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A NEW LMI CRITERION FOR THE REALIZATION OF LIMIT CYCLE-FREE DIRECT FORM DIGITAL FILTERS WITH SATURATION ARITHMETIC

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Abstract. *In this paper, we propose a new criterion in the form of linear matrix inequality (LMI) for the elimination of limit cycles in direct form digital filters with saturation arithmetic and external interference. The proposed criterion not only guarantees asymptotic stability, but also reduces the effect of external interference to an $l_2 - l_\infty$ induced norm constraint. The criterion can be checked readily by using some standard numerical packages. An illustrative example is given to demonstrate the effectiveness of the proposed criterion.*

Keywords: $l_2 - l_\infty$ approach, Asymptotic stability, Digital filters, Finite wordlength effects, Linear matrix inequality (LMI)

1. **Introduction.** When designing a digital filter using fixed-point arithmetic, quantization and overflow nonlinearities are produced due to finite wordlength. Because of these nonlinearities, the filter may exhibit unstable behavior. Such nonlinearities may also become the cause of the occurrence of phenomena like zero-input limit cycles. The occurrence of limit cycles in digital filters is undesirable and represents an unstable behavior. When designing a digital filter using fixed-point arithmetic, therefore, one needs to know the ranges of the values of the filter parameters for the nonexistence of limit cycles and choose the values so that the designed filter is free of limit cycles. If one can choose the values of the filter parameters so as to ensure the asymptotic stability of the filter, then this automatically implies that the filter will be free of limit cycles. If one can choose the values of the filter parameters so as to ensure the asymptotic stability of the filter, then this automatically implies that the filter will be free of limit cycles. The criteria for the nonexistence of limit cycles in digital filters employing saturation overflow arithmetic have attracted much attention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this paper, direct form digital filters involving single overflow nonlinearity [1, 2, 6, 8, 9, 10] are considered. Quantization and overflow nonlinearities may interact with each other. However, if the total number of quantization steps is large, or in other words, the internal wordlength is sufficiently long, then the effects of these nonlinearities can be regarded as decoupled or noninteracting, and can therefore be investigated separately. Under this decoupling approximation, quantization effects may be neglected when studying the effects of overflow [11]. For further study of effects of finite wordlength, the reader is referred to [11].

In the hardware implementation of a high-order digital filter, it is usually broken down into several biquad filters before implementation. Then, interference between the biquad filters becomes a factor: This interference ultimately results in malfunctioning and destruction of the device [12, 13]. However, most existing criteria for the stability of digital filters are only relevant to specific conditions; in unfavorable environments with external interference, these criteria will be of little use. Therefore, it is desirable to obtain an alternative criterion that can overcome the shortcomings of existing criteria.

In real physical systems, one is faced with model uncertainties and a lack of statistical information on the signals. This has led to an interest in $l_2 - l_\infty$ control and state estimation in recent years, with the belief that an $l_2 - l_\infty$ approach (or $\mathcal{L}_2 - \mathcal{L}_\infty$ approach for continuous-time systems) is more robust and less sensitive to disturbance variances and model uncertainties [14, 15, 16, 17, 18, 19]. A natural question arises: can we obtain an $l_2 - l_\infty$ stability criterion for digital filters with saturation arithmetic and external interference? This paper gives an answer to this interesting question. To the best of our knowledge, in terms of the $l_2 - l_\infty$ stability of digital filters with saturation arithmetic and external interference, there has been no result given in the literature so far; this remains an open and challenging problem.

In this paper, we propose a new linear matrix inequality (LMI)-based criterion for the $l_2 - l_\infty$ stability of direct form digital filters with saturation arithmetic and external interference. The proposed criterion guarantees that the digital filter is asymptotically stable and the $l_2 - l_\infty$ induced norm from the external interference to the state vector is reduced to an interference attenuation level. The criterion can be checked easily by recently developed convex optimization algorithms [20, 21].

