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CONTROLLABILITY IMPROVEMENT FOR LINEAR TIME-INVARIANT DYNAMICAL MULTI-AGENT SYSTEMS

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Abstract. *For two types of linear time-invariant dynamical multi-agent systems under leader-follower framework, the problem of graph topology adjustment is addressed to improve system controllability. As important concepts and theoretical foundations, the maximum controllability index of square matrices is defined and analyzed, and a generalized controllability canonical form is introduced for single-input systems. Based on these concepts, approaches for adjusting the leader-follower and follower-follower communication architectures are presented respectively.*
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1. **Introduction.** The study of controllability of composite systems started long ago. Gilbert [1], Davison and Wang [2], and Porter [3] were some of the early researchers endeavoring to develop criteria for checking the controllability of composite systems. Later, based on graph theory, Davison [4] defined the concept "connectivity", which is crucial for the controllability of composite systems. Also based on graph theory, Lin [5] introduced the concept "structural controllability". Zazworsky and Knudsen [6] presented conditions for the controllability of compartmental models, focusing on the configuration of interconnections. Kobayashi et al. [7,8] discussed the criteria for controllability of decentralized configurations concerning fixing-mode and graph topology respectively.

During recent years, dynamical multi-agent systems have been extensively studied by scholars in the field of control theory. The consensus problem [9-11,25,26] attracts most attention, which is essentially a stability problem. Nonetheless, only a few researchers have started to notice the controllability problems. Mesbahi [12] proposed the concept "state-dependent graph" and defined "controllability" for state-dependent graphs. Tanner [13] postulated that more information exchange might be detrimental to controllability. Ji et al. [14] gave sufficient conditions for controllability, based on the algebraic characteristics of certain matrices about the graphs. Rahmani et al. [15] extended the work of Ji et al. in [14], concentrating on the relationship between graph symmetry and controllability. Cai et al. [16-19] proposed the concept "formation controllability" and studied the condition for controllability of high-order systems. Liu et al. [20] discussed the controllability of discrete time systems with switching graph topologies. Ji et al. [21] presented conditions for graph controllability, which are analogous to the results in [6]. Liu et al. [22] endeavored to analyze the relation between controllability and topology of large-scale weighted complex networks from multiple backgrounds.

The main contributions of the current paper are summarized as follows. 1) The concept of maximum controllability index for square matrices is proposed and discussed. This concept reflects the potential of controllability of corresponding dynamical systems. 2) The concept of generalized controllability canonical form of LTI systems with single input is proposed and discussed. This concept is an extension of controllability canonical form. 3) Approaches for controllability improvement of two types of LTI dynamical multi-agent models are presented, on the basis of the theoretical preparations on maximum controllability index, generalized controllability canonical form, and along with our previous results on controllability of high-order dynamical multi-agent systems [14].

The motivation of study on controllability improvement of systems comes from practical requirements. For instance, in linear systems theory, the poles of a system (A, B) can be freely assigned iff the system is controllable. Sometimes, one may encounter such a problem [23]: how to seek a proper input matrix B such that (A, B) is controllable? The intrinsic configuration of an isolated system may possibly be unalterable. However, the architecture of communication network in engineering multi-agent systems is usually adjustable. Therefore, it is meaningful to probe into this problem: how to improve the controllability of a multi-agent system by modifying its architecture?

The major concern of nearly all relevant works [13-15,20-22] recently conducted by other scholars is what kind of geometric characteristics a controllable graph should have. They did achieve some theoretical results. However, none of them have really succeeded. From a different viewpoint, we mainly focus on the algebraic characteristics of matrices instead of attempting to reveal the direct relationship between the geometric characteristics of a graph and controllability. Actually, such a direct relationship might not exist at all. Most of the relevant works dealt with first-order multi-agent systems, whereas we consider high-order systems. Wang et al. [24] tried to discuss the effect on controllability by selection of leaders and weights of communication links; however, so far as our knowledge is concerned, papers systematically discussing controllability improvement problems are still absent in the literature.

The rest of this paper is organized as follows. In Section 2, two types of dynamical multi-agent models and the controllability improvement problems are introduced. In Section 3, theoretical analysis about the maximum possible controllability indices of matrices is presented. In Section 4, approaches of how to improve controllability of the first type of system models are discussed for two different scenarios, respectively. In Section 5, approaches on controllability improvement of the second type of systems are discussed. Section 6 provides an example to demonstrate the main technique of controllability improvement. Finally, Section 7 concludes this paper.

Notations: (A, b) denotes a pair of matrices. $\phi = [1\ 1\ \cdots\ 1]^T$.

