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# FEEDFORWARD NEURAL NETWORK AND FEEDBACK LINEARIZATION CONTROL DESIGN OF BILSAT-1 SATELLITE SYSTEM

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Abstract. *The paper presents a novel feedforward neural network and feedback linearization control of BILSAT-1 satellite system for the almost disturbance decoupling performance. The proposed controller guarantees exponentially global uniform ultimate bounded stability and the almost disturbance decoupling performance without using any learning or adaptive algorithms. The proposed approach provides the architecture of the neural network and the weights among the layers in order to guarantee stability of the system. Moreover, the new approach renders the system to be stable with the almost disturbance decoupling property at each step of selecting weights to enhance the performance if the proposed sufficient conditions are maintained. One example, which cannot be solved by the first paper on the almost disturbance decoupling problem, is proposed in this paper to exploit the fact that the tracking and the almost disturbance decoupling performances are easily achieved by the proposed approach.*
**Keywords:** BILSAT-1 satellite system, Feedforward neural network, Almost disturbance decoupling, Multi-input multi-output system, Feedback linearization approach, Composite Lyapunov approach

1. **Introduction.** A considerable amount of research focused on the field of nonlinear control using neural network [8,21,33]. Neural Network has been motivated by their potential of improving system performance through learning using parallel approach. Some of these learning algorithms [2,30,39] apply the error back propagation approach to minimizing an objective function described as the sum of the square errors. However, the learning algorithms investigated so far are not capable of solving the inherent drawbacks of neural network. These drawbacks include how to determine the range within which the weights of the neural network should be adjusted to guarantee the stability of the controlled system and the stability types of the overall system, i.e., if the overall system is asymptotically stable or exponentially stable, or if it is locally stable or globally stable.

In the famous NN-based adaptive algorithms, since the desired or ideal weights are unknown, they are adapted real-time utilizing appropriately constructed adaptation rules. Gradient approaches are widely adopted [8,33]. However, in [8], there is no guarantee of error and weight convergence. [33] guarantees the weight convergence by a dead-zone scheme. However, setting of an optimal dead-zone parameter to fit unexpected situations is difficult and smaller parameter may result in divergence [21]. Our proposed new approach renders the system to be stable with the almost disturbance decoupling property at each step of selecting weights to enhance the performance via Matlab software if the proposed sufficient conditions are maintained.

Many approaches to stabilizing and tracking tasks have been proposed including feedback linearization, variable structure control (sliding mode control), backstepping, regulation control, non-linear $H^\infty$ control, internal model principle and $H^\infty$ adaptive fuzzy control. [19] has proposed the use of variable structure control to deal with non-linear system. However, chattering behaviour that is caused by discontinuous switching and imperfect implementation that can drive the system into unstable regions is inevitable for variable structure control schemes. Backstepping has proven to be a powerful tool for synthesizing controllers for non-linear systems. However, a disadvantage of this approach is an explosion in the complexity which is a result of repeated differentiations of non-linear functions [35,41]. An alternative approach is to utilize output regulation control [14] in which the outputs are assumed to be excited by an exosystem. However, the non-linear regulation approach requires the solution of difficult partial-differential algebraic equations. Another difficulty is that the exosystem states need to be switched to describe changes in the output and this creates transient tracking errors [28]. In general, non-linear $H^\infty$ control requires the solution of the Hamilton-Jacobi equation, which is a difficult non-linear partial-differential equation [4,15,37]. Only for some particular non-linear systems it is possible to derive a closed-form solution [13]. The control approach that is based on the internal model principle converts the tracking problem into a non-linear output regulation problem. This approach depends on solving a first-order partial-differential equation of the center manifold [14]. For some special non-linear systems and desired trajectories, the asymptotic solutions of this equation have been developed using ordinary differential equations [9,11]. Recently, $H^\infty$ adaptive fuzzy control has been proposed to systematically deal with non-linear systems [6]. The drawback with $H^\infty$ adaptive fuzzy control is that the complex parameter update law makes this approach impractical in real-world situations. During the past decade significant progress has been made in researching control approaches for non-linear systems based on the feedback linearization theory [12,19,27,32]. Moreover, feedback linearization approach has been applied successfully to many real control systems. These include the control of an electromagnetic suspension system [16], pendulum system [7], spacecraft [31], electrohydraulic servosystem [1], car-pole system [5] and bank-to-turn missile system [22]. The main contribution of this study is to solve the linearized and PDC shortcomings by using non-linear feedback linearization approach.

The almost disturbance decoupling problem, i.e., that is the design of a controller that attenuates the effect of the disturbance on the output terminal to an arbitrary degree of accuracy, was originally developed for linear and non-linear control systems by [25,40] respectively. The problem has attracted considerable attention and many significant results have been developed for both linear and non-linear control systems [26,29,38]. The almost disturbance decoupling problem of non-linear single-input single-output (SISO) systems was investigated in [25] by using a state feedback approach and solved in terms of sufficient conditions for systems with non-linearities that are not globally Lipschitz and disturbances being linear but possibly actually being multiples of non-linearities. The resulting state feedback control is constructed following a singular perturbation approach. The sufficient conditions in [25] require that the non-linearities multiplying the disturbances satisfy structural triangular conditions. [25] shows that for non-linear SISO systems the almost disturbance decoupling problem may not be solvable, as is case for

$$\dot{x}_1(t) = \tan^{-1} x_2 + \theta(t), \quad |\theta(t)| > \frac{\pi}{2}, \quad \dot{x}_2(t) = u, \quad y = x_1$$

where $u$, $y$ denoted the input and output respectively and $\theta(t)$ was the disturbance of the system. On the contrary, this example can be easily solved via the proposed approach in this paper. On the contrary, this example can be easily solved via the proposed approach in this paper. It is worth noting that the sufficient conditions given in [25] are not necessary in this study where a nonlinear state feedback control is explicitly designed which solves the almost disturbance decoupling problem. The almost disturbance decoupling problem is solvable for the system by a nonlinear state feedback control, according to our proposed approach, while the sufficient conditions given in [25] fail when applied to the system. The design techniques in this study are also entirely different from those in [25] since the singular perturbation tools are not used.

The goal of this study is to propose a theoretical structure for MIMO nonlinear systems based on feedback linearization and neural network approaches. This study presents sufficient conditions for achieving the exponentially global uniform ultimate bounded stability and the almost disturbance decoupling performances without using any learning or adaptive algorithms. The efficiency of the proposed controller is illustrated by application to the BILSAT-1 satellite system. In this study, we have solved the satellite attitude-tracking problem using the MRP (Modified Rodriguez Parameters) [23] attitude kinematics with the following properties: (1) Since the decoupling matrix of the feedback-linearizing controller depends explicitly on the attitude kinematics, the use of MRPenables one to have an always invertible decoupling matrix, thus an always existing control law; (2) The simulations showed that the parameter $\varepsilon$ has an important influence on the disturbance attenuating characteristics of the controller design by increasing the attenuation parameter $NN_2$; (3) For this purpose, a recently developed feedback linearization technique, which does not require a selection of a complex Lyapunov function and is independent of the initial conditions, is utilized on the satellite dynamical model; (4) From a practical point of view, a matrix inversion algorithm will be necessary but by most of the modern digital signal processors that is quite simple; (5) Also, the reaction wheel used in the model and simulation is a realistic mechanism which is designed by Surrey Satellite Technology Limited (SSTL) and it has a torque limit of 0.02 $N.m$ where the simulation results are far below this value.

2. **Structure of Multilayered Feedforward Neural Network.** The structure of the multilayered feedforward neural network which will be involved later in the controller design is investigated in this section. The neural network used in controller design is three-layered as shown in Figure 1. It consists of an input node vector $X \in \Re^p$, a hidden layer of $p$ activity functions and an output node vector $Y \in \Re^m$. The input-output

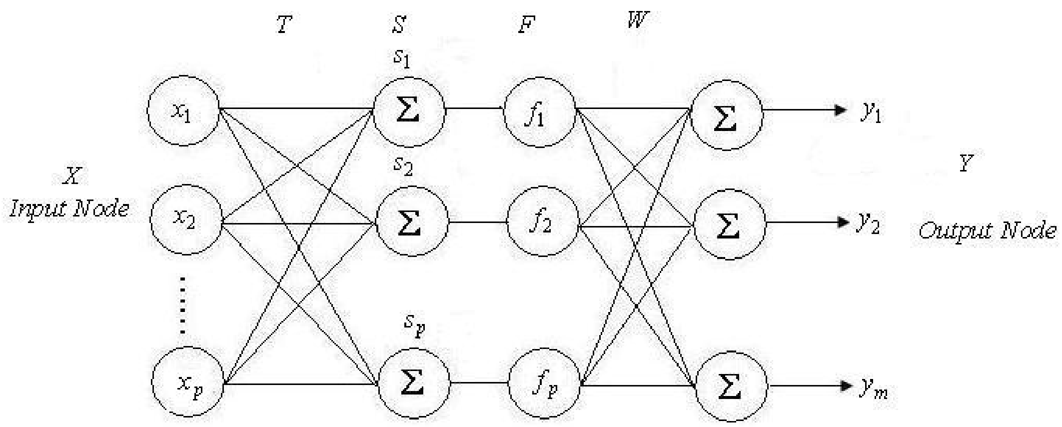


FIGURE 1. Three-layered neural network

relationship can be described as

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_p \end{bmatrix}^T = \begin{bmatrix} \sum_{i=1}^{p} t_{1i}x_i & \sum_{i=1}^{p} t_{2i}x_i & \cdots & \sum_{i=1}^{p} t_{pi}x_i \end{bmatrix}^T \tag{2.1}$$

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T = \begin{bmatrix} \sum_{i=1}^{p} \omega_{1i}f_i(s_i) & \sum_{i=1}^{p} \omega_{2i}f_i(s_i) & \cdots & \sum_{i=1}^{p} \omega_{mi}f_i(s_i) \end{bmatrix}^T \tag{2.2}$$

where $t_{mn}$ is the weight to the hidden node $s_m$ from other input node $x_n$ and $\omega_{mn}$ is the weight to the output node $y_m$ from the hidden node $s_n$. The above input-output relationship can be written in matrix form as

$$S_{p\times 1} = T_{p\times p}X_{p\times 1} \tag{2.3}$$

$$Y_{m\times 1} = W_{m\times p}F_{p\times 1} \tag{2.4}$$

where

$$S \equiv \begin{bmatrix} s_1 & s_2 & \cdots & s_p \end{bmatrix}^T \tag{2.5}$$

$$X \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix}^T \tag{2.6}$$

$$T \equiv \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{bmatrix} \tag{2.7}$$

$$Y \equiv \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T \tag{2.8}$$

$$F(S) \equiv \begin{bmatrix} f_1(s_1) & f_2(s_2) & \cdots & f_p(s_p) \end{bmatrix}^T \tag{2.9}$$

$$W \equiv \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1p} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2p} \\ \vdots & \vdots & & \vdots \\ \omega_{m1} & \omega_{m2} & \cdots & \omega_{mp} \end{bmatrix} \tag{2.10}$$

