, because of the non-linearity of the problem, the standard optimal control approach cannot be straightforwardly applied, and therefore, other techniques must be developed. On the other hand, the presence of uncertainties is common in practice and becomes another critical point to consider. In recent years, the optimal control techniques and related state-dependent Riccati equations have been successfully developed for nonlinear polynomial systems [3, 4, 6] providing a wide range of applicable feedback control laws. That approach, along with the so-called robust maximum principle, leads to developing in this paper the optimal and robust control laws for the nonlinear circuit control problem. The main problem of the obtained algorithms occurs to be computational complexity, which will be addressed in more detail in the future research, and meanwhile, a feasible algorithm for the two case scenario is explicitly proposed in this paper.

The well-known electrical circuit with a nonlinear element, which is represented as a nonlinear inductor, an alternating source of voltage, a pure resistive element and an electrical capacitor, is shown in Figure 1. Applying the *node law* of the circuit theory, such a circuit can be modeled as

$$\frac{d^2}{dt^2}\Phi + \frac{1}{RC}\frac{d}{dt}\Phi + \frac{\alpha_1}{C}\Phi + \frac{\alpha_3}{C}\Phi^3 = \frac{E_0}{R}\cos\omega t, \tag{1}$$

where Φ is the magnetic flux through the nonlinear inductor, E_0 is the alternating source voltage, R and C are the constants of the capacitor, α_1 and α_3 are some operation constants. The nonlinear term appears because of the nonlinear inductor, which is an inductor with a ferromagnetic core, and is modeled, if an abstraction of the hysteresis phenomenon is made, by an i-Φ nonlinear characteristic. Here i is the electrical current. Such a characteristic is approximated by a constitutive relation of the form [12, 26]:

$$i = \alpha_1\Phi + \alpha_3\Phi^3. \tag{2}$$

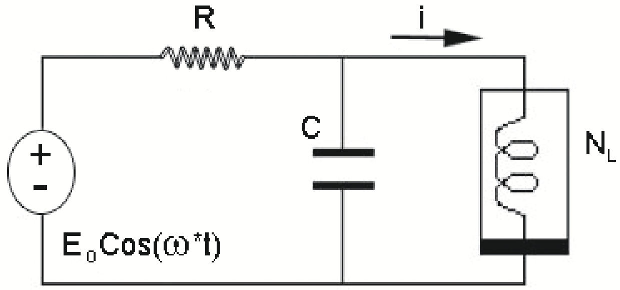

FIGURE 1. Electrical network with a nonlinear element

Defining the variables:

$$p_1 = \frac{1}{RC},\ p_2 = \frac{\alpha_1}{C},\ p_3 = \frac{\alpha_3}{C},\ \zeta(t) = \frac{E_0}{R}\cos\omega t, x_1 = \Phi,\ \dot{x}_1 = x_2,\ \dot{x}_2 = \frac{d^2}{dt^2}\Phi, \tag{3}$$

the following second order system is obtained:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -p_1x_2 - p_2x_1 - p_3x_1^3 + \zeta(t). \tag{4}$$

Depending on the choice of these constants, it is known that solutions of (4) exhibit periodic, almost periodic and chaotic behavior [20]. Note that this system is a nonlinear polynomial system of the third order. Consider this system in a controlled form:

$$\dot{x}(t) = f(t,x) + Bu(t) + d(t), \quad x(t_0) = x_0, \tag{5}$$

where

$$f(t,x) = \begin{pmatrix} x_2 \\ -p_1x_2 - p_2x_1 - p_3x_1^3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad d(t) = \begin{pmatrix} 0 & \zeta(t) \end{pmatrix}^{\mathsf{T}}. \tag{6}$$

Here, $x(t) \in \mathbb{R}^2$ is the state of the circuit, $d(t) \in \mathbb{R}^2$ is a known signal, and the affine control $u(t)$ is a physical control input, which will be designed later on. Consider also a more general form for the nonlinear term (6) represented as $f(t,x,\alpha) = a_0 + a_1x + a_2xx^{\mathsf{T}} + a_3xxx$, where a_1 is a standard 2D matrix, and a_2 and a_3 are 3D and 4D tensors (see [2] for details). Such a representation allows us to design the suggested control law in a compact way.