2. Problem Formulation.

2.1. **Dynamical multi-agent system model I.** The dynamical multi-agent system model I includes $N + N_l$ agents of dth order. The state of agent i is denoted by $x_i = [x_{i1}, x_{i2}, \ldots, x_{id}]^T \in R^d$. The dynamics of each agent can be described as:

$$\dot{x}_i = F \sum\nolimits_{j=1}^{N+N_l} w_{ij} x_j + Bu_i \quad (i = 1, 2, \ldots, N + N_l) \tag{1}$$

where $F \in R^{d\times d}$, $w_{ij} \in R$, $B \in R^{d\times m}$, and $u_i \in R^m$. w_{ij} is the edge or arc weight of a graph G between agents i and j, representing the strength of information link between the two neighbors. For an LTI system, the time-invariant graph G can be denoted by its adjacency matrix W. If the input $u_i(t) \equiv 0$, then agent i is called a *follower*; or if agent i is actuated by some $u_i(t)$ not being zero identically, it is called a *leader*. If all state vectors of agents are stacked together, then the entire state matrix of the system is $X \in R^{d\times(N+N_l)}$. The dynamics of the multi-agent system can be described by

$$\dot{X} = FXW^T + BU \tag{2}$$

where $U \in R^{m\times(N+N_l)}$ is the stack of input vectors. Suppose that the number of leaders is N_l, and the number of followers is N. Without loss of generality, the leaders can be indexed from $N+1$ to $N+N_l$. Thus, $U = [0\ \cdots\ 0\ u_{N+1}\ \cdots\ u_{N+N_l}]$.

A fundamental problem about controllability is: whether the state X of the dynamical multi-agent system is completely controllable? This problem is answered in [16] and some of its results are reviewed here as reference for later discussions.

To discriminate between the followers and the leaders, partition W:

$$W = \begin{bmatrix} W_{ff} & W_{fl} \\ W_{lf} & W_{ll} \end{bmatrix} \tag{3}$$

where $W_{ff} \in R^{N\times N}$ indicates the arcs of G among the followers; $W_{fl} \in R^{N\times N_l}$ the arcs from the leaders to the followers; $W_{lf} \in R^{N_l\times N}$ the arcs from the followers to the leaders; and $W_{ll} \in R^{N_l\times N_l}$.

Definition 2.1. *(Controllable Graph [16,21]) With the last N_l agents as the leaders and a partitioned form of adjacency matrix W as (3), the graph G of the dynamical multi-agent system is controllable if (W_{ff}, W_{fl}) is controllable.*

With the criterion given by the following lemma, complete controllability of an LTI dynamical multi-agent system of high order can be checked.

Lemma 2.1. [16] *The LTI dynamical multi-agent system (1) is completely controllable if and only if the two conditions below are simultaneously satisfied:*
1) The graph G is controllable;
2) (F, B) is controllable.

Evidently, the two conditions in Lemma 2.1 are independent with each other. The following assumption is assumed to be satisfied throughout this paper. Under this assumption, the remainder of the problem is determined by the controllability of graph.

Assumption 2.1. (F, B) *is controllable.*

2.2. **Dynamical multi-agent system model II.** The dynamical multi-agent system model II to be considered is different from (1). Compared with (1), the dynamics of each agent in this system is described as:

$$\dot{x}_i = F\sum_{j=1}^{N+N_l} w_{ij}(x_j - x_i) + Bu_i \quad (i = 1, 2, \ldots, N+N_l) \tag{4}$$

The first order dynamical multi-agent system model with linear consensus algorithm that has been studied extensively in the literature, e.g., in [25,26], is a special case of the model depicted by (4).

Analogous to (2), the dynamics of the entire multi-agent system (4) can be described by:

$$\dot{X} = -FXL^T + BU \tag{5}$$

where $L \in R^{(N+N_l)\times(N+N_l)}$ is the Laplacian matrix [27] of the graph G. The relationship between the adjacency and the Laplacian matrices of a graph G is:

$$L = diag(W\phi) - W \tag{6}$$

From the definition of L, one can see that:

$$L\phi = 0 \tag{7}$$

i.e., ϕ is always an eigenvector corresponding to eigenvalue 0 of L. (7) can be regarded as a criterion to check whether a given matrix is a Laplacian matrix of certain existing graph.

By comparing the dynamic Equation (5) with (2), the partitioned form of L can accordingly be obtained, discriminating between the followers and the leaders:

$$L = \begin{bmatrix} L_{ff} & L_{fl} \\ L_{lf} & L_{ll} \end{bmatrix} \tag{8}$$

It naturally leads to the corresponding definition of controllable graph and the criterion to check complete controllability for dynamical multi-agent system (4), which is similar to the result of Lemma 2.1. Accordingly, in this case a graph G is controllable [16] if (L_{ff}, L_{fl}) is controllable.