3. **Feedback Linearization and Neural Network Controller Design.** The following non-linear control system with disturbances is considered:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix} \\ &+ \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n) & g_2(x_1, x_2, \ldots, x_n) & \cdots & g_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \begin{bmatrix} u_1(x_1, x_2, \ldots, x_n) \\ u_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ u_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \\ &+ \sum_{j=1}^{p} q_j^* \theta_{jd} \end{aligned} \tag{3.1a}$$

$$\begin{bmatrix} y_1(x_1, x_2, \ldots, x_n) \\ y_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ y_m(x_1, x_2, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, \ldots, x_n) \\ h_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ h_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \tag{3.1b}$$

that is

$$\dot{X}(t) = f(X(t)) + g(X(t))u + \sum_{j=1}^{p} q_j^* \theta_{jd}$$
$$y(t) = h(X(t))$$

where $X(t) \equiv [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \Re^n$ is the state vector, $u \equiv [u_1\ u_2\ \cdots\ u_m]^T \in \Re^m$ is the input vector, $y \equiv [y_1\ y_2\ \cdots\ y_m]^T \in \Re^m$ is the output vector, $\theta_d \equiv [\theta_{1d}(t)\ \theta_{2d}(t)\ \cdots\ \theta_{pd}(t)]^T$ is a bounded time-varying disturbances vector, $f \equiv [f_1\ f_2\ \cdots\ f_n]^T \in \Re^n$, $g \equiv [g_1\ g_2\ \cdots\ g_m] \in \Re^{n\times m}$ and $h \equiv [h_1\ h_2\ \cdots\ h_m]^T \in \Re^m$ are smooth vector fields. The nominal system is then defined as follows:

$$\dot{X}(t) = f(X(t)) + g(X(t))u \tag{3.2a}$$
$$y(t) = h(X(t)) \tag{3.2b}$$

The nominal system of the form (3.2) is assumed to have the vector relative degree $\{r_1, r_2, \cdots, r_m\}$ [12], i.e., the following conditions are satisfied for all $X \in \Re^n$:
⟨1⟩

$$L_{g_j} L_f^k h_i(X) = 0 \tag{3.3}$$

for all $1 \le i \le m$, $1 \le j \le m$, $k < r_i - 1$, where the operator $L$ is the Lie derivative [12] and $r_1 + r_2 + \cdots + r_m = r$.
⟨2⟩ The $m \times m$ matrix

$$A \equiv \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(X) & \cdots & L_{g_m} L_f^{r_1-1} h_1(X) \\ L_{g_1} L_f^{r_2-1} h_2(X) & \cdots & L_{g_m} L_f^{r_2-1} h_2(X) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(X) & \cdots & L_{g_m} L_f^{r_m-1} h_m(X) \end{bmatrix} \tag{3.4}$$

non-singular.

The desired output trajectory $y_d^i(t)$, $1 \le i \le m$ and its first $r_i$ derivatives are all uniformly bounded and

$$\left\| \left[ y_d^i, \quad y_d^{i^{(1)}}, \quad \cdots, \quad y_d^{i^{(r_i)}} \right] \right\| \le B_d^i, \quad 1 \le i \le m \tag{3.5}$$

where $B_d^i$ is some positive constant. Under the assumption of well-defined vector relative degree, it has been shown [12] that the mapping

$$\phi:\ \Re^n \to \Re^n \tag{3.6}$$

defined as

$$\xi_i \equiv \begin{bmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} \equiv \begin{bmatrix} \phi_1^i \\ \phi_2^i \\ \vdots \\ \phi_{r_i}^i \end{bmatrix} \equiv \begin{bmatrix} L_f^0 h_i(X) \\ L_f^1 h_i(X) \\ \vdots \\ L_f^{r_i-1} h_i(X) \end{bmatrix}, \quad i = 1, 2, \ldots, m \tag{3.7}$$

$$\phi_k(X(t)) \equiv \eta_k(t), \quad k = r+1, r+2, \ldots, n \tag{3.8}$$

and satisfying

$$L_{g_j} \phi_k(X(t)) = 0,\ k = r+1, r+2, \ldots, n,\ 1 \le j \le m \tag{3.9}$$

is a diffeomorphism onto image, if the following hold
⟨1⟩ The distribution

$$G \equiv span\{g_1, g_2, \ldots, g_m\} \tag{3.10}$$

is involutive.

⟨2⟩ The vector fields

$$Y_j^k, \quad 1 \le j \le m, \quad 1 \le k \le r_j \tag{3.11}$$

are complete, where

$$Y_j^k \equiv (-1)^{k-1} ad_{\tilde{f}}^{k-1} \tilde{g}_j, \quad 1 \le j \le m, \quad 1 \le k \le r_j \tag{3.12}$$

$$\tilde{f}(X) \equiv f(X) - g(X)A^{-1}(X)b(X) \tag{3.13}$$

$$b(X) \equiv \begin{bmatrix} L_f^{r_1} h_1(X) \\ L_f^{r_2} h_2(X) \\ \vdots \\ L_f^{r_m} h_m(X) \end{bmatrix} \tag{3.14}$$

$$\tilde{g} \equiv \begin{bmatrix} \tilde{g}_1 & \tilde{g}_2 & \cdots & \tilde{g}_m \end{bmatrix} \equiv g(X)A^{-1}(X) \tag{3.15}$$

$$ad_f^k g \equiv \begin{bmatrix} f & ad_f^{k-1} g \end{bmatrix} \tag{3.16}$$

$$\begin{bmatrix} f & g \end{bmatrix} \equiv \frac{\partial g}{\partial X} f(X) - \frac{\partial f}{\partial X} g(X) \tag{3.17}$$

For the sake of convenience, define the trajectory error to be

$$e_j^i \equiv \xi_j^i - y_d^{i(j-1)}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, r_i \tag{3.18}$$

$$e^i \equiv \begin{bmatrix} e_1^i & e_2^i & \cdots & e_{r_i}^i \end{bmatrix}^T \in \Re^{r_i} \tag{3.19}$$

and the trajectory error to be multiplied with some adjustable positive constant $\varepsilon$

$$\overline{e_j^i} \equiv \varepsilon^{j-1} e_j^i, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, r_i \tag{3.20}$$

$$\overline{e^i} \equiv \begin{bmatrix} \overline{e_1^i} & \overline{e_2^i} & \cdots & \overline{e_{r_i}^i} \end{bmatrix}^T \in \Re^{r_i} \tag{3.21}$$

$$\overline{e} \equiv \begin{bmatrix} \overline{e^1} \\ \overline{e^2} \\ \vdots \\ \overline{e^m} \end{bmatrix} \in \Re^r \tag{3.22}$$

and

$$\xi \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{bmatrix} \in \Re^r, \tag{3.23}$$

$$\eta(t) \equiv [\eta_{r+1}(t) \quad \eta_{r+2}(t) \quad \cdots \quad \eta_n(t)]^T \in \Re^{n-r} \tag{3.24}$$

$$\begin{aligned} q(\xi(t), \eta(t)) &\equiv [L_f \phi_{r+1}(t) \quad L_f \phi_{r+2}(t) \quad \cdots \quad L_f \phi_n(t)]^T \\ &\equiv \begin{bmatrix} q_{r+1} & q_{r+2} & \cdots & q_n \end{bmatrix}^T \end{aligned} \tag{3.25}$$

Define a phase-variable canonical matrix $A_c^i$ to be

$$A_c^i \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1^i & -\alpha_2^i & -\alpha_3^i & \cdots & -\alpha_{r_i}^i \end{bmatrix}_{r_i \times r_i}, \quad 1 \le i \le m \tag{3.26}$$

where $\alpha_1^i$, $\alpha_2^i$, $\cdots$, $\alpha_{r_i}^i$ are any chosen parameters such that $A_c^i$ is Hurwitz and the vector $B^i$ to be

$$B^i \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{r_i \times 1}^T, \quad 1 \le i \le m \tag{3.27}$$

Let $P^i$ be the positive definite solution of the following Lyapunov equation

$$(A_c^i)^T P^i + P^i A_c^i = -I, \quad 1 \le i \le m \tag{3.28}$$

$$\lambda_{\max}(P^i) \equiv \text{the maximum eigenvalue of } P^i,\ 1 \le i \le m \tag{3.29}$$

$$\lambda_{\min}(P^i) \equiv \text{the minimum eigenvalue of } P^i,\ 1 \le i \le m \tag{3.30}$$

$$\lambda_{\max}^* \equiv \min\left\{\lambda_{\max}(P^1),\ \lambda_{\max}(P^2), \dots, \lambda_{\max}(P^m)\right\} \tag{3.31}$$

$$\lambda_{\min}^* \equiv \min\left\{\lambda_{\min}(P^1),\ \lambda_{\min}(P^2), \dots, \lambda_{\min}(P^m)\right\} \tag{3.32}$$

**Assumption 1.** For all $t \ge 0$, $\eta \in \Re^{n-r}$ and $\xi \in \Re^r$, there exists a positive constant $M$ such that the following inequality holds:

$$\|q_{22}(t,\eta,\bar{e}) - q_{22}(t,\eta,0)\| \le M(\|\bar{e}\|) \tag{3.33}$$

where $q_{22}(t,\eta,\bar{e}) \equiv q(\xi,\eta)$.