1.2. **Contribution.** To control such a circuit, taking a different approach to that in the above mentioned papers, we propose, as we already said, setting an optimal regulator control problem to drive the nonlinear state dynamic to the origin. Observe that the resulting nonlinear system is actually one of the polynomial types. Although the optimal controller for nonlinear systems has to be determined using the nonlinear filtering theory [18] and the general principles of maximum [19] or dynamic programming [7], which do not provide an explicit form for the optimal control in most cases, there is actually a long tradition of the optimal control design for nonlinear systems (see, for example, [1, 29]) and the optimal closed-form filter design for nonlinear [28], and in particular, polynomial systems [3, 4, 6]. In this paper, we propose an optimal control law for (5), based on [3, 4]. The tensor technique mentioned above allows one to find the optimal control law in the feedback form with a gain matrix being the solution of a state-dependent Riccati equation, which is proven to be a powerful technique to solve the optimal control problem for nonlinear polynomial systems. Then, at the second step, a robust optimal control technique based on the RMP [9, 23] for a circuit subject to uncertainties is developed, and we combine the RMP with the SDRE to solve the robust optimal control for the nonlinear circuit. It is important to note that the *"optimal"* quadratic controller problem for nonlinear, in particular, polynomial, systems with parameters belonging to an uncertain *finite* set, to the best of the authors' knowledge, has not even been consistently treated. All of the above mentioned papers, which design an optimal control, study this problem if the model of the considered dynamics is known with accuracy, but for many applications this assumption is unrealistic, because in practice it is common to find some sort of uncertainties. Hence, it is desirable to develop some kind of robustness for the optimal control problems to deal with such possible uncertainties which may have an impact on the nonlinear dynamics of the system.

2. **Optimal Control Design.** Our objective is to drive the circuit variables to the origin. To that end, consider the following quadratic performance index as the control performance measure:

$$g(x(t), u(t)) = \frac{1}{2}x^{\mathsf{T}}(T)Qx(T) + \frac{1}{2}\int_{t_0}^{T} (x^{\mathsf{T}}(t)Lx(t) + u^{\mathsf{T}}(t)Ru(t))\,dt. \tag{7}$$

The performance index is given in standard Bolza form, where it is assumed that R is a strictly positive definite and symmetric matrix, L and Q are two non-negative definite symmetric matrices, $T > t_0$ is a certain time moment, and a^T denotes transpose to a vector (matrix) a. The solution of the optimal regulator (control) problem for polynomial systems with linear control input and a quadratic criterion is given by the following feedback control law that realizes the optimal control with respect to the quadratic criterion given in (7), for the polynomial system (5):

$$u = -R^{-1}B^{\mathsf{T}}[M(t) + p(t)], \tag{8}$$

where the matrix function M is the solution of the state depended Riccati type equation:

$$\begin{aligned}-\dot{M}(t) =& L + \left[a_1 + 2a_2 x(t) + 3a_3 x(t) x^\intercal(t)\right]^\intercal M(t) \\ &+ M(t)\left[a_1 + a_2 x(t) + a_3 x(t) x^\intercal(t)\right] - M(t) B R^{-1} B^\intercal M(t),\end{aligned} \tag{9}$$

with terminal condition $M(T) = Q$, and the parameterized vector function p is solution of the linear equation:

$$-\dot{p}(t) = \left[a_1 + 2a_2 x(t) + 3a_3 x(t) x^\intercal(t)\right]^\intercal p(t) - M(t) B R^{-1} B^\intercal p(t) + M(t) d(t), \tag{10}$$

with boundary conditions $p(T) = 0$. This result follows from the application of the maximum principle; the proof can be found in [2].

Remark 2.1. *The presence of the states in the right-hand sides of the Equations (9) and (10) requires solving them simultaneously with the circuit Equation (5), where the closedloop control (8) is applied to. Feasible numerical algorithms may be employed to solve this problem, such as "shooting", which is the one used for simulation of the examples in this paper.*

3. **Robust Optimal Case.** To solve a regulator problem of the most general type for that circuit, we introduce a different type of control concept for a polynomial system subject to uncertainties. Consider the uncertain circuit as follows:

$$\dot{x}(t;\alpha) = f(t,x,\alpha) + B(t;\alpha)u(t) + d(t;\alpha), \quad x(t_0;\alpha) = x_0, \tag{11}$$

where α is a parameter which belongs to a given parametric set $\mathcal{A}$. In this paper, we consider $\mathcal{A}$ as a finite set, that is $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, each one representing a possible realization or a possible model of the system. The time variable varies in an interval $t \in [t_0, T]$. By instance, the parameters of the polynomial system (11) belong to:

$$a_0 = \{a_{0,1}, a_{0,2}, \ldots, a_{0,N}\}, \quad B = \{B_1, \ldots, B_N\}, \quad d = \{d_1, \ldots, d_N\}.$$