This paper is organized as follows. In Section 2, an LMI criterion for the $l_2 - l_\infty$ stability of direct form digital filters with saturation arithmetic and external interference is proposed. In Section 3, a numerical example is given, and finally, conclusions are presented in Section 4.

2. **New Criterion.** The system under consideration has a linear part described by the transfer function $G(z)$:

$$G(z) = h_1 z^{-n} + h_2 z^{-(n-1)} + h_3 z^{-(n-2)} + \cdots + h_n z^{-1}. \tag{1}$$

The output of $G(z)$ is $y(r)$ and the input to $G(z)$ is $f(y(r))$. The following condition is imposed:

$$z^n - h_n z^{n-1} - h_{n-1} z^{n-2} \cdots - h_2 z - h_1 \neq 0 \tag{2}$$

for all $|z| \geq 1$, which implies that the infinite-precision counterpart of the filter is stable. We assume that the saturation overflow arithmetic is given by

$$f(y(r)) = \begin{cases} 1, & \text{if } \ y(r) > 1, \\ y(r), & \text{if } \ -1 \leq y(r) \leq 1, \\ -1, & \text{if } \ y(r) < -1. \end{cases} \tag{3}$$

Note that the saturation overflow arithmetic is confined to the sector $[0,\ 1]$, i.e.,

$$f(0) = 0, \quad -1 \leq \frac{f(y(r))}{y(r)} \leq 1. \tag{4}$$

The system (1) can be represented by the following state equations:

$$\begin{aligned} x_1(r+1) &= x_2(r), \\ x_2(r+1) &= x_3(r), \\ &\ \vdots \\ x_{n-1}(r+1) &= x_n(r), \\ x_n(r+1) &= f(h_1 x_1(r) + h_2 x_2(r) + \cdots + h_n x_n(r)), \end{aligned} \tag{5}$$

which can be put in the following matrix form:

$$x(r+1) = Ax(r) + Bf\left(H^T x(r)\right), \tag{6}$$

where

$$A = \begin{bmatrix} 0 & & & \\ 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} h_1 \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix}, \quad x(r) = \begin{bmatrix} x_1(r) \\ \vdots \\ x_{n-1}(r) \\ x_n(r) \end{bmatrix}, \tag{7}$$

and I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix. Equation (6) with the condition (3) may be used to describe a class of discrete-time dynamical systems with state saturation, which include digital filters using saturation overflow arithmetic, digital control systems with saturation nonlinearities, a class of neural networks, and so on.

In this paper, we consider the following digital filter with external interference:

$$x(r+1) = Ax(r) + Bf\left(H^T x(r)\right) + w(r), \tag{8}$$

where $w(r) \in R^n$ is the external interference. Let $z(r) \in R^p$ be a linear combination of the states, which is given by

$$\begin{aligned} z(r) &= [z_1(r)\ z_2(r)\ \cdots\ z_p(r)]^T \\ &= Kx(r), \end{aligned} \tag{9}$$

where $K \in R^{p\times n}$ is a known constant matrix.

Given a level $\gamma > 0$, the purpose of this paper is to obtain a new $l_2 - l_\infty$ stability criterion such that the system (8) and (9) with $w(r) = 0$ is asymptotically stable and

$$\sup_{r\geq 0}\left\{z^T(r)z(r)\right\} < \gamma^2 \sum_{r=0}^{\infty} w^T(r)w(r) \tag{10}$$

under zero-initial conditions for all nonzero $w(r)$. The parameter γ is called the $l_2 - l_\infty$ induced norm bound or the interference attenuation level. In this case, the system (8) and (9) is said to be asymptotically stable with $l_2 - l_\infty$ performance γ.

Now we are ready to state a new $l_2 - l_\infty$ stability criterion for direct form digital filters with saturation arithmetic and external interference.

Theorem 2.1. *For a given level $\gamma > 0$, if we assume that there exist a symmetric positive definite matrix P and positive scalars δ, m such that*

$$\begin{bmatrix} \delta HH^T + A^T PA - P & A^T PB + mH & A^T P \\ B^T PA + mH^T & B^T PB - 2m - \delta & B^T P \\ PA & PB & P - I \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} -P & K^T \\ K & -\gamma^2 I \end{bmatrix} < 0, \tag{12}$$

then the system (8) and (9) is asymptotically stable with $l_2 - l_\infty$ performance γ.