For the sake of stating precisely the investigated problem, define

$$d_{ij} \equiv L_{g_j} L_f^{r_i - 1} h_i(X), \quad 1 \le i \le m, \quad 1 \le j \le m \tag{3.34}$$

$$c_i \equiv L_f^{r_i} h_i(X), \quad 1 \le i \le m \tag{3.35}$$

and

$$\overline{\overline{e^i}} \equiv \alpha_1^i \overline{e_1^i} + \alpha_2^i \overline{e_2^i} + \cdots + \alpha_{r_i}^i \overline{e_{r_i}^i}, \quad 1 \le i \le m \tag{3.36}$$

**Definition 3.1.** [19] *Consider the system $\dot{x} = f(t,x,\theta)$, where $f : [0,\infty) \times \Re^n \times \Re^n \to \Re^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$ and $\theta$. This system is said to be input-to-state stable if there exists a class $KL$ function $\beta$, a class $K$ function $\gamma$ and positive constants $k_1$ and $k_2$ such that for any initial state $x(t_0)$ with $\|x(t_0)\| < k_1$ and any bounded input $\theta(t)$ with $\sup_{t \ge t_0} \|\theta(t)\| < k_2$, the state exists and satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} \|\theta(\tau)\|\right) \tag{3.37}$$

*for all $t \ge t_0 \ge 0$.*

Now we formulate the tracking problem with almost disturbance decoupling as follows:

**Definition 3.2.** [26] *The tracking problem with almost disturbance decoupling is said to be globally solvable by the state feedback controller $u$ for the transformed-error system by a global diffeomorphism (3.6), if the controller $u$ enjoys the following properties.*
*⟨1⟩ It is input-to-state stable with respect to disturbance inputs.*
*⟨2⟩ For any initial value $\bar{x}_{e0} \equiv [\bar{e}(t_0) \quad \eta(t_0)]^T$, for any $t \ge t_0$ and for any $t_0 \ge 0$*

$$|y(t) - y_d(t)| \le \beta_{11}(\|x(t_0)\|, t - t_0) + \frac{1}{\sqrt{\beta_{22}}}\beta_{33}\left(\sup_{t_0 \le \tau \le t} \|\theta(\tau)\|\right) \tag{3.38a}$$

*and*

$$\int_{t_0}^{t} [y(\tau) - y_d(\tau)]^2 \, d\tau \le \frac{1}{\beta_{44}}\left[\beta_{55}(\|\bar{x}_{e0}\|) + \int_{t_0}^{t} \beta_{33}\left(\|\theta(\tau)\|^2\right) d\tau\right] \tag{3.38b}$$

*where $\beta_{22}$, $\beta_{44}$ are some positive constants, $\beta_{33}$, $\beta_{55}$ are class $K$ functions and $\beta_{11}$ is a class $KL$ function.*

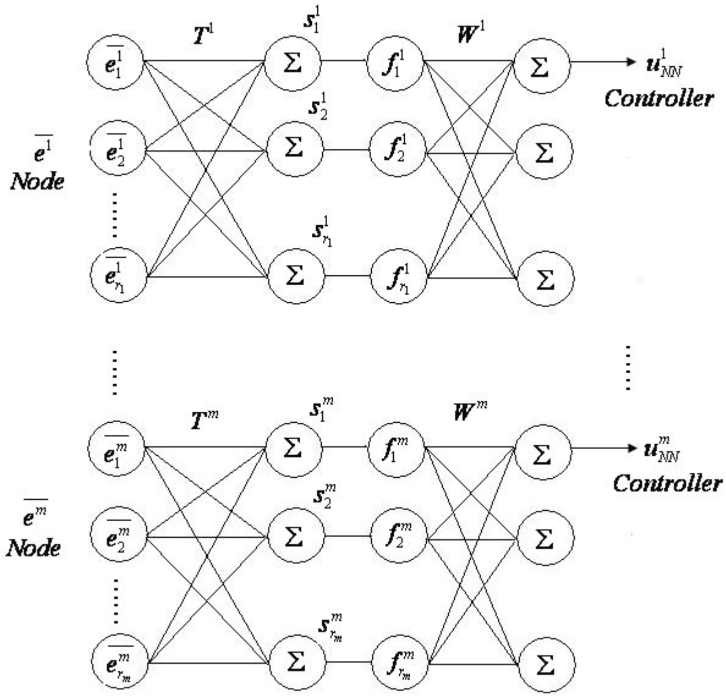


Figure 2. The interconnect structure of the neural network

We immediately presents sufficient conditions for achieving the exponentially global uniform ultimate bounded stability and the almost disturbance decoupling performance. The interconnect structure of the neural network is shown in Figure 2.

The transformation matrices discussed earlier in Section 2 are chosen as

$$S^i = \begin{bmatrix} s_1^i & s_2^i & \cdots & s_{r_i}^i \end{bmatrix}^T,\ 1 \le i \le m \tag{3.39a}$$

$$X^i = \begin{bmatrix} \overline{e_1^i} & \overline{e_2^i} & \cdots & \overline{e_{r_i}^i} \end{bmatrix}^T,\ 1 \le i \le m \tag{3.39b}$$

$$T^i = \begin{bmatrix} t_{11}^i & 0 & \cdots & 0 \\ 0 & t_{11}^i & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & t_{11}^i \end{bmatrix},\ 1 \le i \le m \tag{3.39c}$$

$$Y^i = y_1^i = u_{NN}^i,\ 1 \le i \le m \tag{3.39d}$$

$$F^i = \begin{bmatrix} f_1^i(s_1^i) & f_2^i(s_2^i) & \cdots & f_{r_i}^i(s_{r_i}^i) \end{bmatrix}^T = \begin{bmatrix} s_1^i & s_2^i & \cdots & s_{r_i}^i \end{bmatrix}^T,\ 1 \le i \le m \tag{3.39e}$$

$$W^i = \begin{bmatrix} \omega_{11}^i & \omega_{12}^i & \cdots & \omega_{1r_i}^i \end{bmatrix},\ 1 \le i \le m \tag{3.39f}$$

Thus, the input-output relationship for the neural network controller is obtained as

$$u_{NN} = \begin{bmatrix} u_{NN}^1 & u_{NN}^2 & \cdots & u_{NN}^m \end{bmatrix}^T \tag{3.39g}$$

$$u_{NN}^i = \omega_{11}^i t_{11}^i \overline{e_1^i} + \omega_{12}^i t_{22}^i \overline{e_2^i} + \cdots + \omega_{1r_i}^i t_{r_i r_i}^i \overline{e_{r_i}^i}, \quad 1 \le i \le m \tag{3.39h}$$

**Theorem 3.1.** *Suppose that there exists a continuously differentiable function* $V : \Re^{n-r} \to \Re^+$ *such that the following three inequalities hold for all* $\eta \in \Re^{n-r}$*:*

*(a)* $\omega_1 \|\eta\|^2 \le V(\eta) \le \omega_2 \|\eta\|^2, \quad \omega_1,\ \omega_2 > 0$ (3.40a)

*(b)* $\nabla_t V + (\nabla_\eta V)^T q_{22}(t, \eta, 0) \le -2\alpha_x \|\eta\|^2, \quad \alpha_x > 0$ (3.40b)

*(c)* $\|\nabla_\eta V\| \le \omega_3 \|\eta\|, \quad \omega_3 > 0,$ (3.40c)

*then the tracking problem with almost disturbance decoupling performance is globally solvable by the controller defined by*

$$u = A^{-1}\{-b + v + u_{NN}\} \tag{3.41}$$

$$b \equiv \begin{bmatrix} L_f^{r_1} h_1 & L_f^{r_2} h_2 & \cdots & L_f^{r_m} h_m \end{bmatrix}^T \tag{3.42}$$

$$v \equiv \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^T \tag{3.43}$$

$$\begin{aligned} v_i \equiv y_d^{i(r_i)} &- \varepsilon^{-r_i}\alpha_1^i \left[L_f^0 h_i(X) - y_d^i\right] - \varepsilon^{1-r_i}\alpha_2^i \left[L_f^1 h_i(X) - y_d^{i(1)}\right] \\ &- \cdots - \varepsilon^{-1}\alpha_{r_i}^i \left[L_f^{r_i-1} h_i(X) - y_d^{i(r_i-1)}\right], \quad 1 \le i \le m \end{aligned} \tag{3.44}$$

*Moreover, the influence of disturbances on the $L_2$ norm of the tracking error can be arbitrarily attenuated by increasing the following adjustable parameter $N_2 > 1$:*

$$\omega_{\max}^i = \max(|\omega_{11}^i t_{11}^i|, |\omega_{12}^i t_{22}^i|, \cdots, |\omega_{1r_i}^i t_{r_i r_i}^i|), \quad 1 \le i \le m \tag{3.45a}$$

$$\begin{aligned} k_{11} \equiv \frac{k}{2\varepsilon} &- \varepsilon^{r_1-1}\omega_{\max}^1 r_1 k \left\|B_{r_1}^T P^1\right\| - \cdots - \varepsilon^{r_m-1}\omega_{\max}^m r_m k \left\|B_{r_m}^T P^m\right\| \\ &- \frac{3}{2}\frac{k^2 \left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2}{\varepsilon^2} - \cdots - \frac{3}{2}\frac{k^2 \left\|\phi_\xi^m\right\|^2 \left\|P^m\right\|^2}{\varepsilon^2} - 4 \end{aligned} \tag{3.45b}$$

$$k_{22} \equiv 2\alpha_x - \frac{\omega_3^2 M^2}{16} - \omega_3^2 \left\|\phi_\eta\right\|^2 \tag{3.45c}$$

$$N_2 \equiv \min\{k_{11},\ k_{22}\} \tag{3.45d}$$

$$N_1 \equiv \frac{m+1}{6}\left(\sup_{t_0 \le \tau \le t} \left\|\theta_d(\tau)\right\|\right)^2 \tag{3.45e}$$

$$\phi_\xi^i(\varepsilon) \equiv \begin{bmatrix} \varepsilon \frac{\partial}{\partial X} h_i q_1^* & \cdots & \varepsilon \frac{\partial}{\partial X} h_i q_p^* \\ \vdots & & \vdots \\ \varepsilon^{r_i} \frac{\partial}{\partial X} L_f^{r_i-1} h_i q_1^* & \cdots & \varepsilon^{r_i} \frac{\partial}{\partial X} L_f^{r_i-1} h_i q_q^* \end{bmatrix}, \quad 1 \le i \le m \tag{3.45f}$$

$$\phi_\eta(\varepsilon) \equiv \begin{bmatrix} \frac{\partial}{\partial X}\phi_{r+1} q_1^* & \cdots & \frac{\partial}{\partial X}\phi_{r+1} q_p^* \\ \vdots & & \vdots \\ \frac{\partial}{\partial X}\phi_n q_1^* & \cdots & \frac{\partial}{\partial X}\phi_n q_q^* \end{bmatrix} \tag{3.45g}$$

