International Journal of Innovative Computing, Information and Control
Volume **8**, Number **3(A)**, March **2012**
ICIC International ©2012 ISSN 1349-4198
pp. 1897–1905

OPTIMAL REPETITIVE CONTROL BASED ON TWO-DIMENSIONAL MODEL

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Received December 2010; revised April 2011

Abstract. *This paper presents a new design method for repetitive-control systems based on two-dimensional (2D) system theory and optimal control. First, a 2D model is established that describes the control action within a repetition period and the learning process between periods. Next, the problem of designing a repetitive controller is formulated as an optimization problem for the 2D model. Then, one-dimensional optimal control theory is used to find an optimal controller. Unlike other methods, this one allows the control and learning actions to be preferentially adjusted. Finally, a numerical example demonstrates the validity of this approach.*

Keywords: Repetitive control, Optimal control, Linear quadratic regulator (LQR), Two-dimensional model

1. Introduction.

Periodic signals are very common in engineering. They are associated with engines, electrical motors and generators, converters, machines that perform a cyclic task and many other things. To handle them, repetitive control was originated by Inoue et al. [1] and subsequently developed by many researchers, such as Hara et al. [2] and Tomizuka et al. [3]. It has proven to be a very practical and effective way for a system to track a periodic reference and reject periodic disturbances.

Over the last few decades, a great deal of research has been devoted to the theory and application of repetitive control, and various structures and algorithms have been devised [4-6]. Although repetitive control is closely related to a repetitive process (RP) [7] and to iterative learning control (ILC) [8,9], it differs from them in a fundamental way, namely, the boundary condition. For an RP and ILC, the initial state of each pass is set to the same value. In contrast, for repetitive control the initial state of a period is the final state of the previous period. This difference gives rise to a difference regarding stability. For an RP and ILC, it is relatively easy to guarantee stability; but for repetitive control, it is very difficult. Research on repetitive control has focused mainly on this issue.

A close examination of repetitive control shows that it actually involves two independent types of actions: control and learning. Their characteristics are completely different:

control is a continuous process within one repetition period, while learning is discrete behavior between periods. Information propagation occurs in both continuous and discrete domains. Since there is no mathematical model that rigorously describes these two types of actions, all the analysis and design methods for repetitive control that have so far been developed ignore the difference between them. Moreover, they deal with the two different actions equally in the time domain. As a result, they are based only on the overall effect of the control and learning actions, and cannot produce a repetitive controller with a sophisticated design that handles the two types of actions independently.

Previous studies on repetitive control have generally employed a feedback controller to improve the closed-loop stability. They consider the problem of designing a feedback controller separately from that of designing a repetitive controller. This may result in stability conditions that are complex and conservative. The new configuration for a repetitive-control system in this paper solves the problem. It contains a dynamic compensator and a state-feedback controller. The design method employs two-dimensional (2D) system theory [7,10-12] and a linear-quadratic regulator [13-15]. The parameters of both the feedback controller and dynamic compensator are designed simultaneously. First, a 2D model is derived that describes a repetitive-control system, and the design of the system is formulated as the problem of stabilizing a 2D system. Next, an optimal controller for the 2D system is obtained by solving a one-dimensional optimal control problem. The optimal repetitive-control law features the preferential adjustment of control and learning. Finally, a numerical example demonstrates the validity of this approach.

Throughout this paper, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices.

2. Problem Formulation and 2D Model of Repetitive-Control System.

In the basic configuration of a repetitive-control system (Figure 1), $G(s)$ is the compensated plant, $r(t)$ is a periodic reference input with a period of L, K_e is a static compensator, and

$$C_R(s) = \frac{1}{1 - e^{-sL}} \tag{1}$$

is a repetitive controller. Since

$$C_R(j\omega_k) = \frac{1}{1 - e^{-j\omega_k L}} = \infty, \quad \omega_k = \frac{2k\pi}{L}, \quad k = 0, 1, \cdots,$$

the gain of the controller is infinite at the angular frequencies of the fundamental and harmonic waves of a periodic signal, and $C_R(s)$ is an internal model of a periodic signal. So, inserting $C_R(s)$ into the control system guarantees that the output of the system, $y(t)$, tracks the periodic reference signal, $r(t)$, without steady-state tracking error when the closed-loop system is stable.