The circuit dynamics is given by a family of N different possible non-linear differential equations sometimes called Multi-Model problem, with no information about the trajectory that is realized. So, it appears that for this type of problems we have N possible state dynamic equations, each of them describing a model and there is no a priory information which will be the active one, but of course, it will be at least one. The system (5) is assumed to be uniformly controllable and observable for each fixed parameter α; the definitions of uniform controllability and observability for nonlinear systems can be found in [13]. For each fixed parameter α, the nonlinear function of the circuit, $f(t,x,\alpha)$, is a polynomial of 2 variables, which are the components of the state vectors $x(t;\alpha) \in \mathbb{R}^2$. Following the previous work [5], a p-degree polynomial of a vector $x(t;\alpha) \in \mathbb{R}^n$ is regarded as a p-linear form of n components of $x(t)$, that is:

$$f(t,x,\alpha) = a_0(t;\alpha) + a_1(t;\alpha)x + a_2(t;\alpha)xx^\intercal + \cdots + a_3(t;\alpha)xxx. \tag{12}$$

Here, the involved parameters are: a_0 is a vector of dimension 2, a_1 is a matrix of dimension 2×2, a_2 is a 3D tensor of dimension $2 \times 2 \times 2$, etc.

Remark 3.1. *Clearly, the uncertainty on the realized parameters is represented by the value of α. Such a parameter belongs, as we said, to the finite set $\mathcal{A}$ that contains all the possible scenarios or parametric realizations of the nonlinear plant, which is fixed during the actual process, with no possibility of change once the process has started. So, each trajectory is uniquely determined by such a set of parameters. Nevertheless, there is no information on the trajectory that is realized. In this way, the proposed control should deal with all of the parameters and should provide an acceptable behavior for such a class of systems.*

The quadratic cost criteria to be minimized are:

$$g\left(x(t;\alpha),u(t),\alpha\right)=\frac{1}{2}x^{\intercal}(T;\alpha)Qx(T;\alpha)+\frac{1}{2}\int_{t_1}^{T}\left(x^{\intercal}(t;\alpha)Lx(t;\alpha)+u^{\intercal}(t)Ru(t)\right)dt. \quad (13)$$

That type of the criteria with uncertain dynamics have been studied in the past [14, 15]. The problem is stated and solved applying the min-max concept, that is, taking the worst case scenario of the functional and then minimizing it by the control. Therefore, the problem to solve is:

$$\min\nolimits_{u(t)}\max\nolimits_{\alpha\in\mathcal{A}}g\left(x(t;\alpha),u,\alpha\right). \quad (14)$$

Therefore, the philosophy of design here is really based on the concept of min–max control where the operation of maximization is taken over a set of uncertainties (in our case, a parameter from a finite set) and the operation of minimization is taken over a set of admissible control strategies. Thus, this paper focuses on the design of a control, which exhibits some sort of robustness property for a class of multi-plant polynomial systems given by a system of ordinary differential equations with parameters from a given finite set. As well-known [22], the Bolza problem can be simplified expressing the original cost function (7) as a Mayer-type functional, that is, a cost function depending only on the terminal values of the state vector; such a transformation involves the extension of the state space. In what follows, we define:

$$x_{n+1}(t;\alpha):=\frac{1}{2}\int_{t_1}^{t}\left(x^{\intercal}(\xi;\alpha)Lx(\xi;\alpha)+u^{\intercal}(\xi)Ru(\xi)\right)d\xi.$$

Taking the derivative in time

$$\dot{x}_{n+1}(t;\alpha)=x^{\intercal}(t;\alpha)Lx(t;\alpha)+u^{\intercal}(t)Ru(t),$$

the new cost function is given by

$$g\left(x(t;\alpha),u(t),\alpha\right)=\frac{1}{2}x^{\intercal}(T;\alpha)Qx(T;\alpha)+x_{n+1}(T;\alpha).$$

Evidently, the term $x^{\intercal}(T;\alpha)Qx(T;\alpha)$ does not depend on the new introduced coordinate, that is:

$$\frac{\partial}{\partial x_{n+1}(T;\alpha)}x^{\intercal}(T;\alpha)Qx(T;\alpha)=0.$$

We proceed with finding the necessary condition for mini-max optimality, for the new Mayer problem. Introduce the following Hamiltonian function:

$$\begin{aligned}H(t,x,q,u,\alpha):=&\,q^{\intercal}(t;\alpha)\left[f(t,x,\alpha)+B(t;\alpha)u(t)+d(t;\alpha)\right]\\&+q_{n+1}^{\intercal}(t;\alpha)\frac{1}{2}\left(x^{\intercal}(t;\alpha)Lx(t;\alpha)+u^{\intercal}(t)Ru(t)\right),\end{aligned}$$

which depends on the given uncertain vector. Under the general mini-max necessary conditions given in the original work [9], the vectors $q(t;\alpha)$ satisfy, for each parameter α, the following equation

$$\begin{aligned}-\frac{d}{dt}q\left(t;\alpha\right)&=\frac{\partial}{\partial x(t;\alpha)}\left[H\left(t,x,q,u,\alpha\right)\right]\\&=q_{n+1}^{\intercal}\left(t;\alpha\right)Lx\left(t;\alpha\right)+\left(\frac{\partial f\left(t,x,\alpha\right)}{\partial x(t;\alpha)}\right)^{\intercal}q\left(t;\alpha\right).\end{aligned}$$