*Moreover, the output tracking error of system (3.1) is exponentially attracted into a sphere $B_{\underline{r}}$, $\underline{r} = \sqrt{\frac{N_1}{N_2}}$, with an exponential rate of convergence*

$$\frac{1}{2}\left(\frac{N_2}{\Delta_{\max}} - \frac{N_1}{\Delta_{\max}\underline{r}^2}\right) = \frac{1}{2}\alpha^* \tag{3.45h}$$

*where*

$$\Delta_{\max} = \max\left\{\omega_2, \frac{k}{2}\lambda_{\max}^*\right\} \tag{3.45i}$$

*where $k(\varepsilon) : \Re^+ \to \Re^+$ is any continuous function satisfying*

$$\lim_{\varepsilon \to 0} k(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{\varepsilon}{k(\varepsilon)} = 0 \tag{3.45j}$$

**Proof:** Applying the coordinate transformation (3.6) yields

$$\begin{aligned}\dot{\xi}_1^1 &= \frac{\partial \phi_1^1}{\partial X}\frac{dX}{dt} = \frac{\partial h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^* \theta_{jd}\right]\\ &= \frac{\partial h_1}{\partial X} f + \frac{\partial h_1}{\partial X} g_1 u_1 + \cdots + \frac{\partial h_1}{\partial X} g_m u_m + \sum_{j=1}^{p} \frac{\partial h_1}{\partial X} q_j^* \theta_{jd}\\ &= \frac{\partial h_1}{\partial X} f + \sum_{j=1}^{p} \frac{\partial h_1}{\partial X} q_j^* (\theta_{jd})\\ &= \xi_2^1 + \sum_{j=1}^{p} \frac{\partial h_1}{\partial X} q_j^* (\theta_{jd})\end{aligned} \tag{3.46}$$

$\vdots$

$$\begin{aligned}\dot{\xi}_{r_1-1}^1 &= \frac{\partial \phi_{r_1-1}^1}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_1-2} h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^* \theta_{jd}\right]\\ &= \frac{\partial L_f^{r_1-2} h_1}{\partial X} f + \frac{\partial L_f^{r_1-2} h_1}{\partial X} g_1\ u_1 + \cdots + \frac{\partial L_f^{r_1-2} h_1}{\partial X} g_m\ u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-2} h_1}{\partial X} q_j^* \theta_{jd}\\ &= \frac{\partial L_f^{r_1-2} h_1}{\partial X} f + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-2} h_1}{\partial X} q_j^* \left(\theta_{jd}\right)\\ &= L_f^{r_1-1} h_1 + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-2} h_1}{\partial X} q_j^* \left(\theta_{jd}\right)\end{aligned} \tag{3.47}$$

$$\begin{aligned}\dot{\xi}_{r_1}^1 &= \frac{\partial \phi_{r_1}^1}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_1-1} h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^* \theta_{jd}\right]\\ &= \frac{\partial L_f^{r_1-1} h_1}{\partial X} f + \frac{\partial L_f^{r_1-1} h_1}{\partial X} g_1 u_1 + \cdots + \frac{\partial L_f^{r_1-1} h_1}{\partial X} g_m u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-1} h_1}{\partial X} q_j^* \theta_{jd}\\ &= L_f^{r_1} h_1 + L_{g_1} L_f^{r_1-1} h_1\ u_1 + \cdots + L_{g_m} L_f^{r_1-1} h_1\ u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-1} h_1}{\partial X} q_j^* \left(\theta_{jd}\right)\\ &= c_1 + d_{11} u_1 + \cdots + d_{1m} u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_1-1} h_1}{\partial X} q_j^* \left(\theta_{jd}\right)\end{aligned} \tag{3.48}$$

$\vdots$

$$\begin{aligned}\dot{\xi}_1^m &= \frac{\partial \phi_1^m}{\partial X}\frac{dX}{dt} = \frac{\partial h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^* \theta_{jd}\right]\\ &= \frac{\partial h_m}{\partial X} f + \frac{\partial h_m}{\partial X} g_1 u_1 + \cdots + \frac{\partial h_m}{\partial X} g_m u_m + \sum_{j=1}^{p} \frac{\partial h_m}{\partial X} q_j^* \theta_{jd}\\ &= L_f^1 h_m + \sum_{j=1}^{p} \frac{\partial h_1}{\partial X} q_j^* (\theta_{jd}) = \xi_2^m + \sum_{j=1}^{p} \frac{\partial h_m}{\partial X} q_j^* (\theta_{jd})\end{aligned} \tag{3.49}$$

$\vdots$

$$\begin{aligned}
\dot{\xi}^m_{r_m-1} &= \frac{\partial \phi^m_{r_m-1}}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_m-2}h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd}\right] \\
&= \frac{\partial L_f^{r_m-2}h_m}{\partial X}f + \frac{\partial L_f^{r_m-2}h_m}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_m-2}h_m}{\partial X}g_mu_m \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\theta_{jd} \\
&= L_f^{r_m-1}h_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\left(\theta_{jd}\right) \\
&= \xi^m_{r_m} + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\left(\theta_{jd}\right)
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
\dot{\xi}^m_{r_m} &= \frac{\partial \phi^m_{r_m}}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_m-1}h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd}\right] \\
&= \frac{\partial L_f^{r_m-1}h_m}{\partial X}f + \frac{\partial L_f^{r_m-1}h_m}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_m-1}h_m}{\partial X}g_mu_m \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-1}h_m}{\partial X}q_j^*\theta_{jd} \\
&= L_f^{r_m}h_m + L_{g_1}L_f^{r_m-1}h_mu_1 + \cdots + L_{g_m}L_f^{r_m-1}h_mu_m \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-1}h_m}{\partial X}q_j^*\left(\theta_{jd}\right) \\
&= c_m + d_{m1}u_1 + \cdots + d_{mm}u_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-1}h_m}{\partial X}q_j^*\left(\theta_{jd}\right)
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
\dot{\eta}_k(t) &= \frac{\partial \phi_k}{\partial X}\frac{dX}{dt} = \frac{\partial \phi_k}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd}\right] \\
&= \frac{\partial \phi_k}{\partial X}f + \frac{\partial \phi_k}{\partial X}g_1u_1 + \cdots + \frac{\partial \phi_k}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial \phi_k}{\partial X}q_j^*\theta_{jd} \\
&= L_f\phi_k + \sum_{j=1}^{p}\frac{\partial \phi_k}{\partial X}q_j^*(\theta_{jd}) = q_k + \sum_{j=1}^{p}\frac{\partial \phi_k}{\partial X}q_j^*(\theta_{jd}), \\
&\quad k = r+1, r+2, \ldots, n
\end{aligned} \tag{3.52}$$

Since

$$c_i(\xi(t), \eta(t)) \equiv L_f^{r_i}h_i(X(t)), \quad 1 \le i \le m \tag{3.53}$$

$$d_{ij} \equiv L_{g_j}L_f^{r_i-1}h_i(X), \quad 1 \le i \le m, \quad 1 \le j \le m \tag{3.54}$$

$$q_k(\xi(t), \eta(t)) = L_f\phi_k(X), \quad k = r+1, r+2, \ldots, n \tag{3.55}$$

the dynamic equations of system (3.1) in the new co-ordinates are as follows:

$$\dot{\xi}_i^1(t) = \xi_{i+1}^1(t) + \sum_{j=1}^{p}\frac{\partial}{\partial X}L_f^{i-1}h_1q_j^*\left(\theta_{jd}\right), \qquad i = 1, 2, \ldots, r_1 - 1 \tag{3.56}$$

$$\begin{aligned}\dot{\xi}_{r_1}(t) =& c_1(\xi(t),\eta(t)) + d_{11}(\xi(t),\eta(t))u_1 + \cdots + d_{1m}(\xi(t),\eta(t))u_m \\ &+ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_1-1} h_1 q_j^*(\theta_{jd})\end{aligned} \tag{3.57}$$

$$\vdots$$

$$\dot{\xi}_i^m(t) = \xi_{i+1}^m(t) + \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{i-1} h_m q_j^*(\theta_{jd}), \quad i = 1, 2, \ldots, r_m - 1 \tag{3.58}$$

$$\begin{aligned}\dot{\xi}_{r_m}^m(t) =& c_m(\xi(t),\eta(t)) + d_{m1}(\xi(t),\eta(t))u_1 + \cdots + d_{mm}(\xi(t),\eta(t))u_m \\ &+ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_m-1} h_m q_j^*(\theta_{jd})\end{aligned} \tag{3.59}$$

$$\dot{\eta}_k(t) = q_k(\xi(t),\eta(t)) + \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_k(X) q_j^*(\theta_{jd}), \quad k = r+1, \ldots, n \tag{3.60}$$

$$y_i(t) = \xi_1^i(t), \quad 1 \le i \le m \tag{3.61}$$

According to Equations (3.18) (3.44) (3.53) and (3.54), the tracking controller can be rewritten as

$$u = A^{-1}[-b + v + u_{NN}] \tag{3.62}$$

Substituting Equation (3.62) into (3.57) and (3.59), the dynamic equations of system (3.1) can be shown as follows:

$$\begin{aligned}\begin{bmatrix} \dot{\xi}_1^i(t) \\ \dot{\xi}_2^i(t) \\ \vdots \\ \dot{\xi}_{r_i-1}^i(t) \\ \dot{\xi}_{r_i}^i(t) \end{bmatrix} =& \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi_1^i(t) \\ \xi_2^i(t) \\ \vdots \\ \xi_{r_i-1}^i(t) \\ \xi_{r_i}^i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_{NN}^i \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v_i + \begin{bmatrix} \sum_{j=1}^{p} \frac{\partial}{\partial X} h_i q_j^*(\theta_{jd}) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^1 h_i q_j^*(\theta_{jd}) \\ \vdots \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_i-1} h_i q_j^*(\theta_{jd}) \end{bmatrix}\end{aligned} \tag{3.63}$$

$$\begin{bmatrix} \dot{\eta}_{r+1}(t) \\ \dot{\eta}_{r+2}(t) \\ \vdots \\ \dot{\eta}_{n-1}(t) \\ \dot{\eta}_n(t) \end{bmatrix} = \begin{bmatrix} q_{r+1}(t) \\ q_{r+2}(t) \\ \vdots \\ q_{n-1}(t) \\ q_n(t) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{r+1} q_j^*(\theta_{jd}) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{r+2} q_j^*(\theta_{jd}) \\ \vdots \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{n-1} q_j^*(\theta_{jd}) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_n q_j^*(\theta_{jd}) \end{bmatrix} \tag{3.64}$$

$$y_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{1\times r_i} \begin{bmatrix} \xi_1^i(t) \\ \xi_2^i(t) \\ \vdots \\ \xi_{r_i-1}^i(t) \\ \xi_{r_i}^i(t) \end{bmatrix}_{r_i\times 1} = \xi_1^i(t), \quad 1 \le i \le m \tag{3.65}$$

Combining Equations (3.18), (3.20), (3.21), (3.26) and (3.44), it can be easily verified that Equations (3.63)-(3.65) can be transformed into the following form.