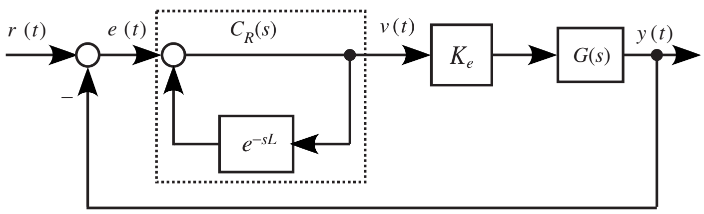


FIGURE 1. Basic configuration of repetitive-control system

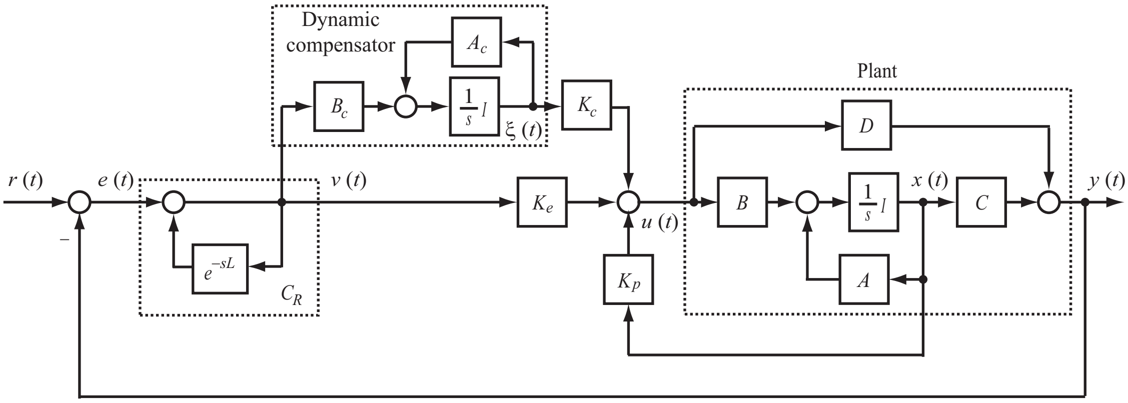


FIGURE 2. Repetitive-control system

When the repetitive-control system in Figure 1 has a pure-delay element and unit positive feedback, it is called a neutral-type delay system [2][1]. Since it contains an infinite number of poles on the imaginary axis, it is very difficult to stabilize. The system in Figure 1 is stabilizable only when $G(s)$ has a direct path from the input to the output; that is, the relative degree of the plant must be zero. If that is not the case, then a low-pass filter must be inserted into the delay path to facilitate stabilization. This structure is called a modified repetitive-control system. The problem with it is that the low-pass filter prevents the internal model principle from holding, which leads to steady-state tracking error. In other words, stability is guaranteed at the cost of tracking precision [2]. Note that we employ a dynamic compensator in this study. It is possible to use a static compensator, K_e; but that gives rise to a trade-off between transient tracking performance and stability. A dynamic compensator solves the problem.

As a preliminary step, we first consider the repetitive control of a plant with a relative degree of zero. For the configuration in Figure 2, we consider the problem of designing the dynamic compensator

$$\dot{\xi}(t) = A_c\xi(t) + B_c v(t), \quad \xi(t) \in \mathbb{R}^{n_c} \tag{2}$$

and the control law

$$u(t) = K_p x(t) + K_e v(t) + K_c \xi(t). \tag{3}$$

We take the hybrid nature of repetitive control into account by introducing two domains: a continuous domain, τ, for the control process within one repetition period; and a discrete domain, k, for the learning behavior between periods. The relationship between τ, k, and time, t, is

$$t = kL + \tau, \quad \tau \in [0, L], \quad k \in \{0, 1, \cdots\}. \tag{4}$$

Any variable, $\chi(t)$, in the time domain can be written as

$$\chi(t) = \chi(kL + \tau) := \chi(k, \tau) \tag{5}$$

in the 2D domain. Also, we assume that $\chi(t) = 0$ for $t < 0$ and denote

$$\Delta\chi(t) := \chi(t) - \chi(t - L).$$

With the above preparation, let us construct a 2D model for repetitive control.

[1]A neutral-type delay system has the form $\dot{x}(t) - \Gamma\dot{x}(t - L_1) = Ax(t) + Bx(t - L_2)$, where $\Gamma \neq 0$ and $L_1 \neq 0$.