2.3. **Controllability improvement problems for dynamical multi-agent systems.** In this paper, the agent dynamics and the interactive effect among the neighbors are regarded as intrinsic characteristic of a dynamical multi-agent system and are unalterable, while it is assumed that some of the arc weights of the graph are physically adjustable. The main aim here is to improve graph controllability by adjusting these arcs.

Remark 2.1. *For uncontrollable systems, the controllability index can be regarded as a measure of controllability. Sometimes, complete controllability of full state is not necessary, e.g., in formation control [16]. It is usually better if an uncontrollable system achieves higher controllability.*

For each kind of dynamical multi-agent system, controllability improvement approaches dealing with the following two scenarios will be discussed in the rest of this paper:

1) *Fixed follower-follower graph topology*

In this case, the submatrix W_{ff} is fixed and unalterable. One needs to seek some appropriate information links from the leaders to the followers W_{fl} to achieve higher controllability, i.e., to increase the controllability index of graph G.

2) *Fixed leader-follower graph topology*

In this case, on the contrary, the submatrix W_{fl} is unalterable. One should adjust W_{ff} on the basis of its original configuration to achieve higher controllability.

There should be two considerations for these approaches: on one hand, controllability is desired to be improved as much as possible; on the other hand, less change should be brought to the original graph.

Dynamical multi-agent systems with a single leader are typical in nature and engineering applications, and it is reasonable for these systems to be stressed. In this paper, for the case that the leader-follower arcs are fixed, only the instance with a single leader is considered.

3. The Maximum Controllability Index of Matrix.

This section will provide the mathematical preparations for controllability improvement. The first important concept is *Maximum Controllability Index*.

Definition 3.1 (Maximum Controllability Index). *Consider a single-input LTI system $\dot{z} = Az + bv$ with $z \in R^n$ the state and $v \in R$ the input. Suppose $A \in R^{n\times n}$ is given and $b \in R^n$ can be freely selected in R^n, then the maximum possible dimension for the* *controllable subspaces of systems among the set* $\{(A, b)|b \in R^n\}$ *is defined as the maximum controllability index of matrix* A*, denoted by* $\gamma(A)$*.*

The maximum controllability index of a given matrix A can be easily computed. Meanwhile an appropriate vector b can be found *s.t.* the controllability index of (A, b) equals to $\gamma(A)$. The theorem and lemmas that follow will contain relevant discussions.

Theorem 3.1. *The maximum controllability index* $\gamma(A)$ *equals the degree of the minimum polynomial* $\phi_A(\lambda)$ *of matrix* A*.*

With the following two lemmas, Theorem 3.1 will naturally be proved.

Lemma 3.1. *The maximum controllability index* $\gamma(A)$ *is equal to or less than the degree of the minimum polynomial* $\phi_A(\lambda)$ *of matrix* $A \in R^{n\times n}$*.*

Proof: Please refer to [18].

Lemma 3.2. *1) The maximum controllability index* $\gamma(A)$ *is equal to or greater than the degree of the minimum polynomial* $\phi_A(\lambda)$ *of* A*. 2) Suppose that the given* $A \in R^{n\times n}$ *with Jordan canonical form* $\hat{A} = PAP^{-1}$ *has* μ *distinct eigenvalues* $\lambda_1, \ldots, \lambda_\mu$*. Let* $b = P^{-1}\hat{b}$*, where* $\hat{b}$ *is any vector in* R^n *with at least* μ *non-zero entries, each corresponding to the last row of the Jordan block that has the maximum dimension among the Jordan blocks of* λ_i $(i = 1, 2, \ldots, \mu)$*. The dimension of controllable subspace of* (A, b) *is* $\gamma(A)$*.*

Proof: Please refer to [18].

Remark 3.1. *The statement 2) of Lemma 3.2 also provides a method to seek feasible candidate of* b *to maximize the controllable dimension of* (A, b) *for a given* A*.*

Denote the indexes of the necessarily non-zero entries in such a $\hat{b} = Pb$ by $i_1, i_2, \ldots, i_\mu$. If all elements in P^{-1} are real, simply, a $\hat{b}$ with μ necessary nonzero elements being 1 and all the other elements being 0 is feasible. Otherwise, if P and P^{-1} are complex valued, because $\hat{b} = Pb$, a feasible b should not be orthogonal to all the μ rows in P indexed respectively by $i_1, i_2, \ldots, i_\mu$. With the following theorem, one will see that such a b must exist in R^n.

Theorem 3.2. *For a given* $A \in R^{n\times n}$*, almost any* $b \in R^n$ *makes the dimension of controllable subspace of* (A, b) *be equal to* $\gamma(A)$*.*

Proof: Please refer to [18]. It can be shown that the measure of the set for infeasible b in the parameter space R^n is zero.