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + \phi_\eta(\theta_d) \equiv q_{22}(t, \eta(t), \overline{e}) + \phi_\eta(\theta_d) \tag{3.66a}$$

$$\varepsilon\dot{\overline{e^i}}(t) = A_c^i\overline{e^i} + B_{r_i}\varepsilon^{r_i}u_{NN}^i + \phi_\xi^i(\theta_d), \quad 1 \le i \le m \tag{3.66b}$$

$$y_i(t) = \xi_1^i(t), \quad 1 \le i \le m \tag{3.67}$$

We consider $L(\bar{e}, \eta)$ defined by a weighted sum of $V(\eta)$ and $W(\bar{e})$,

$$L(\bar{e}, \eta) \equiv V(\eta) + k(\varepsilon)W(\bar{e}) \equiv V(\eta) + k(\varepsilon)\left(W^1\left(\overline{e^1}\right) + \cdots + W^m\left(\overline{e^m}\right)\right) \tag{3.68}$$

where

$$W(\bar{e}) \equiv W^1\left(\overline{e^1}\right) + \cdots + W^m\left(\overline{e^m}\right) \tag{3.69}$$

as a composite Lyapunov function of the subsystems (3.66a) and (3.66b) [20,24], where $W\left(\overline{e^i}\right)$ satisfies

$$W^i\left(\overline{e^i}\right) \equiv \frac{1}{2}\overline{e^i}^T P^i \overline{e^i} \tag{3.70}$$

In view of Equations (3.18), (3.33) and (3.40), the derivative of $L$ along the trajectories of (3.66a) and (3.66b) is given by

$$\begin{aligned}
\dot{L} &= \left[\nabla_t V + (\nabla_\eta V)^T\dot{\eta}\right] \\
&\quad + \frac{k}{2}\left[\left(\dot{\overline{e^1}}\right)^T P^1\overline{e^1} + \left(\overline{e^1}\right)^T P^1\left(\dot{\overline{e^1}}\right) + \cdots + \left(\dot{\overline{e^m}}\right)^T P^m\overline{e^m} + \left(\overline{e^m}\right)^T P^m\left(\dot{\overline{e^m}}\right)\right] \\
&= \left[\nabla_t V + (\nabla_\eta V)^T\dot{\eta}\right] + \frac{k}{2}\Bigg[\left(\frac{1}{\varepsilon}A_c^1\overline{e^1} + B_{r_1}\varepsilon^{r_1}u_{NN}^1 + \frac{1}{\varepsilon}\phi_\xi^1(\theta_d)\right)^T P^1\overline{e^1} \\
&\quad + \left(\overline{e^1}\right)^T P^1\left(\frac{1}{\varepsilon}A_c^1\overline{e^1} + B_{r_1}\varepsilon^{r_1}u_{NN}^1 + \frac{1}{\varepsilon}\phi_\xi^1(\theta_d)\right) + \cdots \\
&\quad + \left(\frac{1}{\varepsilon}A_c^m\overline{e^m} + B_{r_m}\varepsilon^{r_m}u_{NN}^m + \frac{1}{\varepsilon}\phi_\xi^m(\theta_d)\right)^T P^m\overline{e^m} \\
&\quad + \left(\overline{e^m}\right)^T P^m\left(\frac{1}{\varepsilon}A_c^m\overline{e^m} + B_{r_m}\varepsilon^{r_m}u_{NN}^m + \frac{1}{\varepsilon}\phi_\xi^m(\theta_d)\right)\Bigg] \\
&\le \left[\nabla_t V + (\nabla_\eta V)^T q_{22}(t, \eta(t), \overline{e}) + (\nabla_\eta V)^T\phi_\eta(\theta_d)\right] - \frac{k}{2\varepsilon}\left[\left(\overline{e^1}\right)^T\overline{e^1} + \cdots + \left(\overline{e^m}\right)^T\overline{e^m}\right] \\
&\quad + k\varepsilon^{r_1-1}\left\|B_{r_1}^T P^1\right\|\left\|\overline{e^1}\right\|\left\|u_{NN}^1\right\| + \cdots + k\varepsilon^{r_m-1}\left\|B_{r_m}^T P^m\right\|\left\|\overline{e^m}\right\|\left\|u_{NN}^m\right\| \\
&\quad + \frac{k}{\varepsilon}\left[\left\|(\theta_d)\right\|\left\|\phi_\xi^1\right\|\left\|P^1\right\|\left\|\overline{e^1}\right\| + \cdots + \left\|(\theta_d + \theta_u)\right\|\left\|\phi_\xi^m\right\|\left\|P^m\right\|\left\|\overline{e^m}\right\|\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \left[\nabla_t V + (\nabla_\eta V)^T q_{22}(t,\eta(t),0)\right] + (\nabla_\eta V)^T \left[q_{22}(t,\eta(t),\bar{e}) - q_{22}(t,\eta(t),0)\right] \\
&\quad + \|\nabla_\eta V\| \|\phi_\eta\| \|(\theta_d)\| - \frac{k}{2\varepsilon}\left[\left\|\overline{e^1}\right\|^2 + \left\|\overline{e^2}\right\|^2 + \cdots + \|\overline{e^m}\|^2\right] \\
&\quad + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2 \left\|\overline{e^1}\right\|^2 + k\varepsilon^{r_1-1}\omega_{\max}^1 r_1 \left\|B_{r_1}^T P^1\right\| \left\|\overline{e^1}\right\|^2 + \cdots \\
&\quad + k\varepsilon^{r_m-1}\omega_{\max}^m r_m \left\|B_{r_m}^T P^m\right\| \|\overline{e^m}\|^2 + \frac{1}{6}\|(\theta_d)\|^2 + \cdots \\
&\quad + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2 \|P^m\|^2 \|\overline{e^m}\|^2 + \frac{1}{6}\|(\theta_d)\|^2 \\
&\le \left[\nabla_t V + (\nabla_\eta V)^T q_{22}(t,\eta(t),0)\right] + \|\nabla_\eta V\| \|q_{22}(t,\eta(t),\bar{e}) - q_{22}(t,\eta(t),0)\| \\
&\quad + \|\nabla_\eta V\| \|\phi_\eta\| \|(\theta_d)\| - \frac{k}{2\varepsilon}\left[\|\bar{e}\|^2\right] + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2 \left\|\overline{e^1}\right\|^2 + \frac{1}{6}\|(\theta_d)\|^2 + \cdots \\
&\quad + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2 \|P^m\|^2 \|\overline{e^m}\|^2 + \frac{1}{6}\|(\theta_d)\|^2 + \varepsilon^{r_1-1}\omega_{\max}^1 r_1 k \left\|B_{r_1}^T P^1\right\| \left\|\overline{e^1}\right\|^2 \\
&\quad - \varepsilon^{r_m-1}\omega_{\max}^m r_m k \left\|B_{r_m}^T P^m\right\| \|\overline{e^m}\|^2 \\
&\le -2\alpha_x \|\eta\|^2 + \omega_3 \|\eta\| M \|\bar{e}\| + \omega_3 \|\eta\| \|\phi_\eta\| \|(\theta_d)\| - \frac{k}{2\varepsilon}\|\bar{e}\|^2 \qquad (3.71) \\
&\quad + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2 \left\|\overline{e^1}\right\|^2 + \frac{1}{6}\|(\theta_d)\|^2 + \cdots + \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2 \|P^m\|^2 \|\overline{e^m}\|^2 \\
&\quad + \frac{1}{6}\|(\theta_d)\|^2 + \varepsilon^{r_1-1}\omega_{\max}^1 r_1 k \left\|B_{r_1}^T P^1\right\| \left\|\overline{e^1}\right\|^2 - \varepsilon^{r_m-1}\omega_{\max}^m r_m k \left\|B_{r_m}^T P^m\right\| \|\overline{e^m}\|^2 \\
&\le -2\alpha_x \|\eta\|^2 + \frac{1}{16}\omega_3^2 M^2 \|\eta\|^2 + 4\|\bar{e}\|^2 + \omega_3^2 \|\phi_\eta\|^2 \|\eta\|^2 + \frac{1}{6}\|(\theta_d)\|^2 - \frac{k}{2\varepsilon}\|\bar{e}\|^2 \\
&\quad + \frac{1}{6}\|(\theta_d)\|^2 + \left(\frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2 + \varepsilon^{r_1-1}\omega_{\max}^1 r_1 k \left\|B_{r_1}^T P^1\right\|\right)\|\bar{e}\|^2 + \cdots \\
&\quad + \left(\frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2 \|P^m\|^2 + \varepsilon^{r_m-1}\omega_{\max}^m r_m k \left\|B_{r_m}^T P^m\right\|\right)\|\bar{e}\|^2 + \frac{1}{6}\|(\theta_d)\|^2 \\
&= -\|\eta\|^2 \left[2\alpha_x - \frac{1}{16}\omega_3^2 M^2 - \omega_3^2 \|\phi_\eta\|^2\right] + \frac{m+1}{6}\|(\theta_d)\|^2 \\
&\quad - \|\bar{e}\|^2 \left[\frac{k}{2\varepsilon} - \varepsilon^{r_1-1}\omega_{\max}^1 r_1 k \left\|B_{r_1}^T P^1\right\| - \varepsilon^{r_m-1}\omega_{\max}^m r_m k \left\|B_{r_m}^T P^m\right\|\right. \\
&\quad \left. - \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2 \left\|P^1\right\|^2 - \cdots - \frac{3}{2}\frac{k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2 \|P^m\|^2\right]
\end{aligned}
$$

i.e.,

$$\dot{L} \le -k_{11}\|\bar{e}\|^2 - k_{22}\|\eta\|^2 + \frac{m+1}{6}\|(\theta_d)\|^2 \tag{3.72}$$