Consider the following controllable and observable plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{6}$$

where $D \neq 0$; $x(t) \in \mathbb{R}^n$ is the state of the plant; and $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$ are the control input and output, respectively. For simplicity, this paper considers only the single-input single-output case ($m = 1$). We also make the standard assumption in servo-system design that the plant (6) has no zeros on the imaginary axis.

For a given periodic reference input, $r(t)$, the tracking error is

$$e(t) = r(t) - y(t). \tag{7}$$

(6) and (7) yield

$$\Delta\dot{x}(k,\tau) = A\Delta x(k,\tau) + B\Delta u(k,\tau) \tag{8}$$

and

$$e(k,\tau) = e(k-1,\tau) - C\Delta x(k,\tau) - D\Delta u(k,\tau). \tag{9}$$

(8) and (9) constitute a 2D model of the repetitive-control system. While the conventional model (6) only describes the overall effect of control and learning, (8) describes the control process during the k-th period, and (9) describes the learning behavior between the k-th and $(k-1)$-th periods. Since (8) does not contain the term $x(k-1,\tau)$, control actions are independent of learning behavior. On the other hand, (9) shows that learning behavior is strongly affected by control results. This is because the amount of learning required decreases as the speed of convergence for control increases.

Combining (8) and (9) yields

$$\begin{bmatrix} \Delta\dot{x}(k,\tau) \\ e(k,\tau) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \begin{bmatrix} \Delta x(k,\tau) \\ e(k-1,\tau) \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} \Delta u(k,\tau). \tag{10}$$

Note that

$$e(k,\tau) = \Delta v(k,\tau). \tag{11}$$

The 2D representation of the dynamic compensator (2) is

$$\Delta\dot{\xi}(k,\tau) = A_c\Delta\xi(k,\tau) + B_c e(k,\tau), \tag{12}$$

and the control law (3) becomes

$$\Delta u(k,\tau) = K_p\Delta x(k,\tau) + K_e e(k,\tau) + K_c\Delta\xi(k,\tau). \tag{13}$$

Now, the design problem can be stated as a 2D regulation problem: Find a dynamic compensator (12) and a stabilizing control law (13) for (10) that adjust the control and learning actions.

3. **Design of Optimal Repetitive Controller.** The problem of designing a repetitive controller can formulated as the problem of stabilizing the 2D system (10). Note that $\Delta x(t) = 0$, $e(t) = 0$ and $\Delta u(t) = 0$ in the steady state. Here, we employ the linear quadratic regulator (LQR) method [13-15].

The quadratic performance index is chosen to be

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\int_{kL}^{(k+1)L} \left[\Delta x^T(k,\tau)Q_x\Delta x(k,\tau) + Q_e e^2(k,\tau) + R\Delta u^2(k,\tau)\right] d\tau. \tag{14}$$

The semi-positive-definite matrix $Q_x \in \mathbb{R}^{n\times n}$, the nonnegative real number $Q_e \in \mathbb{R}$, and the positive real number $R \in \mathbb{R}$ are given weights that adjust control, learning, and the control input, respectively. The problem is to find (12) and (13) for the 2D system (10) that minimize (14).

Remark 3.1. *For learning, the quadratic performance index should be*

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\int_{kL}^{(k+1)L}\left[\Delta x^T(k,\tau)Q_x\Delta x(k,\tau) + Q_e e^2(k-1,\tau) + R\Delta u^2(k,\tau)\right] d\tau.$$

However, since optimization with this index produces the same results as using (14), we use (14) for simplicity.

Remark 3.2. *Note that the LQR method is widely used for tracking control in ILC* [8,16]. *Since $\Delta e(t) = -C\Delta x(t) - D\Delta u(t)$, the terms $Q_e e^2(t)$ and $\Delta x^T(t)Q_x\Delta x(t)$ are mingled in the performance index. As a result, (14) just assesses either the state or the tracking error together with the control input. However, the 2D model allows $\Delta e(t)$ and $\Delta x(t)$ to be assessed separately so that, not only can the overall control performance be adjusted, but the control and learning actions can each be tuned.*

On the other hand, if we rewrite the performance index (14) in the time domain, we obtain

$$J = \frac{1}{2}\int_0^{\infty}\left[\Delta x^T(t)Q_x\Delta x(t) + Q_e e^2(t) + R\Delta u^2(t)\right] dt. \tag{15}$$

This is an LQR problem that is easy to solve using standard optimization theory [13] in the time domain, as explained below.