4. Controllability Improvement of Dynamical Multi-agent System Model I.

4.1. Fixed follower-follower graph topology.

Suppose there are N followers in a dynamical multi-agent system (1) and the arcs among them are unalterable, i.e., the W_{ff} in (3) is fixed. Suppose that new agents are added as leaders in order to control the multi-agent system. The communication architecture from the leaders to the followers, i.e., the W_{fl}, should be determined to make the multi-agent system controllable.

Whether or not a single leader is sufficient to control the system? If a single leader is sufficient, how can a proper $W_{fl} \in R^N$ be designed? Actually, the analysis in the last section has already answered these questions theoretically.

When $\gamma(W_{ff}) < N$, the system is impossible to be controllable for any $W_{fl} \in R^N$ if there is only one leader. It is necessary to increase the number of leaders to achieve complete controllability. What is the least number of leaders required? Theorem 4.1 will answer this question.

Theorem 4.1. *If the fixed adjacency submatrix $W_{ff} \in R^{N\times N}$ of the N followers in a dynamical multi-agent system (1) is derogatory, then at least $\alpha(W_{ff})$ leaders are required for complete controllability, where $\alpha(W_{ff})$ denotes the maximum geometric multiplicity of eigenvalues of W_{ff}.*

Proof: Suppose that the Jordan canonical form of W_{ff} is $\hat{W}_{ff} = PW_{ff}P^{-1}$. Without loss of generality, assume that λ_1 is the eigenvalue of W_{ff} that possesses the maximum geometric multiplicity among the μ distinct eigenvalues. The submatrix in $\hat{W}_{ff}$ associated with λ_1 is J_1, comprising $\alpha(W_{ff})$ Jordan blocks:

$$J_1 = \begin{bmatrix} J_{11} & & \\ & \ddots & \\ & & J_{1,\alpha(W_{ff})} \end{bmatrix}$$

Now consider the controllability of $(\hat{W}_{ff}, \hat{W}_{fl})$, where $\hat{W}_{fl}$ is what to be designed. According to the controllability test based on Jordan canonical form, if $(\hat{W}_{ff}, \hat{W}_{fl})$ is completely controllable, then the $\alpha(W_{ff})$ row vectors in $\hat{W}_{fl}$ respectively corresponding to the bottom rows in $J_{11}, \ldots, J_{1,\alpha(W_{ff})}$ must be linearly independent. This implicates that $rank(\hat{W}_{fl}) \geq \alpha(W_{ff})$. Therefore, the number of columns in $\hat{W}_{fl}$ should at least be $\alpha(W_{ff})$. If $\hat{W}_{fl} \in R^{N\times\alpha(W_{ff})}$, for the rest of submatrices $J_2, J_3, \ldots, J_\mu$, the requirement of controllability test can easily be satisfied with appropriately selected rows in $\hat{W}_{fl}$ because each geometric multiplicity of $\lambda_2, \lambda_3, \ldots, \lambda_\mu$ is less than or equal to $\alpha(W_{ff})$. With such a $\hat{W}_{fl}$, the arcs from the leaders to the followers can be physically realized according to $W_{fl} = P^{-1}\hat{W}_{fl}$. Note that in multi-agent system (1), the number of columns in W_{fl} represents the number of leaders.

4.2. **Fixed leader-follower graph topology.** Consider a dynamical multi-agent system comprising one leader and N followers with fixed communication architecture from the single leader to each follower. How can the subgraph associated with the followers be adjusted to notably improve the controllability of the system? The answer to this question will be given in this subsection.

In the partitioned form (3) of the adjacency matrix, submatrix $W_{ff} \in R^{N\times N}$ is the object to be adjusted whereas vector $W_{fl} \in R^N$ is fixed. Remember that the controllability of graph G of dynamical multi-agent system (1) is equivalent to the controllability of the following matrix pair:

$$(W_{ff}, W_{fl}) \tag{9}$$

The generalized controllability canonical form of such a system will be introduced as follows, which is the foundation for the controllability improvement.

Definition 4.1 (Generalized Controllability Canonical Form)**.** *The generalized controllability canonical form (M, h) of a single-input LTI system of nth order that could be uncontrollable possess the following structure*

$$\dot{x} = \left[\begin{array}{cc|c|c|c} \Theta_{m+1} & \ddots & O_3 & O_2 & \\ \ddots & \ddots & & & \\ \hline \multicolumn{2}{c|}{\times} & \Theta_3 & & O_1 \\ \hline \multicolumn{3}{c|}{\times} & \Theta_2 & \\ \hline \multicolumn{4}{c|}{\times} & \Theta_1 \end{array}\right] x + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{10}$$

where the diagonal blocks $\Theta_i \in R^{\gamma_i \times \gamma_i}$ $(i = 1, \cdots, m+1)$ *denote companion matrices [29] as*

$$\Theta_i = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \\ * & * & \cdots & * \end{bmatrix}$$

with '$*$' denoting the elements that can be any value; '$\times$' any submatrix; and O_i $(i = 1, 2, \cdots, m)$ *the zero matrices.*