Proof: First, to establish the $l_2 - l_\infty$ performance for the system (8) and (9), consider the following Lyapunov function:

$$V(x(r)) = x^T(r)Px(r). \tag{13}$$

Along the trajectory of (8), we have

$$\begin{aligned}
\Delta V(x(r)) =& V(x(r+1)) - V(x(r)) \\
=& \left[Ax(r) + Bf\left(H^T x(r)\right) + w(r)\right]^T P \left[Ax(r) + Bf\left(H^T x(r)\right) + w(r)\right] \\
& - x^T(r)Px(r) \\
=& x^T(r)\left[A^T PA - P\right] x(r) + x^T(r)A^T PBf\left(H^T x(r)\right) + x^T(r)A^T Pw(r) \\
& + f\left(H^T x(r)\right) B^T PAx(r) + \left\{f\left(H^T x(r)\right)\right\}^2 B^T PB + f\left(H^T x(r)\right) B^T Pw(r) \\
& + w^T(r)PAx(r) + w^T(r)PBf\left(H^T x(r)\right) + w^T(r)Pw(r) \\
& + f(H^T x(r))\left[2mH^T x(r) - 2mf\left(H^T x(r)\right)\right] - 2mf(y(r))[y(r) - f(y(r))].
\end{aligned}$$

From (4), it is clear that

$$\{f(H^T x(r))\}^2 \le \{H^T x(r)\}^2 = x^T(r)HH^T x(r). \tag{14}$$

Then, for a positive scalar δ, we have

$$\delta\left[x^T(r)HH^T x(r) - \left\{f\left(H^T x(r)\right)\right\}^2\right] \ge 0. \tag{15}$$

Using (15), a new bound for $\Delta V(x(r))$ can be obtained as

$$\begin{aligned}
\Delta V(x(r)) \le& x^T(r)\left[A^T PA - P\right] x(r) + x^T(r)\left[A^T PB + mH\right] f\left(H^T x(r)\right) \\
& + x^T(r)A^T Pw(r) + f(H^T x(r))\left[B^T PA + mH^T\right] x(r) \\
& + \left\{f\left(H^T x(r)\right)\right\}^2 \left[B^T PB - 2m\right] + f\left(H^T x(r)\right) B^T Pw(r) \\
& + w^T(r)PAx(r) + w^T(r)PBf\left(H^T x(r)\right) + w^T(r)Pw(r) \\
& - 2mf(y(r))[y(r) - f(y(r))] + \delta\left[x^T(r)HH^T x(r) - \left\{f\left(H^T x(r)\right)\right\}^2\right] \\
=& \begin{bmatrix} x(r) \\ f\left(H^T x(r)\right) \\ w(r) \end{bmatrix}^T \begin{bmatrix} \delta HH^T + A^T PA - P & A^T PB + mH & A^T P \\ B^T PA + mH^T & B^T PB - 2m - \delta & B^T P \\ PA & PB & P - I \end{bmatrix} \\
& \times \begin{bmatrix} x(r) \\ f\left(H^T x(r)\right) \\ w(r) \end{bmatrix} + w^T(r)w(r) + \Phi(r),
\end{aligned} \tag{16}$$

where $\Phi(r) = -2mf(y(r))[y(r) - f(y(r))]$. Note that $\Phi(r)$ is nonpositive in view of (3).