Theorem 3.1. *For the plant (6), the optimal control law that yields* $\min J$ *for the system in Figure 2 is*

$$\Delta u^*(k,\tau) = [K_p\ K_e\ K_c]\begin{bmatrix}\Delta x(k,\tau)\\ e(k,\tau)\\ \Delta\xi(k,\tau)\end{bmatrix}, \tag{16}$$

$$K_p = -R^{-1}B^TP, \tag{17}$$

$$K_e = \frac{1}{2}R^{-1}DQ_e, \tag{18}$$

$$K_c = -R^{-1}B^T, \tag{19}$$

where the matrices P in (17), and A_c and B_c in (2) are

$$0 = A^TP + PA - PBR^{-1}B^TP + Q_x, \tag{20}$$

$$A_c = -(A + BK_p)^T, \tag{21}$$

$$B_c = \left(\frac{1}{2}C^T + K_p^TD\right)Q_e. \tag{22}$$

Furthermore, the optimal performance index is

$$J_{\min} = \frac{1}{2}\left[x^T(0)Q_x x(0) + Q_e e^2(0)\right]. \tag{23}$$

Note that, since the repetitive-control system employs a dynamic compensator, the input matrix of the compensator, B_c, contains the weight Q_e in (15), and the Riccati Equation (20) contains the weights Q_x and R in (15).

Introducing the following Hamiltonian makes the proof easy:

$$\begin{aligned} H = {} & \frac{1}{2}\Delta x(t)^TQ_x\Delta x(t) + \frac{1}{2}Q_e e^2(t) + \frac{1}{2}R\Delta u^2(t) \\ & + \lambda^T\left[A\Delta x(t) + B\Delta u(t)\right] + \frac{1}{2}\Delta x^T(T)S_x\Delta x(T) + \frac{1}{2}S_e e^2(T), \end{aligned} \tag{24}$$

where λ is a function of $\Delta x(t)$ and $e(t)$, and thus is omitted.

Clearly, (16) is equivalent to

$$u^*(t) = [K_p \; K_e \; K_c] \begin{bmatrix} x(t) \\ v(t) \\ \xi(t) \end{bmatrix}. \tag{25}$$

The optimal repetitive-control law, (25), contains three variables: the state of the plant, $x(t)$; the output of the internal model, $v(t)$; and the state of the dynamic compensator, $\xi(t)$. They serve different purposes: The state feedback part, $K_p x(t)$, improves the control performance during one repetition period; and the state of the dynamic compensator, $K_e v(t)$, and the state feedback part of the dynamic compensator, $K_c \xi(t)$, improve the learning performance between repetition periods. For a given R, adjusting Q_x changes the gain matrix K_p, as can be seen in (17) and (20). So, the control action during one period is regulated very simply by Q_x. On the other hand, the learning action between periods is related to K_e and K_c. It is clear from (18) that changing Q_e directly affects K_e, and thus the learning process. However, when D is very small, K_e cannot directly adjust the learning process very much. In this case, it is regulated indirectly by the state $\xi(t)$.

(10)-(13) yield a closed-loop repetitive-control system in 2D form:

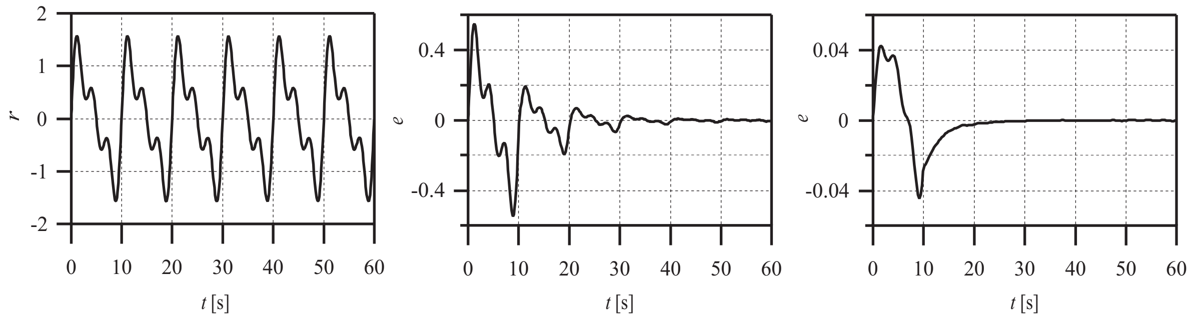
$$\left[\begin{array}{c} \Delta\dot{x}(k,\tau) \\ \Delta\dot{\xi}(k,\tau) \\ \hdashline e(k,\tau) \end{array}\right] = A_{cl} \left[\begin{array}{c} \Delta x(k,\tau) \\ \Delta\xi(k,\tau) \\ \hdashline e(k-1,\tau) \end{array}\right], \tag{26}$$