Remark 4.1. *The matrix* M *in the generalized controllability canonical form (10) has a hierarchical structure, delineated by dotted lines.*

Definition 4.2 (Linkage Element)**.** *Let* $\eta_i = 0$ $(i = 1, 2, \cdots, m)$ *respectively denote the element at the left lower corner of* O_i *in (10). These elements are called linkage elements in the generalized controllability canonical form.*

One will see that these elements are rather important to controllability improvement.

Theorem 4.2. *Any nth order LTI single-input system* $\dot{x} = Ax + bu$ *can be transformed into its generalized controllability canonical form* $\dot{\hat{x}} = M\hat{x} + hu$ *by a nonsingular similarity transformation* P*, with* $\hat{x} = Px$*.*

Proof: If the system is controllable, it can be transformed into controllability canonical form, which is a specific case of generalized controllability canonical form. Suppose the controllability index is $\gamma_1 < n$. For simplicity of the notations, matrix pairs will be used to represent the corresponding LTI systems.

Step 1. Decompose the entire system into controllable/uncontrollable subsystems via similarity transformation $P_1 \in R^{\gamma_1 \times \gamma_1}$:

$$\left(\begin{bmatrix} A_{11} \in R^{(n-\gamma_1)\times(n-\gamma_1)} & 0 \\ A_{21} & A_{22} \in R^{\gamma_1 \times \gamma_1} \end{bmatrix}, \begin{bmatrix} 0 \\ b_2 \in R^{\gamma_1} \end{bmatrix} \right) = (P_1 A P_1^{-1}, P_1 b) \tag{11}$$

where the matrix pair (A_{22}, b_2) is controllable.

Step 2. Transform the controllable subsystem (A_{22}, b_2) into controllability canonical form via similarity transformation $P_2 \in R^{\gamma_1 \times \gamma_1}$:

$$\begin{aligned} &P_2 A_{22} P_2^{-1} = \Theta_1 \\ &P_2 b_2 = [0 \ \cdots \ 0 \ 1]^T \end{aligned}$$

The entire system is currently transformed into:

$$\left(\begin{bmatrix} A_{11} & 0 \\ \times & \Theta_1 \end{bmatrix}, h \right) \tag{12}$$

Step 3. Decompose A_{11}. For this end, a virtual matrix pair (A_{11}, b_1) is required, with b_1 an auxiliary vector constructed according to Lemma 3.2 such that the controllability index of (A_{11}, b_1) equals γ_2, which is the maximum controllability index of A_{11}. Decomposing (A_{11}, b_1) into controllable/uncontrollable subsystems by nonsingular transformation $P_3 \in \gamma_2$ yields:

$$\left(\begin{bmatrix} A_{11}^{(11)} & 0 \\ A_{11}^{(21)} & A_{11}^{(22)} \in R^{\gamma_2 \times \gamma_2} \end{bmatrix}, \begin{bmatrix} 0 \\ b_1^{(2)} \in R^{\gamma_2} \end{bmatrix} \right)$$

Step 4. Transform the virtual controllable subsystem $(A_{11}^{(22)}, b_1^{(2)})$ into controllability canonical form by nonsingular transformation $P_4 \in R^{(n-\gamma_1-\gamma_2)\times(n-\gamma_1-\gamma_2)}$:

$$\begin{aligned} &P_4 A_{11}^{(22)} P_4^{-1} = \Theta_2 \\ &P_4 b_1^{(2)} = [0 \ \cdots \ 0 \ 1]^T \end{aligned}$$

So far, the entire system can be transformed into

$$\left(\begin{bmatrix} A_{11}^{(11)} & 0 & \\ \times & \Theta_2 & 0 \\ & \times & \Theta_1 \end{bmatrix}, h\right) \tag{13}$$

by the series of similarity transformations:

$$P = \begin{bmatrix} I & & \\ & P_4 & \\ & & I \end{bmatrix} \begin{bmatrix} P_3 & \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & P_2 \end{bmatrix} P_1 \tag{14}$$

where I in (14) may represent any identity matrix with certain dimension.

Evidently, $A_{11}^{(11)}$ in (13) can further be transformed by repeating the similar operations Step 1 $\sim$ Step 4 ... until a generalized controllability canonical form is finally achieved.

The above proof not only proved Theorem 4.2 but also demonstrated the algorithm for deriving a generalized controllability canonical form for uncontrollable system.