For the last coordinate the time derivative is:

$$\dot{q}_{n+1}\left(t;\alpha\right)=0,$$

and the transversality conditions for these vectors at the terminal time are:

$$\begin{aligned} q(T;\alpha) &= -\mu(\alpha)\, grad\left[x^{\intercal}(T;\alpha)\, Qx(T;\alpha) + x_{n+1}(t;\alpha)\right] \\ &= -\mu(\alpha)\, QxT;\alpha]\, q_{n+1}(T;\alpha_i) \\ &= -\mu(\alpha), \end{aligned}$$

where the constant $\mu(\alpha)$ is a non-negative real value appearing from the general case of the mini-max problem (for more details on this conditions, see also [23]). For each fixed parameter the partial derivative of $f(t,x,\alpha)$ in x is given by:

$$\frac{\partial f(t,x,\alpha)}{\partial x(t;\alpha)} = a_1(t;\alpha) + 2a_2(t;\alpha)\, x + 3a_3(t;\alpha)\, xx^{\intercal}. \tag{15}$$

Consider the following technique of summation of the individual Hamiltonian functions for each α_i $(i = 1, \ldots, N)$, where the summation is taken over all the possible uncertainty parameters. Indeed, it is possible to introduce a *generalized* Hamiltonian function encompassing all of the family plants:

$$\begin{aligned} H^{\Diamond}(t,x,q,u) = \sum_{i=1}^{N} &\left[\left(q^{\intercal}(t;\alpha_i)\left[f(t,x,\alpha_i) + B(t;\alpha_i)\, u(t) + d(t;\alpha_i)\right]\right)\right. \\ &\left. + q_{n+1}^{\intercal}(t;\alpha_i)\, \tfrac{1}{2}\left(x^{\intercal}(t;\alpha_i)\, Lx(t;\alpha_i) + u^{\intercal}(t)\, Ru(t)\right)\right], \end{aligned}$$

which allows us to find the minimax control as

$$u(t) = R^{-1} \sum_{i=1}^{N} \lambda_i B^{\intercal}(t;\alpha_i)\, q(t;\alpha_i), \tag{16}$$

where we take the vector of parameters $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^{\intercal}$ from the set

$$S^N := \left\{\lambda \in \mathbb{R}^N : \lambda_i \geq 0; \sum_{i=1}^{N} \lambda_i = 1\right\}. \tag{17}$$

Similarly to the linear-quadratic case, the mini-max control appears as a mixture of the controls that are the optimal strategies for each fixed parameter value α and the controls are to be found in a multi-dimensional simplex set (17). Now the mini-max strategies are to be found in a finite dimensional space instead of the original functional space.

4. **Parameterized Mini-Max Control for Duffing Equation.** Let us now introduce the following block-diagonal matrices:

$$\begin{aligned} &\mathbf{f} := \begin{bmatrix} f(t,x,\alpha_1) & \ldots & 0 \\ \vdots & . & \vdots \\ 0 & \ldots & f(t,x,\alpha_N) \end{bmatrix}, \quad \mathbf{L} := \begin{bmatrix} L & \ldots & 0 \\ \vdots & . & \vdots \\ 0 & \ldots & L \end{bmatrix}, \\ &\Lambda := \begin{bmatrix} \lambda_1 I_{n\times n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_2 I_{n\times n} \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} Q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q \end{bmatrix}, \\ &\mathbf{B}^{\intercal} := \left[B^{\intercal}(t,\alpha_1) \cdots B^{\intercal}(t,\alpha_N)\right], \quad I \in \mathbb{R}^{n\times n}, \quad i = 1,2, \end{aligned} \tag{18}$$

the following vectors for the extended state and the known external signal:

$$\begin{aligned} \mathbf{d}^{\intercal} &:= \left(d^{1\intercal}, \ldots, d^{M\intercal}\right) \in \mathbb{R}^{n\cdot M} \\ \mathbf{x} &:= \left(x^{1\intercal}, \ldots, x^{M\intercal}\right)^{\intercal} \in \mathbb{R}^{n\cdot M} \end{aligned}$$

and the following tensor matrix for the coefficients of the polynomial:

$$\mathbf{a}_i := \begin{bmatrix} a_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_i \end{bmatrix}, \\ i = 0, \ldots, s.$$

The extended dynamics for the polynomial systems appears as:

$$\dot{\mathbf{x}} = \mathbf{f}\left(t, \mathbf{x}\left(t\right)\right) + \mathbf{B}\left(t\right) u\left(t\right) + \mathbf{d}\left(t\right).$$