From Equation (3.45c), we obtain

$$\dot{L} \le -N_2\left(\|\bar{e}\|^2 + \|\eta\|^2\right) + \frac{m+1}{6}\|(\theta_d)\|^2 \tag{3.73}$$

Define

$$\bar{e} \equiv \begin{bmatrix} \overline{e^1} \\ \overline{e^2} \\ \vdots \\ \overline{e^m} \end{bmatrix} \equiv \begin{bmatrix} \overline{e_1^1} \\ \overline{e_{rem}^1} \end{bmatrix}, \quad \overline{e_{rem}^1} \in \Re^{r-1} \tag{3.74}$$

Hence

$$\dot{L} \leq -N_2 \left( \|\eta\|^2 + \left\| \overline{e_1^1} \right\|^2 + \left\| \overline{e_{rem}^1} \right\|^2 \right) + \frac{m+1}{6} \left\| (\theta_d) \right\|^2 \tag{3.75}$$

Utilizing Equation (3.75) easily yields

$$\int_{t_0}^{t} \left( y_1(\tau) - y_d^1(\tau) \right)^2 d\tau \leq \frac{L(t_0)}{N_2} + 6\frac{m+1}{4N_2} \int_{t_0}^{t} \left\| (\theta_d(\tau)) \right\|^2 d\tau \tag{3.76}$$

Similarly, it is easy to prove that

$$\int_{t_0}^{t} \left( y_i(\tau) - y_d^i(\tau) \right)^2 d\tau \leq \frac{L(t_0)}{N_2} + \frac{m+1}{6N_2} \int_{t_0}^{t} \left\| (\theta_d(\tau)) \right\|^2 d\tau, \quad 2 \leq i \leq m \tag{3.77}$$

so that statement Equation (3.38b) is satisfied. From Equation (3.73), we get

$$\dot{L} \leq -N_2 \left( \|y_{total}\|^2 \right) + \frac{m+1}{6} \left\| (\theta_d) \right\|^2 \tag{3.78a}$$

where

$$\|y_{total}\|^2 \equiv \|\bar{e}\|^2 + \|\eta\|^2 . \tag{3.78b}$$

By virtue of Theorem 5.2 [19], Equation (3.78a) implies the input-to-state stability for the closed-loop system. Furthermore, it is easy to see that

$$\Delta_{\min} \left( \|\bar{e}\|^2 + \|\eta\|^2 \right) \leq L \leq \Delta_{\max} \left( \|\bar{e}\|^2 + \|\eta\|^2 \right) \tag{3.79}$$

that is

$$\Delta_{\min} \left( \|y_{total}\|^2 \right) \leq L \leq \Delta_{\max} \left( \|y_{total}\|^2 \right) \tag{3.80}$$

where $\Delta_{\min} \equiv \min \left\{ \omega_1, \frac{k}{2} \lambda_{\min}^* \right\}$ and $\Delta_{\max} \equiv \max \left\{ \omega_2, \frac{k}{2} \lambda_{\max}^* \right\}$. From Equation (3.73) and Equation (3.80) we yield that

$$\dot{L} \leq -\frac{N_2}{\Delta_{\max}} L + \frac{m+1}{6} \left( \sup_{t_0 \leq \tau \leq t} \left\| (\theta_d(\tau)) \right\| \right)^2 \tag{3.81}$$

Hence,

$$L(t) \leq L(t_0) e^{-\frac{N_2}{\Delta_{\max}}(t-t_0)} + \frac{\Delta_{\max}(m+1)}{6N_2} \left( \sup_{t_0 \leq \tau \leq t} \left\| (\theta_d(\tau)) \right\| \right)^2, \quad t \geq t_0 \tag{3.82}$$

which implies

$$\left| y_1(t) - y_d^1(t) \right| \leq \sqrt{\frac{2L(t_0)}{k\lambda_{\min}^*}} e^{-\frac{N_2}{2\Delta_{\max}}(t-t_0)} + \sqrt{\frac{\Delta_{\max}(m+1)}{3k\lambda_{\min}^* N_2}} \left( \sup_{t_0 \leq \tau \leq t} \left\| (\theta_d(\tau)) \right\| \right) \tag{3.83}$$

Similarly, it is easy to prove that

$$\left| y_i(t) - y_d^i(t) \right| \leq \sqrt{\frac{2L(t_0)}{k\lambda_{\min}^*}} e^{-\frac{N_2}{2\Delta_{\max}}(t-t_0)} + \sqrt{\frac{\Delta_{\max}(m+1)}{3k\lambda_{\min}^* N_2}} \left( \sup_{t_0 \leq \tau \leq t} \left\| (\theta_d(\tau)) \right\| \right), \ 2 \leq i \leq m \tag{3.84}$$

So that Equation (3.38a) is proved and then the tracking problem with almost disturbance decoupling is globally solved. Finally, we will prove that the sphere $B_{\underline{r}}$ is a global attractor for the output tracking error of system (3.1). From Equations (3.78a) and (3.45e), we get

$$\dot{L} \leq -N_2 \left( \|y_{total}\|^2 \right) + N_1 \tag{3.85}$$

For $\|y_{total}\| > \underline{r}$, we have $\dot{L} < 0$. Hence, any sphere defined by

$$B_{\underline{r}} = \left\{ \begin{bmatrix} \bar{e} \\ \eta \end{bmatrix} : \|\bar{e}\|^2 + \|\eta\|^2 \leq \underline{r} \right\} \tag{3.86}$$

is a global final attractor for the tracking error system of the nonlinear control systems (3.1). Furthermore, it is easy routine to see that, for $y \notin B_{\underline{r}}$, we have

$$\begin{aligned} \frac{\dot{L}}{L} &\leq \frac{-N_2 \|y_{total}\|^2 + N_1}{L} \leq \frac{-N_2 \|y_{total}\|^2 + N_1}{\Delta_{\max} \|y_{total}\|^2} \leq \frac{-N_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max} \|y_{total}\|^2} \\ &\leq \frac{-N_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max} \underline{r}^2} = -\alpha^* \end{aligned} \tag{3.87}$$

that is,

$$\dot{L} \leq -\alpha^* L$$

According to the comparison theorem, we get

$$L(t) \leq L(t_0) \exp\left[-\alpha^*(t - t_0)\right]$$

Therefore,

$$\begin{aligned} \Delta_{\min} \|y_{total}\|^2 &\leq L(y_{total}(t)) \leq L(y_{total}(t_0)) \exp\left[-\alpha^*(t - t_0)\right] \\ &\leq \Delta_{\max} \|y_{total}(t_0)\|^2 \exp\left[-\alpha^*(t - t_0)\right] \end{aligned} \tag{3.88}$$

Consequently, we get $\|y_{total}\| \leq \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}} \|y_{total}(t_0)\| \exp\left[-\frac{1}{2}\alpha^*(t - t_0)\right]$, i.e., the convergence rate toward the sphere $B_{\underline{r}}$ is equal to $\alpha^*/2$. This completes our proof.

In order to easily develop the main results via computational tool, an efficient computational algorithm for deriving the almost disturbance decoupling control is proposed as follows:

1) Calculate the vector relative degree $r_1, r_2, \cdots, r_m$ of the given control system.
2) Choose the diffeomorphism $\phi$ such that the Assumption 1 is satisfied.
3) Adjust some parameters $\alpha_1^i, \alpha_2^i, \cdots, \alpha_{r_i}^i$ such that the matrices $A_c^i$ are Hurwitz and calculate the positive definite matrices $P^i$ of the Lyapunov equations (3.28) by software package MATLAB.
4) Based on the famous Lyapunov approach, design a Lyapunov function to solve the conditions (3.40a)-(3.40c). If the relative degree $r_1 + r_2 + \cdots + r_m$ is equal to the system dimension $n$, then this step should be omitted and immediately go to the next step.
5) Appropriately tune the parameters $\omega_{\max}^i$, $\omega_{1r_i}^i$, $t_{r_i r_i}^i$, $1 \leq i \leq m$, $k$, $k_{11}$, $k_{22}$, $\varepsilon$ such that $N_2 > 1$ and go to the next step. Otherwise, we go to the Step 3 and repeat the overall designing procedures.
6) According to Equations (3.39g) and (3.41), the desired feedback linearization controller $u$ can be constructed such that the uniform ultimate bounded stability is guaranteed. That is, the system dynamics enter a neighborhood of zero state and remain within it thereafter.

4. **Illustrative Example.** Consider the BILSAT-1 satellite of the Turkish Scientific and Technological Research Council. From the detailed discussion of the satellite models presented in [3,17,18,36], the dynamic equations are given as follows:

$$\dot{x}_1 = (x_4)\left(\frac{1 + x_1^2 - x_2^2 - x_3^2}{4}\right) + (x_5)\left(\frac{-x_3 + x_1 x_2}{2}\right) + (x_6)\left(\frac{-x_2 + x_1 x_3}{2}\right) \tag{4.1a}$$

$$\dot{x}_2 = (x_4)\left(\frac{-x_2 + x_1x_2}{2}\right) + (x_5)\left(\frac{1 - x_1^2 + x_2^2 - x_3^2}{4}\right) + (x_6)\left(\frac{-x_1 + x_2x_3}{2}\right) \quad (4.1b)$$

$$\dot{x}_3 = (x_4)\left(\frac{-x_2 + x_1x_3}{2}\right) + (x_5)\left(\frac{x_1 + x_2x_3}{2}\right) + (x_6)\left(\frac{1 - x_1^2 - x_2^2 + x_3^2}{4}\right) \quad (4.1c)$$

$$\dot{x}_4 = 0.2303\Delta_{11} - s_{11} - 0.2303u_1 + \left(\frac{1}{4.35}\right)\tau_x^d + \left(\frac{1}{4.337}\right)\tau_y^d + \left(\frac{1}{3.664}\right)\tau_z^d \quad (4.1d)$$

$$\dot{x}_5 = 0.231\Delta_{22} - s_{22} - 0.231u_2 \quad (4.1e)$$

$$\dot{x}_6 = 0.2735\Delta_{33} - s_{33} - 0.2735u_3 \quad (4.1f)$$

$$\dot{x}_7 = -0.2303\Delta_{11} + 125.2303u_1 - \left(\frac{1}{4.35}\right)\tau_x^d - \left(\frac{1}{4.337}\right)\tau_y^d - \left(\frac{1}{3.664}\right)\tau_z^d \quad (4.1g)$$