Controllability improvement will be based on generalized controllability canonical form. Actually, the linkage elements η_i $(i = 1, 2, \cdots, m)$ are the keys of the approach. It is easy to verify the validity of the following theorem and the detail is omitted here.

Theorem 4.3. *If the value of η_1 in the generalized controllability canonical form (10) is altered from 0 to 1, then the dimension of controllable subspace for the corresponding dynamic system is increased from γ_1 to $\gamma_1 + \gamma_2$. If all the linkage elements η_i $(i = 1, 2, \cdots, m)$ are altered from 0 to 1, then the dynamic system is completely controllable.*

The linkage elements are the bridges that may connect the previously uncontrollable state variables with the input. For instance, if $\eta_1 = 0$, the first $n-\gamma_1$ state variables in (10) are separated from the input information; otherwise, the influence of input information could reach these variables.

Remark 4.2. *If $\eta_1 = 1$, Θ_i and Θ_i combine into one companion matrix. If all $\eta_i = 1$ $(i = 1, 2, \cdots, m)$, all $\Theta_i \in R^{\gamma_i \times \gamma_i}$ $(i = 1, \cdots, m + 1)$ combine into one companion matrix and (M, h) become a controllability canonical form.*

The algorithm to improve the controllability of system (A, b) by adjusting A is summarized as follows. First, transform it into generalized controllability canonical form (M, h) by nonsingular matrix P, with $M = PAP^{-1}$. Second, improve the controllability of (M, h) by altering the values of linkage elements into 1 and get $(M + \Delta M, h)$. Third, retrieve an improved A by the same but reversed transformation P^{-1}, with $A + \Delta A = P^{-1}(M + \Delta M)P$. Finally, a controllable matrix pair $(A + \Delta A, b)$ is derived.

For dynamical multi-agent system (1), the above algorithm is directly applicable to improve the controllability of (W_{ff}, W_{fl}) by adjusting W_{ff}. The adjustment can be regarded as an offset ΔW_{ff} mounted upon W_{ff}. There also exist methods [18] to reduce the magnitude of this offset.

5. Controllability Improvement of Dynamical Multi-agent System Model II.

5.1. **Fixed follower-follower graph topology.** Consider the controllability improvement problem of dynamical multi-agent system (4). Suppose that W_{ff} is intrinsic and unalterable while the arcs from the leaders to each follower, i.e., entries of W_{fl}, are to be designed. The problem to be dealt with in this subsection is: How can one seek some appropriate W_{fl} to guarantee the controllability of (L_{ff}, L_{fl})?

If without any leader, (6) shows that $L_{ff} = diag(W_{ff}\phi) - W_{ff}$, determined merely by W_{ff}. If a leader exists, L_{ff} will be affected by the leader even if W_{ff} is fixed:

$$L_{ff} = diag(W_{ff}\phi) - W_{ff} + diag(W_{fl}) \tag{15}$$

Besides, $L_{fl} = -W_{fl}$.

The next theorem will be helpful for solving this problem. Its proof relies on the following lemma about matrix eigenvalue perturbation.

Lemma 5.1. [29] *Let $A \in R^{n\times n}$ be diagonalizable with $A = S\Lambda S^{-1}$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$. Let $E \in R^{n\times n}$ and let $\|\cdot\|$ be some matrix norm s.t. $\|D\| = \max_{1\le i\le n} |d_i|$ for all diagonal matrices $D = diag(d_1, \cdots, d_n) \in R^{n\times n}$. If $\hat{\lambda}$ is an eigenvalue of $A + E$, then there is some eigenvalue λ_i of A for which*

$$\left|\hat{\lambda} - \lambda_i\right| \le \kappa(S)\,\|E\|$$

where $\kappa(\cdot)$ is the condition number with respect to the matrix norm $\|\cdot\|$.

Theorem 5.1. *For any given $A \in R^{n\times n}$, there always exists some $b = [b_1 \ \cdots \ b_n]^T \in R^n$ s.t. the matrix pair $(A - diag(b), b)$ is controllable.*

Proof: Assume that $|b_i| > \|A\|_\infty$ and $|b_i - b_j| > \|A\|_\infty$ $(\forall i, j \in \{1, 2, \cdots, n\} \quad i \ne j)$, where $\|\cdot\|_\infty$ denotes the maximum row sum matrix norm [29]. Suppose matrix A is the perturbation to matrix $-diag(b)$, with the original eigenvalues $-b_1, \ldots, -b_n$. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $A - diag(b)$. According to the Gersgorin disk Theorem [29], it must be true that:

$$|\Delta\lambda_i| < \sum\nolimits_{j=1}^{n} |a_{ij}| \le \|A\|_\infty \quad (\forall i \in \{1, 2, \cdots, n\}) \tag{16}$$

where $\Delta\lambda_i = -b_i - \lambda_i$.