Note that the dependence on the uncertain parameter α has disappeared, the new dynamics includes the complete family of plants, but the control remains the same for all plants. That means that the same control will be applied for all the dynamics simultaneously. Based on the extended system, the mini-max regulator that realizes (14) with respect to the quadratic criterion given in (13), for the polynomial system (11), takes the form:

$$u = -R^{-1}\mathbf{B}^{\intercal}\left[\mathbf{M}_{\lambda} + \mathbf{p}_{\lambda}\right], \tag{19}$$

where the matrix function $\mathbf{M}_{\lambda}$ is the solution of the Riccati-type equation

$$\begin{aligned} -\dot{\mathbf{M}}_{\lambda} = {} & \Lambda\mathbf{L} + \left[\mathbf{a}_1 + 2\mathbf{a}_2\mathbf{x} + 3\mathbf{a}_3\mathbf{x}\mathbf{x}^{\intercal} + \ldots + s\mathbf{a}_s\mathbf{x}\cdots(s-1)\,times\cdots\mathbf{x}\right]^{\intercal}\mathbf{M}_{\lambda} \\ & + \mathbf{M}_{\lambda}\left[\mathbf{a}_1 + \mathbf{a}_2\mathbf{x} + \mathbf{a}_3\mathbf{x}\mathbf{x}^{\intercal} + \ldots + \mathbf{a}_s\mathbf{x}\cdots(s-1)\,times\cdots\mathbf{x}\right] \\ & - \mathbf{M}_{\lambda}\mathbf{B}R^{-1}\mathbf{B}^{\intercal}\mathbf{M}_{\lambda}, \end{aligned} \tag{20}$$

with terminal condition $\mathbf{M}_{\lambda}\left(T\right) = \Lambda\mathbf{Q}$, and the parameterized vector function $\mathbf{p}_{\lambda}$ is the solution of the linear equation

$$\begin{aligned} -\dot{\mathbf{p}}_{\lambda} = {} & \mathbf{M}_{\lambda}\mathbf{a}_0 + \left[\mathbf{a}_1 + 2\mathbf{a}_2\mathbf{x} + 3\mathbf{a}_3\mathbf{x}\mathbf{x}^{\intercal} + \ldots + s\mathbf{a}_s\mathbf{x}\cdots(s-1)\,times\cdots\mathbf{x}\right]^{\intercal}\mathbf{p}_{\lambda} \\ & - \mathbf{M}_{\lambda}\mathbf{B}R^{-1}\mathbf{B}^{\intercal}\mathbf{p}_{\lambda} + \mathbf{M}_{\lambda}\mathbf{d}, \end{aligned} \tag{21}$$

with terminal condition as $\mathbf{p}_{\lambda}\left(T\right) = 0$. The matrix Λ containing the optimal weight parameters $\lambda = \lambda^{*}$ solves the following optimization problem:

$$\begin{aligned} & \lambda^{*} = \min_{\lambda \in S^N} J\left(\lambda\right), \\ & J\left(\lambda\right) := \max_{\alpha \in \mathcal{A}} g\left(x\left(t;\alpha\right), u, \alpha\right), \end{aligned} \tag{22}$$

with $u\left(t\right)$, given in (19), parameterized by the vector $\lambda = \left(\lambda_1^{*}, \lambda_2^{*}, \ldots, \lambda_N^{*}\right)^{\intercal}$ $\left(\lambda_i^{*} \in S^N\right)$ through (20) and (21).

Remark 4.1. *In the case of a fully known plant* ($\alpha = 1$)*, that is, there is no parametric uncertainty, the above equations collapse into the standard known result of the optimal control for the polynomial system* [2].

Remark 4.2. *The dependence on* λ *in the cost function can be seen through the dependence of the cost on the solutions of the parameterized Equations (20) and (21); an expression for this can be found in* [10].

Finding the optimal weight vector λ^{*}, which solves the problem (7), may not be, an easy task. In what follows, we provide a feasible numerical procedure to find the mini-max weights in the case of two scenarios ($\alpha = 2$).

Algorithm:

Step 1) Select an initial condition for the control weights Λ in the extended SDRE (20).

Step 2) Apply the control action equal to the combination of standard optimal strategies using the solution of the individual SDRE (19).

Step 3) Determine the corresponding possible dynamics by means of the numerical method mentioned in the Remark 2.1 solving (14) and (15), along with (11), and calculate the

corresponding cost functional.
Step 4) Using the corresponding cost functional values, and considering that $\lambda := (\lambda_1, 1-\lambda_1)$, perform a search to find the minimizing values of λ.

It is still possible to explore numerical methods to solve the problem for more than two scenarios, this will be the object of future research.

5. **Simulation Example.** In this section, we present two numerical examples for the controlling of the Duffing equation. In the first case presents the numerical results produced by the robust algorithm for the given two sets of parameters, in the second one, we consider the optimal control of the circuit with no uncertainties.