$$\dot{x}_8 = -0.231\Delta_{22} + 125.231u_2 \quad (4.1h)$$

$$\dot{x}_9 = -0.2735\Delta_{33} + 125.2735u_3 \quad (4.1i)$$

$$y_1 = h_1(X) = x_1 \quad (4.1k)$$

$$y_2 = h_2(X) = x_2 \quad (4.1l)$$

$$y_3 = h_3(X) = x_3 \quad (4.1m)$$

where

$$\sum \equiv 1 - x_1^2 - x_2^2 - x_3^2 \quad (4.1n)$$

$$\Lambda \equiv 1 + x_1^2 + x_2^2 + x_3^2 \quad (4.1o)$$

$$\omega_0 \equiv 0.0010831 \quad (4.1p)$$

$$s_{11} \equiv \frac{-4\omega_0 x_6\left(-x_1^2 + x_2^2 - x_3^2\right) - x_6 \sum^2 \omega_0 + 8\omega_0 x_5 x_2 x_3 - 4x_5 x_1 \sum \omega_0}{\Lambda^2} \quad (4.1q)$$

$$s_{22} \equiv \frac{8\omega_0 x_6 x_2 x_1 - 4x_6 x_3 \sum \omega_0 - 8\omega_0 x_4 x_2 x_3 + 4x_4 x_1 \sum \omega_0}{\Lambda^2} \quad (4.1r)$$

$$s_{33} \equiv \frac{-8\omega_0 x_5 x_2 x_1 + 4x_5 x_3 \sum \omega_0 + 4\omega_0 x_4\left(-x_1^2 + x_2^2 - x_3^2\right) + x_4 \sum^2 \omega_0}{\Lambda^2} \quad (4.1s)$$

$$\begin{aligned}\Delta_{11} \equiv &\left[\frac{x_6\Lambda^2 - \omega_0\left(8x_2x_3 - 4x_1\sum\right)}{\Lambda^2}\right]\left[0.008x_8 + \frac{4.337x_5\Lambda^2 - 4.337\omega_0\left(4\left(-x_1^2 + x_2^2 - x_3^2\right) + \sum^2\right)}{\Lambda^2}\right] \\ &+ \left[\frac{-x_5\Lambda^2 + \omega_0\left(4\left(-x_1^2 + x_2^2 - x_3^2\right) + \sum^2\right)}{\Lambda^2}\right]\left[0.008x_9 + \frac{3.664x_6\Lambda^2 - 3.664\omega_0\left(8x_2x_3 - 4x_1\sum\right)}{\Lambda^2}\right]\end{aligned} \quad (4.1t)$$

$$\begin{aligned}\Delta_{22} \equiv &\left[\frac{-x_6\Lambda^2 + \omega_0\left(8x_2x_3 - 4x_1\sum\right)}{\Lambda^2}\right]\left[0.008x_7 + \frac{4.531x_4\Lambda^2 - 4.35\omega_0\left(8x_2x_1 - 4x_3\sum\right)}{\Lambda^2}\right] \\ &+ \left[\frac{x_4\Lambda^2 - \omega_0\left(8x_2x_1 - 4x_3\sum\right)}{\Lambda^2}\right]\left[0.008x_9 + \frac{3.664x_6\Lambda^2 - 3.664\omega_0\left(8x_2x_3 - 4x_1\sum\right)}{\Lambda^2}\right]\end{aligned} \quad (4.1u)$$

$$\begin{aligned}\Delta_{33} \equiv &\left[\frac{x_5\Lambda^2 - \omega_0\left(4\left(-x_1^2 + x_2^2 - x_3^2\right) + \sum^2\right)}{\Lambda^2}\right]\left[0.008x_7 + \frac{4.531x_4\Lambda^2 - 4.35\omega_0\left(8x_2x_1 - 4x_3\sum\right)}{\Lambda^2}\right] \\ &+ \left[\frac{-x_4\Lambda^2 + \omega_0\left(8x_2x_1 - 4x_3\sum\right)}{\Lambda^2}\right]\left[0.008x_8 + \frac{4.337x_5\Lambda^2 - 4.337\omega_0\left(4\left(-x_1^2 + x_2^2 - x_3^2\right) + \sum^2\right)}{\Lambda^2}\right]\end{aligned} \quad (4.1v)$$

where $\tau_x^d$, $\tau_y^d$ and $\tau_z^d$ are the disturbance torques due to external effects such as aerodynamics, gravity gradient and solar pressure, $\left[\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}\right]^T$ is the Modified Rodriguez

Parameters (MRP) attitude vector, $\begin{bmatrix} x_4 & x_5 & x_6 \end{bmatrix}^T$ is the angular velocity of the satellite body, $\begin{bmatrix} x_7 & x_8 & x_9 \end{bmatrix}^T$ is the velocity vector of the reaction wheel, $\omega_0$ is the orbital velocity and $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T = \begin{bmatrix} \tau_a^x & \tau_a^y & \tau_a^z \end{bmatrix}$ is the input torque vector.

Now we will show how to explicitly construct a controller that tracks the desired orientation of $y_d^1 = 20° = 0.35\text{rad}$, $y_d^2 = 40° = 0.7\text{rad}$, $y_d^3 = 60° = 1.05\text{rad}$ and attenuates the disturbance's effect on the output terminal to an arbitrary degree of accuracy. Let's arbitrarily choose $\alpha_1^1 = \alpha_1^2 = \alpha_1^3 = 1$, $\alpha_2^1 = \alpha_2^2 = \alpha_2^3 = 1$. Using the Matlab software yields

$$P^1 = P^2 = P^3 = \begin{bmatrix} 3.5 & 2.5 \\ 2.5 & 15 \end{bmatrix} \tag{4.2}$$

From Equation (3.41), we obtain the desired tracking controllers

$$u = A^{-1}\left(-\vec{b} + \vec{v} + \begin{bmatrix} u_{NN}^1 \\ u_{NN}^2 \\ u_{NN}^3 \end{bmatrix}\right) \tag{4.3a}$$

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4.3b}$$

$$a_{11} \equiv \frac{0.25 + 0.25\left(x_1^2 - x_2^2 - x_3^2\right)}{-4.35 + 0.008}, \quad a_{12} \equiv \frac{-0.5x_3 + 0.5x_2x_1}{-4.337 + 0.008}, \quad a_{13} \equiv \frac{0.5x_2 + 0.5x_3x_1}{-3.664 + 0.008},$$
$$a_{21} \equiv \frac{0.5x_3 + 0.5x_2x_1}{-4.35 + 0.008}, \quad a_{22} \equiv \frac{0.25 + 0.25\left(-x_1^2 + x_2^2 - x_3^2\right)}{-4.337 + 0.008}, \quad a_{23} \equiv \frac{-0.5x_1 + 0.5x_3x_2}{-3.664 + 0.008},$$
$$a_{31} \equiv \frac{-0.5x_2 + 0.5x_3x_1}{-4.35 + 0.008}, \quad a_{32} \equiv \frac{0.5x_1 + 0.5x_3x_2}{-4.337 + 0.008}, \quad a_{33} \equiv \frac{0.25 + 0.25\left(-x_1^2 - x_2^2 + x_3^2\right)}{-3.664 + 0.008}$$

$$\vec{v} \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{4.3c}$$

$$\begin{aligned} v_1 = &-(0.22)^{-2}(0.2)(x_1 - 0.35) - (0.22)^{-1}(0.2)\left[(x_4)\left(\frac{1 + x_1^2 - x_2^2 - x_3^2}{4}\right)\right. \\ &\left. + (x_5)\left(\frac{-x_3 + x_1x_2}{2}\right) + (x_6)\left(\frac{-x_2 + x_1x_3}{2}\right)\right] \\ v_2 = &-(0.22)^{-2}(0.2)(x_2 - 0.7) - (0.22)^{-1}(0.2)\left[(x_4)\left(\frac{-x_2 + x_1x_2}{2}\right)\right. \\ &\left. + (x_5)\left(\frac{1 - x_1^2 + x_2^2 - x_3^2}{4}\right) + (x_6)\left(\frac{-x_1 + x_2x_3}{2}\right)\right] \\ v_3 = &-(0.22)^{-2}(0.2)(x_3 - 1.05) - (0.22)^{-1}(0.2)\left[(x_4)\left(\frac{-x_2 + x_1x_3}{2}\right)\right. \\ &\left. + (x_5)\left(\frac{x_1 + x_2x_3}{2}\right) + (x_6)\left(\frac{1 - x_1^2 - x_2^2 + x_3^2}{4}\right)\right] \end{aligned}$$

$$\vec{b} \equiv \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{4.3d}$$

$$
\begin{aligned}
f_1 &\equiv (x_4)\left(\frac{1+x_1^2-x_2^2-x_3^2}{4}\right)+(x_5)\left(\frac{-x_3+x_1x_2}{2}\right)+(x_6)\left(\frac{-x_2+x_1x_3}{2}\right)\\
f_2 &\equiv (x_4)\left(\frac{-x_2+x_1x_2}{2}\right)+(x_5)\left(\frac{1-x_1^2+x_2^2-x_3^2}{4}\right)+(x_6)\left(\frac{-x_1+x_2x_3}{2}\right)\\
f_3 &\equiv (x_4)\left(\frac{-x_2+x_1x_3}{2}\right)+(x_5)\left(\frac{x_1+x_2x_3}{2}\right)+(x_6)\left(\frac{1-x_1^2-x_2^2+x_3^2}{4}\right)\\
f_4 &\equiv 0.2303\Delta_{11}-s_{11}\\
f_5 &\equiv 0.231\Delta_{22}-s_{22}\\
f_6 &\equiv 0.2735\Delta_{33}-s_{33}
\end{aligned}
$$