If the LMI (11) is satisfied, we have

$$\Delta V(x(r)) < w^T(r)w(r). \tag{17}$$

Under the zero-initial condition, one has $V(x(r))|_{r=0} = 0$ and $V(x(r)) \ge 0$. Define

$$J(r) = V(x(r)) - \sum_{k=0}^{r-1} w^T(k)w(k). \tag{18}$$

Then, for any nonzero $w(r)$, we obtain

$$\begin{aligned}
J(r) &= V(x(r)) - V(x(r))|_{r=0} - \sum_{k=0}^{r-1} w^T(k)w(k) \\
&= \sum_{k=0}^{r-1} \left[\Delta V(x(k)) - w^T(k)w(k)\right].
\end{aligned}$$

From (17), we have $J(r) < 0$. It means that

$$V(x(r)) < \sum_{k=0}^{r-1} w^T(k)w(k).$$

On the other hand, by the Schur complement, the LMI (12) is equivalent to

$$K^T K - \gamma^2 P < 0. \tag{19}$$

Noting (9) and (13), (19) implies that

$$\begin{aligned} z^T(r)z(r) &= x^T(r)K^T Kx(r) \\ &< \gamma^2 x^T(r)Px(r) \\ &= \gamma^2 V(x(r)) \\ &< \gamma^2 \sum_{k=0}^{r-1} w^T(k)w(k) \\ &\leq \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k). \end{aligned} \tag{20}$$

Taking the supremum over $r \geq 0$ leads to (10).

Next, we show that, under the LMI conditions (11) and (12), the system (8) with $w(r) = 0$ is asymptotically stable. When $w(r) = 0$, we have

$$\Delta V(x(r)) < 0 \tag{21}$$

from (17). This guarantees

$$\lim_{r\to\infty} x(r) = 0 \tag{22}$$

from Lyapunov stability theory [22]. This completes the proof.

Remark 2.1. *The stability condition proposed in this paper is given in terms of LMIs. This LMI-based criterion is computationally efficient and flexible due to the recently developed convex optimization algorithms* [20]. *Various efficient convex optimization algorithms can be used to check whether the LMIs (11) and (12) are feasible. In this paper, in order to solve the LMIs, we utilized MATLAB LMI Control Toolbox, which implements state-of-the-art interior-point algorithms* [21, 23, 24, 25].

Remark 2.2. *Most existing results on stability analysis for digital filters in the literature were restricted to digital filters without external interference. Unfortunately, with the existing works, it is impossible to analyze stability for digital filters with external interference. For the first time, this paper proposes a criterion for stability of digital filters with external interference. The presented result of this paper opens a new path for application of the $l_2 - l_\infty$ approach to stability analysis for digital filters. In contrast to the existing works on stability analysis for digital filters, the advantage of our approach is that it can be applied to digital filters with parametric uncertainty and external interference.*

Remark 2.3. *If we use the following augmented Lyapunov function:*

$$V(x(r)) = x^T(r)P_1 x(r) + x^T(r+1)P_2 x(r+1),$$

where $P_1 = P_1 > 0$ and $P_2 = P_2 > 0$, instead of (13), the potential conservatism of LMI condition (11) and (12) may be reduced. The reduction of the conservatism remains as a future work.