$$\begin{aligned} A_{cl} &= \left[\begin{array}{cc;{2pt/2pt}c} A_1 & \dfrac{BK_eC}{E} & \dfrac{BK_e}{E} \\ A_2 & A_3 & \dfrac{B_c}{E} \\ \hdashline A_4 & -\dfrac{DK_eC}{E} & \dfrac{1}{E} \end{array}\right], \\ A_1 &= A + \frac{B(K_p - K_eC)}{E}, \\ A_2 &= -B_c\left[C + \frac{D(K_p - K_eC)}{E}\right], \\ A_3 &= A_c - \frac{B_cDK_eC}{E}, \\ A_4 &= -\frac{C + D(K_p - K_eC)}{E}, \\ E &= 1 + DK_e. \end{aligned}$$

The eigenvalues of the dynamic compensator are determined by the control part, A_3. However, (22) shows that B_c contains Q_e. So in this case, Q_e adjusts the learning action through the state $\xi(t)$.

Remark 3.3. *Unlike other design methods, it is easy to obtain the parameters of the dynamic compensator and the feedback controller (3) by selecting suitable values for R, Q_x and Q_e. Furthermore, control and learning can be preferentially adjusted by tuning the weights Q_x and Q_e, respectively, in the performance index (14).*

4. **Numerical Example.** This section presents a numerical example that demonstrates the validity of the design method.



(a) Reference input (b) Tracking error using the design parameters in (28) (c) Tracking error using the design parameters in (29)

FIGURE 3. Simulation results for different weighting matrices

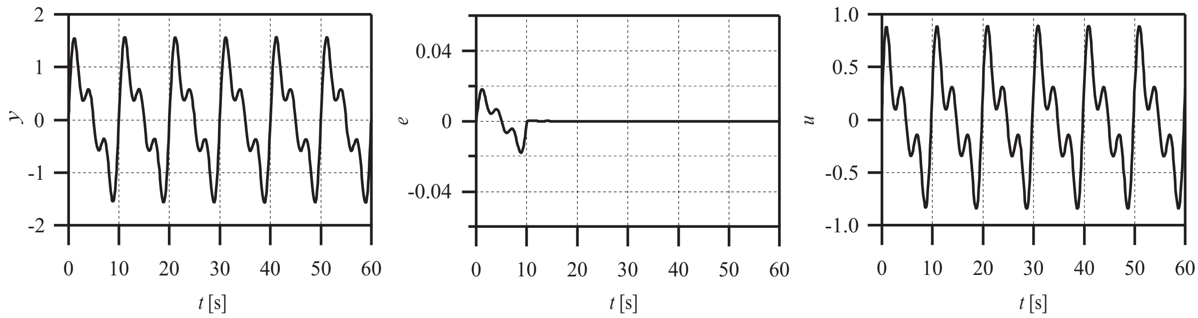


FIGURE 4. Simulation results for design parameters in (30)

Consider the plant (6) with

$$A = \begin{bmatrix} -2 & 3 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = [1\ 0], \quad D = 1.$$