5.1. **Example 1.** Consider the following 2-model circuit

$$\begin{array}{ll} \text{Model 1} & \left\{ \begin{array}{l} \dot{x}_1^1 = x_2^1 \\ \dot{x}_2^1 = -1.1 * x_1^1 - 0.4 * x_2^1 - (x_1^1)^3 + 2.05 * \cos(1.8 * t) + u \end{array} \right. \\ \text{Model 2} & \left\{ \begin{array}{l} \dot{x}_1^2 = x_2^2 \\ \dot{x}_2^2 = -1.15 * x_1^2 - 0.45 * x_2^2 - 1.05 * (x_1^2)^3 + 2 * \cos(1.9 * t) + u \end{array} \right. \end{array} \tag{23}$$

Given the differences between the two models of the parameters, it is impossible to design an individual control for each plant, because it would not work for another plant. We select $R = 1$, $Q = 1$, $L = 1$ and $T = 1.5$. The following table (Table 1) shows the values of the performance index for different values of the parameter λ.

Table 1. Values of the performance index for different values of the parameter λ

λ_1	λ_2	$g(x,u,1)$	$g(x,u,2)$
0	1	2.5688	2.5843
1	0	2.6255	2.6409
0.5000	0.5000	2.5639	2.5796
0.2500	0.7500	2.5577	2.5733
0.3000	0.7000	2.5576	2.5733
0.2700	0.7300	2.5576	2.5732
0.2800	0.7200	2.5576	2.5732

This table presents the corresponding changes in the values of the cost function depending on the values of the parameter λ. We can still perform a search to see for which set of values of the vector λ the cost takes the minimum value, considering that we are minimizing a weighted sum. The minimum is achieved on $\lambda = \begin{pmatrix} 0.2800 & 0.7200 \end{pmatrix}$ and, as a result, the individual cost for each one of the two scenarios has almost the same value. We obtain the performance index as a function of the weighting parameter λ near the extremum point. This is illustrated in Figure 2, where we can see that the performance index has the minimum around $\lambda_1 = 0.28$. The corresponding state variable dynamics is depicted in Figure 3. Actually, it is shown for both plants: the blue and green lines correspond to the states 1 and 2 of the first plant and the red and light blue lines represent states 1 and 2 of the second plant, respectively. Note that both plants demonstrate almost identical behavior, when the robust control is applied, thus proving that the robust design works well for any of the two plants. The control law and the final criterion as functions of time are shown in Figures 4 and 5. Here, the blue line corresponds to plant 1 and the green line represents plant 2. Note again that both lines virtually coincide due to the practically same cost.