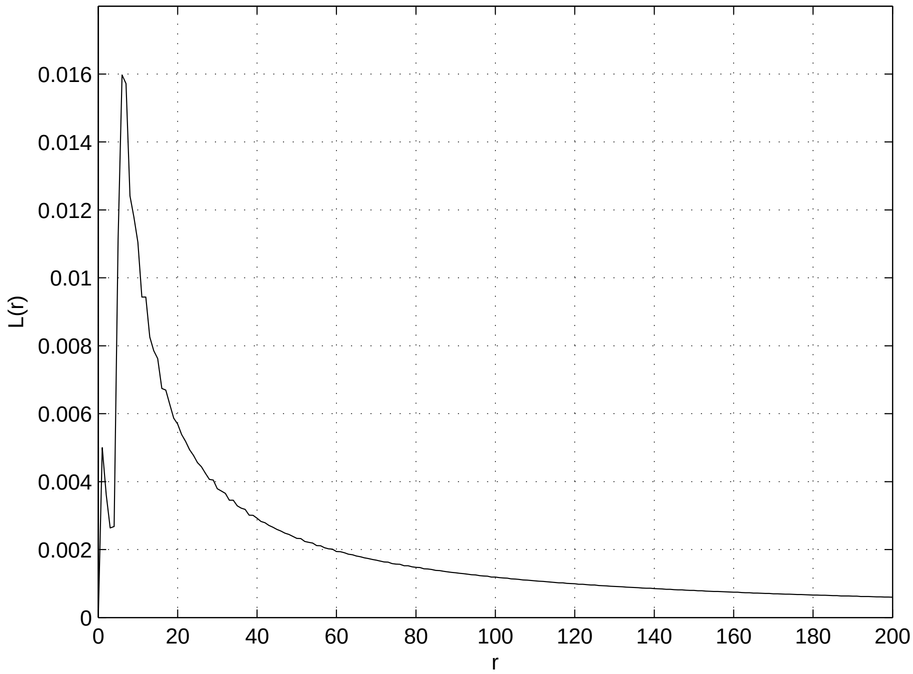

FIGURE 1. The plot of $L(r)$

Remark 2.4. *The $l_2 - l_\infty$ induced norm* [26, 27] *is defined as*

$$\|T_{zw}\|_{l_2-l_\infty} = \frac{\sqrt{\sup_{r\ge 0}\{z^T(r)z(r)\}}}{\sqrt{\sum_{r=0}^{\infty} w^T(r)w(r)}}$$

where T_{zw} is a transfer function matrix from $w(r)$ to $z(r)$. For a given level $\gamma > 0$, $\|T_{zw}\|_{l_2-l_\infty}$ can be restated in the equivalent form (10). If we define

$$L(r) = \frac{\sup_{0\le k\le r}\left\{z^T(k)z(k)\right\}}{\sum_{k=0}^{r} w^T(k)w(k)}, \tag{23}$$

the relation (10) can be represented by

$$L(\infty) < \gamma^2. \tag{24}$$

In the next section, through the plot of $L(r)$, the relation (24) is verified.

3. **Numerical Example.**

Consider a second-order system (8) with

$$h_1 = -0.6, \quad h_2 = 1.1, \quad w(r) = \begin{bmatrix} \cos(5r) \\ \sin^2(2r) \end{bmatrix}, \quad K = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{25}$$

For the design objective (10), let the $l_2 - l_\infty$ performance be specified by $\gamma = 0.45$. Solving the LMIs (11) and (12) by the convex optimization technique of MATLAB software gives

$$P = \begin{bmatrix} 0.0963 & -0.0749 \\ -0.0749 & 0.1771 \end{bmatrix}, \quad m = 0.1595, \quad \delta = 0.0041.$$

Figure 1 shows the plot of $L(r)$ and verifies $L(\infty) < \gamma^2 = 0.2025$. This means that the $l_2 - l_\infty$ induced norm from the external interference $w(r)$ to the state vector $z(r)$ is reduced within the $l_2 - l_\infty$ induced norm bound γ. It can be easily verified that each of the criteria in previous works [1, 2, 6, 8, 9, 10] fails in the example given by (8) and (9) with the parameters (25). On the other hand, the criterion (11) and (12) guarantees