In order to use the PBH test for controllability of $(A - diag(b), b)$, let us first consider the matrix $\lambda_1 I - (A - diag(b))$. It equals:

$$\begin{aligned} & diag\left(\begin{bmatrix} 0 & -b_1 + b_2 & \cdots & -b_1 + b_n \end{bmatrix}\right) - (A - \Delta\lambda_1 I) \\ &= \left[\begin{array}{c:ccc} \Delta\lambda_1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ \hdashline -a_{21} & -b_1 + b_2 + \Delta\lambda_1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -b_1 + b_n + \Delta\lambda_1 - a_{nn} \end{array}\right] \end{aligned} \tag{17}$$

Replacing the first column of (17) by b yields:

$$\begin{aligned} H_1 &\triangleq \left[\begin{array}{c:ccc} b_1 & -a_{12} & \cdots & -a_{1n} \\ \hdashline b_2 & -b_1 + b_2 + \Delta\lambda_1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & -a_{n2} & \cdots & -b_1 + b_n + \Delta\lambda_1 - a_{nn} \end{array}\right] \\ &= \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ b_2 & -b_1 + b_2 & & \vdots \\ \vdots & & \ddots & 0 \\ b_n & 0 & & -b_1 + b_n \end{bmatrix} + E_1 \\ &= \Psi_1 + E_1 \end{aligned} \tag{18}$$

where E_1 is derived by setting the first column of $\Delta\lambda_1 I - A$ to zero. Because of (16), $\Delta\lambda_1$ is bounded. So the right lower block of both (17) and (18), namely,

$$\begin{bmatrix} -b_1 + b_2 + \Delta\lambda_1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \ddots & \vdots \\ -a_{n2} & \cdots & -b_1 + b_n + \Delta\lambda_1 - a_{nn} \end{bmatrix}$$

can be strictly diagonally dominant with appropriately selected $b_1, \cdots, b_n$, then it is nonsingular and the rank is $n-1$. Accordingly,

$$rank(H_1) = rank\left([\lambda_1 I - (A - diag(b))\ b]\right)$$

It is simple to verify the fact that due to the structure of Ψ_1, it must be diagonalizable with its spectrum being

$$\{\xi_1 = b_1,\ \xi_2 = -b_1 + b_2, \ldots, \xi_n = -b_1 + b_n\}$$

Thus, Lemma 5.1 is available to (18), with E_1 regarded as perturbation. Let

$$\Lambda_1 = diag\left([b_1\ -b_1 + b_2\ \cdots\ -b_1 + b_n]\right)$$

and suppose

$$\Psi_1 = S_1 \Lambda_1 S_1^{-1}$$

Now assume that H_1 is singular, then according to Lemma 5.1, zero must be an eigenvalue of H_1, which can be regarded as being displaced from certain eigenvalue ξ_i of Ψ_1 by the perturbation E_1, with

$$|\xi_i| \le \kappa(S_1)\, \|E_1\| \tag{19}$$

It is simple to verify that S_1 is independent of the concrete values of $b_1, b_2, \cdots, b_n$. Besides, according to (16), $\|E_1\|$ is bounded with given A. Thus, if $b_1, b_2, \cdots, b_n$ are selected *s.t.*

$$|\xi_i| > \kappa(S_1)\, \|E_1\| \quad (\forall i \in \{1, 2, \ldots, n\}) \tag{20}$$

then there will be a paradox between inequalities (19) and (20) and H_1 must be nonsingular, i.e., $(\lambda_1 I - (A - diag(b)), b)$ is of rank n.

Similarly, $(\lambda_i I - (A - diag(b)), b)$ $(\forall i \in \{2, \ldots, n\})$ can also be set to be of rank n by properly selected $b_1, b_2, \cdots, b_n$. Finally as a result, $(A - diag(b), b)$ is controllable.

The proof of Theorem 5.1 is comparatively complicated; however, it implies a simple fact that for any given A, $(A - diag(b), b)$ can always be controllable so long as the absolute values of $b_1, b_2, \cdots, b_n$ and the differences among them are sufficiently large.

Theorem 5.1 and (15) naturally yield a conclusion: for any fixed W_{ff}, there exists certain $W_{fl} \in R^N$ *s.t.* (L_{ff}, L_{fl}) is controllable. In other words, Theorem 5.1 theoretically shows that by appropriately designing the leader-follower communication links, a single leader is always sufficient to guarantee the controllability of system (4) with fixed follower-follower communication links.

5.2. **Fixed leader-follower graph topology.** In this subsection, suppose the leader-follower information links are fixed and the original follower-follower information links are to be adjusted to improve the controllability of a dynamical multi-agent system (4). The approach based on a generalized controllability canonical form discussed in the subsection 4.2 is still effective. The main difference is that adjacency matrix 'W' should be replaced by the Laplacian matrix 'L'.