$$
\begin{aligned}
b_1 \equiv\ & (0.5x_4x_1+0.5x_5x_2+0.5x_6x_3)\,f_1+(-0.5x_4x_2+0.5x_5x_1+0.5x_6)\,f_2\\
&+(-0.5x_4x_3-0.5x_5+0.5x_6x_1)\,f_3+\left(0.25+0.25\left(x_1^2-x_2^2-x_3^2\right)\right)f_4\\
&+(-0.5x_3+0.5x_2x_1)\,f_5+(0.5x_2+0.5x_3x_1)\,f_6\\
b_2 \equiv\ & (0.5x_4x_2-0.5x_5x_1-0.5x_6)\,f_1+(0.5x_4x_1+0.5x_5x_2+0.5x_6x_3)\,f_2\\
&+(0.5x_4-0.5x_5x_3+0.5x_6x_2)\,f_3+(0.5x_3+0.5x_2x_1)\,f_4\\
&+\left(0.25+0.25\left(-x_1^2+x_2^2-x_3^2\right)\right)f_5+(-0.5x_1+0.5x_2x_3)\,f_6\\
b_3 \equiv\ & (0.5x_4x_3+0.5x_5-0.5x_6x_1)\,f_1+(-0.5x_4+0.5x_5x_3-0.5x_6x_2)\,f_2\\
&+(0.5x_4x_1+0.5x_5x_2+0.5x_6x_3)\,f_3+(-0.5x_2+0.5x_3x_1)\,f_4\\
&+(0.5x_1+0.5x_2x_3)\,f_5+\left(0.25+0.25\left(-x_1^2-x_2^2+x_3^2\right)\right)f_6
\end{aligned}
$$

$$
\begin{aligned}
u_{NN}^1 =\ & \omega_{11}^1t_{11}^1\varepsilon^0(x_1-0.35)+\omega_{12}^1t_{22}^1\varepsilon^1\Bigg[(x_4)\left(\frac{1+x_1^2-x_2^2-x_3^2}{4}\right)\\
&+(x_5)\left(\frac{-x_3+x_1x_2}{2}\right)+(x_6)\left(\frac{-x_2+x_1x_3}{2}\right)\Bigg]
\end{aligned}
\tag{4.3e}
$$

$$
\begin{aligned}
u_{NN}^2 =\ & \omega_{11}^2t_{11}^2\varepsilon^0(x_2-0.7)+\omega_{12}^2t_{22}^2\varepsilon^1\Bigg[(x_4)\left(\frac{-x_2+x_1x_2}{2}\right)\\
&+(x_5)\left(\frac{1-x_1^2+x_2^2-x_3^2}{4}\right)+(x_6)\left(\frac{-x_1+x_2x_3}{2}\right)\Bigg]
\end{aligned}
\tag{4.3f}
$$

$$
\begin{aligned}
u_{NN}^3 =\ & \omega_{11}^3t_{11}^3\varepsilon^0(x_3-1.05)+\omega_{12}^3t_{22}^3\varepsilon^1\Bigg[(x_4)\left(\frac{-x_2+x_1x_3}{2}\right)\\
&+(x_5)\left(\frac{x_1+x_2x_3}{2}\right)+(x_6)\left(\frac{1-x_1^2-x_2^2+x_3^2}{4}\right)\Bigg]
\end{aligned}
\tag{4.3g}
$$

It can be verified that the relative conditions of Theorem 3.1 are satisfied if $\varepsilon = 0.22$, $t_{11}^1 = t_{22}^1 = t_{11}^2 = t_{22}^2 = t_{11}^3 = t_{22}^3 = 1$, $\omega_{11}^1 = \omega_{22}^1 = \omega_{11}^2 = \omega_{22}^2 = \omega_{11}^3 = \omega_{22}^3 = 0.024$, $\alpha_x = 0.6$, $\omega_1 = \omega_2 = 4$, $\omega_3 = 8$, $M = 0.1$, $r_1 = r_2 = r_3 = 2$, and $k = 6\sqrt{\varepsilon}$. Hence, the tracking controllers will steer the output tracking errors of the closed-loop system, starting from any initial value, to be attenuated to zero by virtue of Theorem 3.1. The complete trajectories of the outputs are depicted in Figure 3. Based on the observation of Figure 3, our proposed controller has successfully driven the the BILSAT-1 satellite to track the desired attitudes in Euler angles (roll, pitch and yaw) $y_d^1 = 20^\circ = 0.35$rad, $y_d^2 = 40^\circ = 0.7$rad, $y_d^3 = 60^\circ = 1.05$rad within short time and attenuates the disturbance's effect on the output terminal to an arbitrary degree of accuracy. It is easy to see that the proposed controller does not employ any learning or adaptive algorithm and the stability and the almost disturbance decoupling performance of the system are guanranteed at each

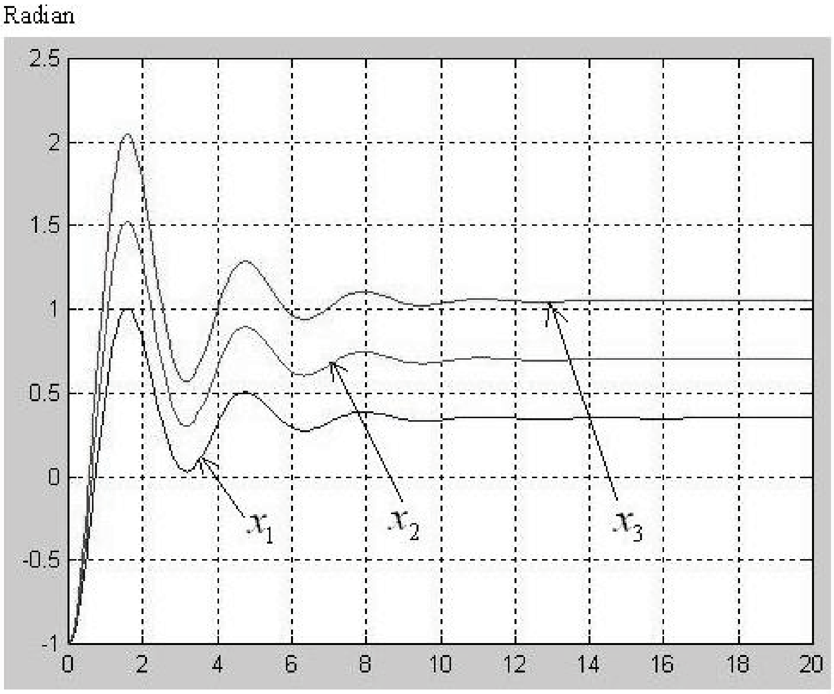


FIGURE 3. The output trajectories of the BILSAT-1 satellite

step of selecting weights to enhance the performance via MATLAB as long as the related sufficient conditions are satisfied.

5. **Comparative Example with Existing Approach.** [25] exploited the fact that the almost disturbance decoupling problem could not be solvable for the following system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \tan^{-1}(x_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta(t) \tag{5.1a}$$

$$y(t) = x_1(t) := h(X(t)) \tag{5.1b}$$

where $u$, $y$ denoted the input and output respectively, $\theta(t) := 0.05 \sin t$ is the disturbance. The almost disturbance decoupling problem can be easily solved via the proposed approach in this paper. Following the same procedures shown in the demonstrated example, we can solve the tracking problem with almost disturbance decoupling problem by the controller $u$ defined as

$$\begin{aligned} u = \left(1 + x_2^2\right) & \left[-\sin t - (0.03)^{-2} (x_1 - \sin t) - (0.03)^{-1} (\tan^{-1} x_2 - \cos t) \right. \\ & \left. + \omega_{11}^1 t_{11}^1 (x_1 - \sin t) + \omega_{12}^1 t_{22}^1 (\tan^{-1} x_2 - \cos t)\right] \end{aligned} \tag{5.2}$$

The output error of feedback-controlled system for (5.1) is depicted in Figure 4 with the neural network weights $\omega_{11}^1 = -0.2$, $t_{11}^1 = -0.2$, $\omega_{12}^1 = -0.2$, $t_{22}^1 = -0.2$.

It is worth noting that the sufficient conditions given in [25] (in particular the structural conditions on nonlinearities multiplying disturbances) are not necessary in this study where a nonlinear state feedback control is explicitly designed which solves the almost disturbance decoupling problem. For instance, the almost disturbance decoupling problem is solvable for the system (5.1) by a nonlinear state feedback control, according to our proposed approach, while the sufficient conditions given in [25] fail when applied to the system (5.1). The design techniques in this study are also entirely different than those in [25] since the singular perturbation tools are not used.

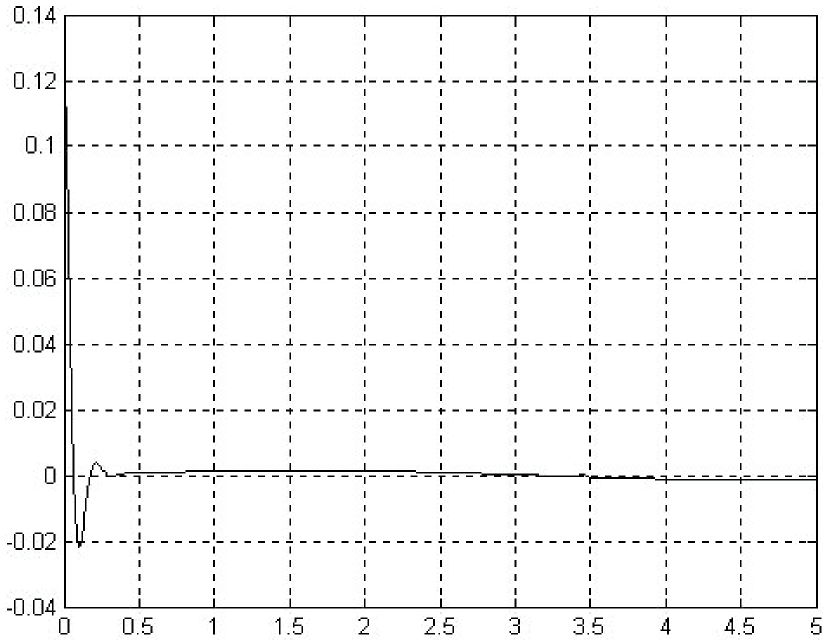


FIGURE 4. The output error of feedback-controlled system for (5.1)

6. **Conclusion.** A novel neural feedback control to globally solve the tracking problem with almost disturbance decoupling for MIMO nonlinear system has been proposed. The new approach to neural network and feedback linearization controller enables the designer to determine the interconnect structure among the layers needed to stabilize the overall system without any learning or adaptive algorithms. One comparative example is proposed to show the significant contribution of this paper with respect to some existing approach. A practical example of BILSAT-1 satellite system demonstrates the applicability of the proposed feedback linearization approach and the composite Lyapunov approach. Simulation results have been presented to show that the proposed methodology can be successfully applied to feedback linearization problem and is able to achieve the desired tracking and almost disturbance decoupling performances of the controlled system.

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