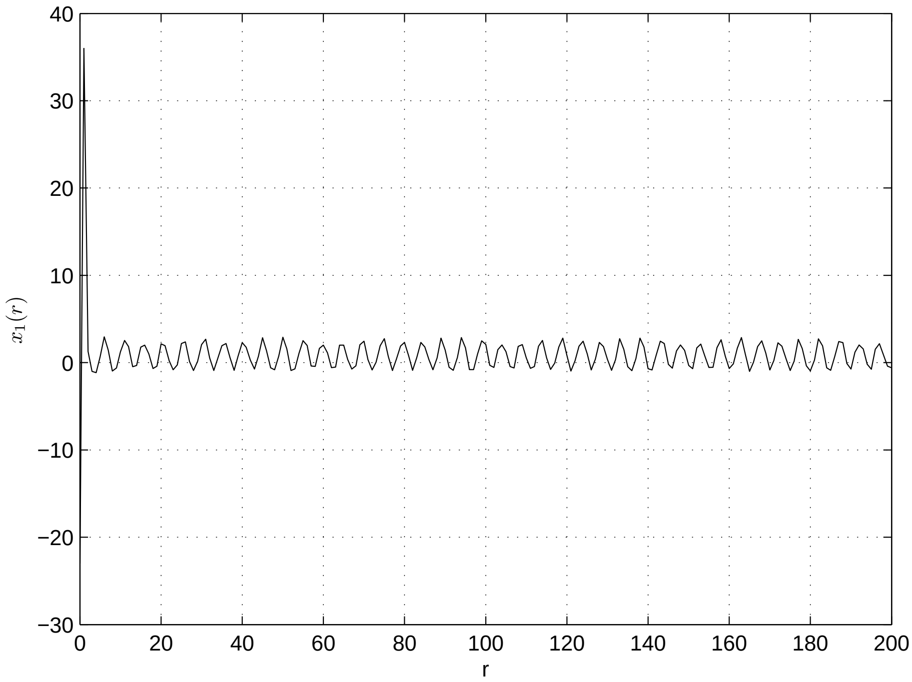

FIGURE 2. Response of the state $x_1(r)$

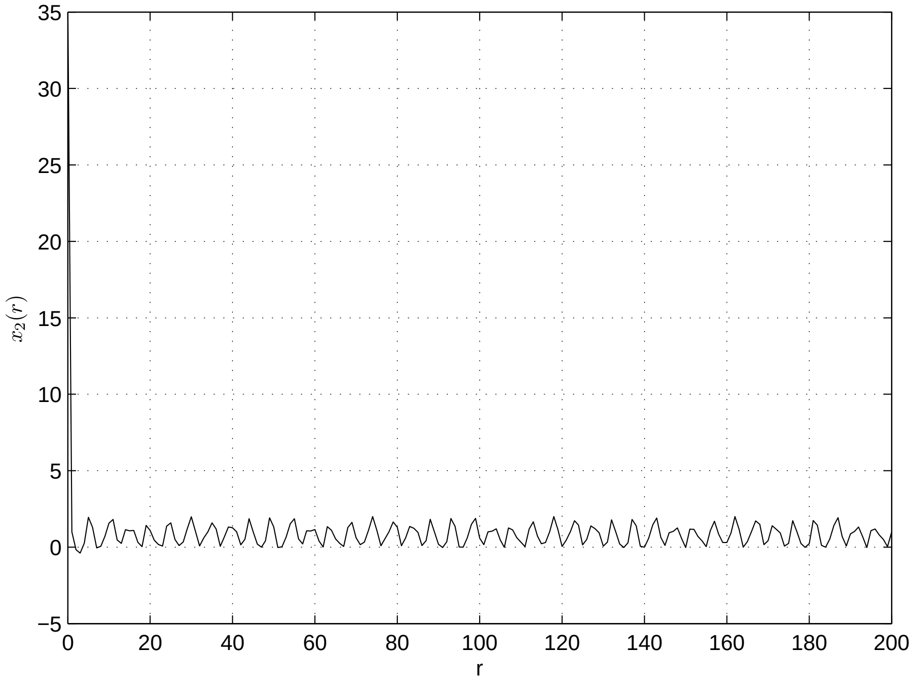

FIGURE 3. Response of the state $x_2(r)$

the asymptotic stability with $l_2 - l_\infty$ performance in this example. Figures 2 and 3 show state trajectories when the initial state vector is given by $x(0) = [-23\ 35]^T$. From these figures, it can be seen that the proposed criterion guarantees to reduce the effect of the external interference $w(r)$ on the state vector $x(r)$ to within a disturbance attenuation level $\gamma = 0.45$.

4. **Conclusions.** In this paper, a new LMI-based criterion for the l_2-l_∞ stability of direct form digital filters with saturation arithmetic and external interference has been proposed. The proposed criterion can guarantee reduction in the effect of external interference to an interference attenuation level. Thus, it overcomes the limitations of existing criteria. A numerical example is presented to demonstrate the validity of the proposed approach.

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