Let the periodic reference input be

$$r(t) = \sin\frac{2\pi}{10}t + 0.5\sin\frac{4\pi}{10}t + 0.5\sin\frac{6\pi}{10}t, \tag{27}$$

and let R in (14) be

$$R = 1.$$

If we choose

$$Q_x = 100I_2, \quad Q_e = 1 \tag{28}$$

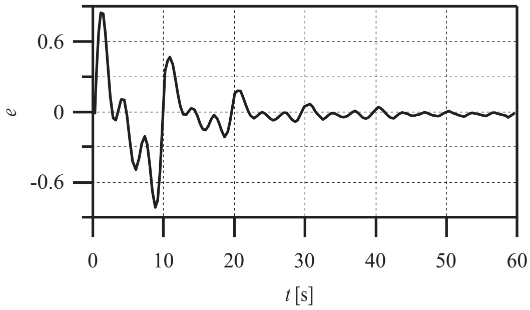
to emphasize control rather than learning in the repetitive-control law, it takes six periods to complete the learning process (See simulation results in Figure 3(b)). However, if we select

$$Q_x = I_2, \quad Q_e = 100 \tag{29}$$

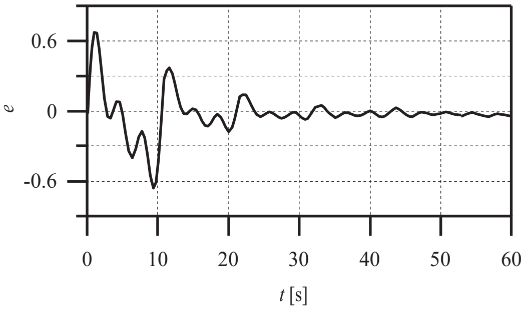
to emphasize learning, it takes just three periods (Figure 3(c)).

We found that

$$Q_x = 10I_2, \quad Q_e = 100. \tag{30}$$



(a) Tracking error using the design parameters in (34)



(b) Tracking error using the design parameters in (35)

FIGURE 5. Results of adjusting Q_x in conventional repetitive-control system

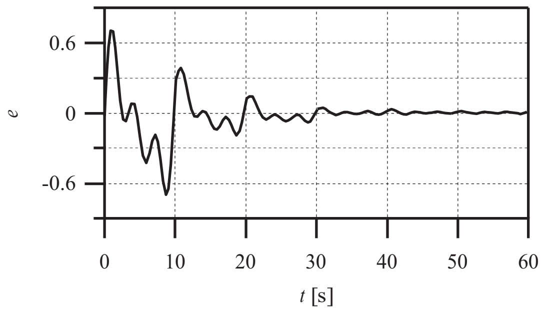


FIGURE 6. Simulation results for conventional repetitive-control system designed using (32), (33) and (35)

provided the best control and learning performance. The parameters of the resulting controller are

$$\begin{cases} A_c = \begin{bmatrix} -3.5753 & -3.1505 \\ 1.4085 & -8.1831 \end{bmatrix}, \; B_c = \begin{bmatrix} -57.5254 \\ -159.1527 \end{bmatrix}, \\ K_p = [-1.5753 \;\; -1.5915], \\ K_c = [-1 \;\; -2], \; K_e = 100. \end{cases} \tag{31}$$

In this case, the system enters the steady state in the third period (Figure 4); and even during the transient response, the largest tracking error is less than 2%.

Note that conventional methods of designing a repetitive-control system cannot separately adjust control and learning. As mentioned in Remark 3.2, they usually set Q_e to

$$Q_e = 0. \tag{32}$$

Here, without loss of generality, we let

$$R = 1 \tag{33}$$

to obtain a conventional repetitive-control system so that we can compare the results with those for our 2D system. Letting

$$Q_x = 10I_2 \tag{34}$$

yields the simulation results in Figure 5(a), and letting

$$Q_x = 200I_2, \tag{35}$$

yields the results in Figure 5(b). We found that $Q_x = 100I_2$ produced the best results (Figure 6). Comparing them with Figure 4, we see that our 2D method yields better transient and steady-state tracking performance than the conventional method does.

Since other methods set $Q_e = 0$ in the performance index (14), they cannot adjust the learning directly. In contrast, our method assesses both $e(t)$ and $\Delta x(t)$ in the performance index, thereby enabling the control and learning actions to each be tuned.

5. **Conclusion.** This paper describes a design method for optimal repetitive control. Unlike other methods, it employs a 2D model to separate control from learning. The problem of designing a repetitive controller is first formulated as an optimization problem for the 2D system. Then, an optimal repetitive controller is obtained by employing one-dimensional optimal control theory. This method features the preferential tuning of control and learning. Simulation results demonstrate the validity of the method.

Acknowledgment. This work was supported in part by the National Science Foundation of China under Grants 60974045, 60674016 and 60425310.

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