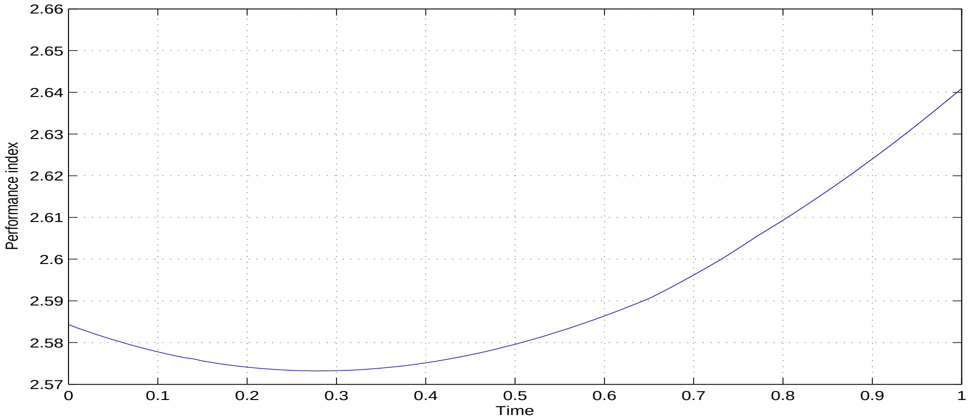

FIGURE 2. 2-Model: Performance index $J(\lambda)$

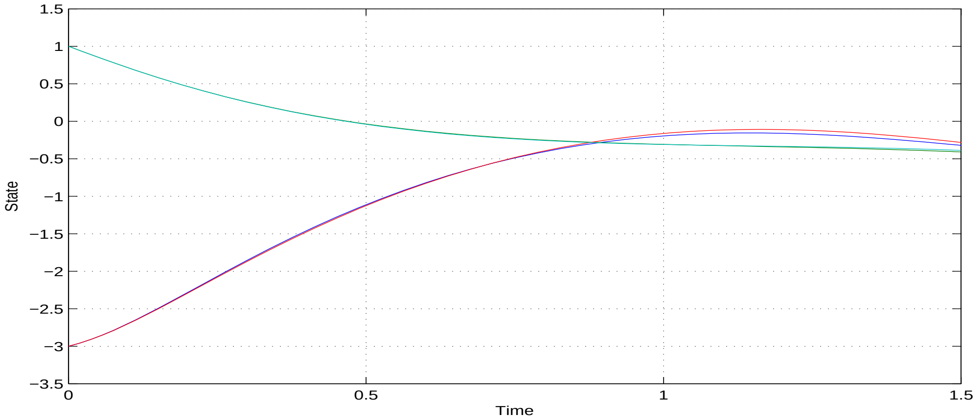

FIGURE 3. 2-Model: Circuit states

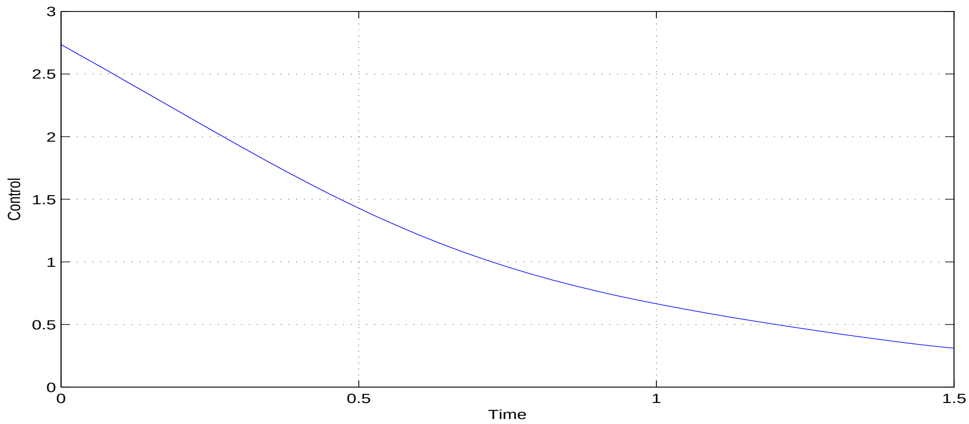

FIGURE 4. 2-Model: Control signal corresponding to λ^*

5.2. **Example 2.** Consider the following one-model circuit:

$$\text{1-Model} \quad \begin{cases} \dot{x}_1^1 = x_2^1 \\ \dot{x}_2^1 = -1.1 * x_1^1 - 0.4 * x_2^1 - (x_1^1)^3 + 2.1 * \cos(1.8 * t) + u \end{cases} \tag{24}$$

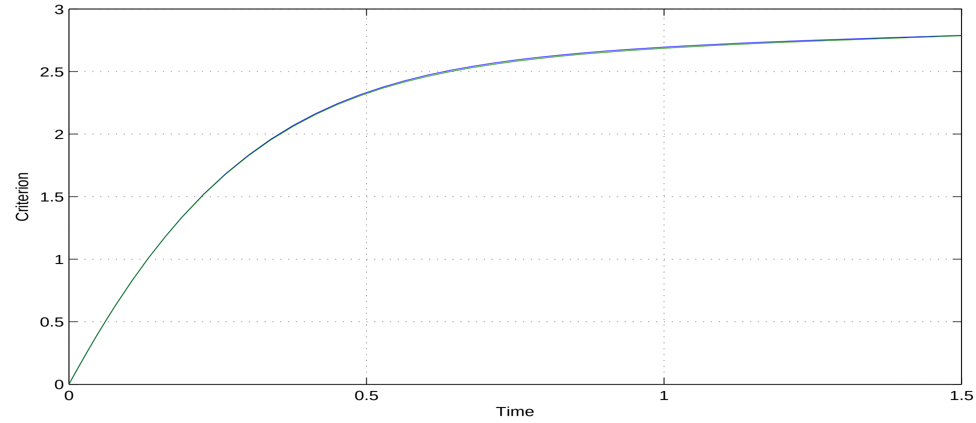

FIGURE 5. 2-Model: Criterion corresponding to λ^*

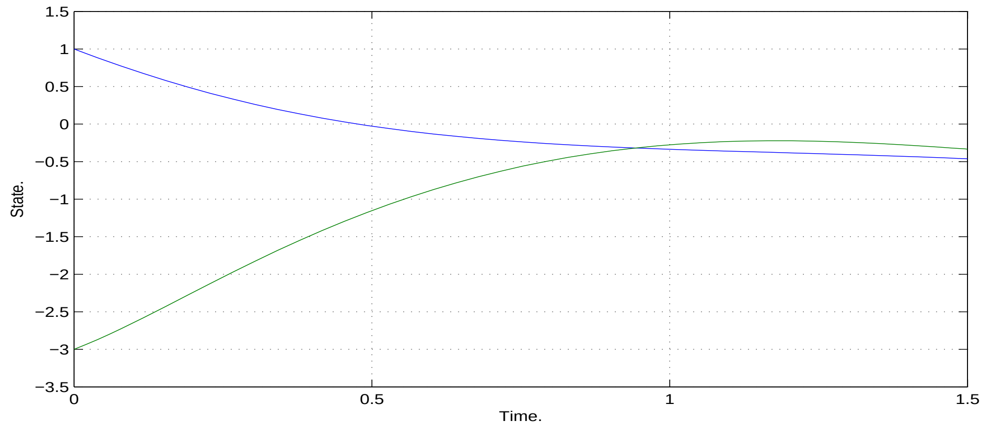

FIGURE 6. 1-Model: Circuit states

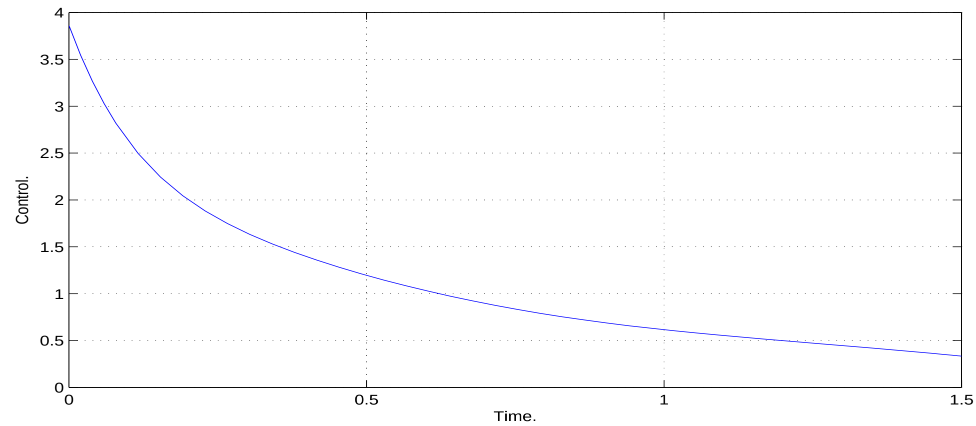

FIGURE 7. 1-Model: Control signal

In this example, we select the parameters of the cost function as $R = 1$, $Q = 1$, $L = 1$ and $T = 1.5$. The corresponding state variable dynamics is depicted in Figure 6, where

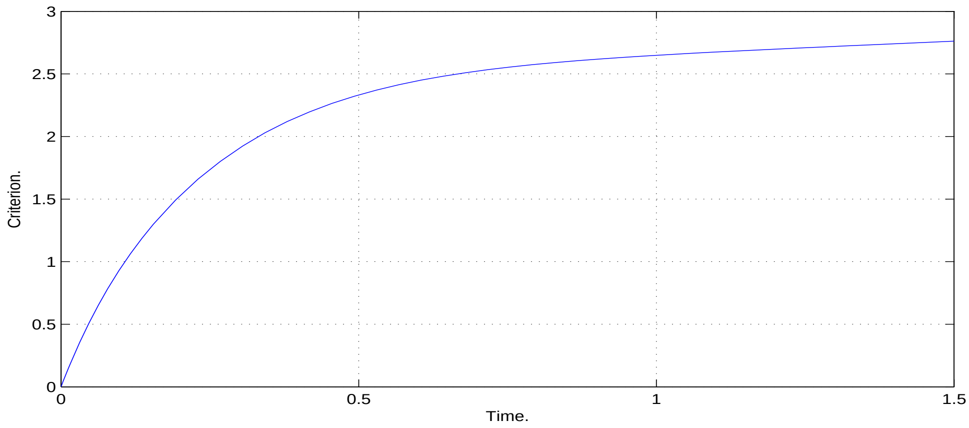

FIGURE 8. 1-Model: Criterion

the blue and green lines correspond to the states 1 and 2, respectively. The control law and the criterion are shown in Figures 7 and 8. We can see that the SDRE technique-based control law efficiently works to stabilize the states around zero, upon being applied to the model (24). In this example, the time is set to 1.5; however, it can be extended as much as needed.

6. **Conclusion.** This paper studied the optimal and the robust optimal control problems for a nonlinear electrical circuit modeled by a Duffing equation with a quadratic cost as a performance index. The control laws are designed for a system with and without parameter uncertainties in the right-hand side of this nonlinear differential equation. The obtained robust strategy allows one to efficiently solve the problem of controlling the Duffing equation with uncertainties in the parameters. The simulations examples show good performance of both controllers for a given set of parameters.

Acknowledgment. Manuel Jiménez-Lizárragawas supported by project PAICyT CE403-10 and CONACYT 169734. Michael Basin was supported by project CONACYT 129081.

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