The main part of controllability improvement algorithm in this scenario is summarized as follows. First, transform (L_{ff}, L_{fl}) into generalized controllability canonical form (M, h) by nonsingular transformation P, with $M = PL_{ff}P^{-1}$. Second, improve the controllability of (M, h) by altering the values of linkage elements into 1 and get

$(M + \Delta M, h)$. Third, retrieve an improved L_{ff} by the same but reversed transformation P^{-1}, with $L_{ff} + \Delta L_{ff} = P^{-1}(M + \Delta M)P$. A controllable matrix pair $(L_{ff} + \Delta L_{ff}, L_{fl})$ is thus derived. Fourth, compute a new adjacency matrix W according to L.

Assume that graph G is simple, i.e., there is no loop [27], then when an improved L is derived, a unique W corresponding to such an L can be computed by Formula (21) below, which is a direct inference from the definition of Laplacian matrix (6):

$$W = -L \circ (\phi\phi^T - I) \tag{21}$$

In (21), symbol '$\circ$' denotes a Hardamard product [29]. Ultimately, the corresponding arc weights of the system can be physically adjusted with such a new adjacency matrix W.

The approach is more restrictive for dynamical multi-agent system model (4). The original L always satisfy the rule (7), but after adjustment, $L_{ff} + \Delta L_{ff}$ may possibly break it and not be physically realizable.

In order to avoid this consistency problem, let us further observe the configuration of generalized controllability canonical form (M, h). We shall find that the diagonal of M and all the elements beneath it are free to be altered without any effect on controllability. These elements can be utilized to eliminate inconsistency.

Evidently, the amount of the free elements is $r = N(1 + N)/2$. Let $y_1, y_2, \ldots, y_r \in R$ denote the variations of these free elements respectively, which are unknown variables to be determined. Let $E_{(i,j)} \in R^{N\times N}$ denote the matrix with a single nonzero element 1 at index (i, j). ΔL_{ff} can be decomposed:

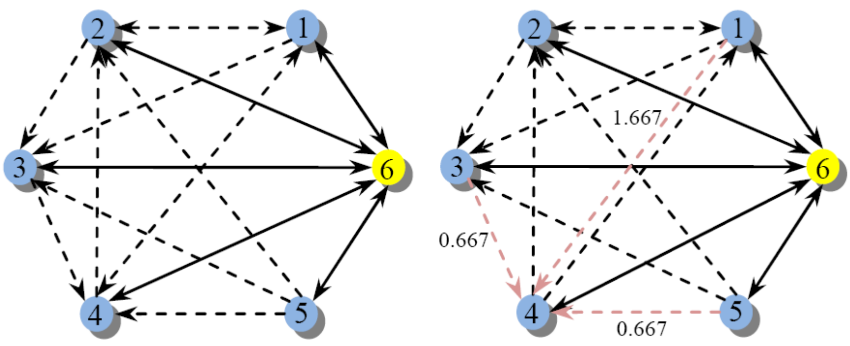
$$\Delta L_{ff} = P^{-1}\Delta MP = y_1\Phi_1 + \cdots + y_r\Phi_r + \Psi \tag{22}$$

where $\Phi_1, \ldots, \Phi_r \in R^{N\times N}$ denote the additional variations aroused by $y_1 = 1, \ldots, y_r = 1$, respectively, and Ψ the fundamental variations aroused by the linkage elements $\eta_1 = 1, \ldots, \eta_m = 1$. As an example, suppose y_1 corresponds to $E_{(1,1)}$, then $\Phi_1 = P^{-1}E_{(1,1)}P$.

To meet the rule (7), for each row of ΔL_{ff}, a scalar linear equation in $y_1, y_2, \ldots, y_r$ can be derived from the relationship $(\Delta L_{ff})\phi = 0$, which means that the sum of elements in each row is zero. The number of such an additional series of equations is N.

6. **Example.** The following example may be helpful to illustrate the technique about controllability improvement based on controllability canonical form.

Consider a system depicted by (1) with graph topology G, shown in Figure 1. In both of the two graphs of Figure 1, agents $1 \sim 5$ are the followers and agent 6 is the leader. The solid lines denote the fixed arcs between the leader and the followers while the dotted lines the adjustable arcs among the followers. The unlabeled weight of arc is 1. In Figure 1(b), the tinged lines denote the arcs adjusted.



(a) Original uncontrollable graph (b) Adjusted graph

FIGURE 1. Original and adjusted graphs

The original adjacency matrix is

$$W = \begin{bmatrix} W_{ff} & W_{fl} \\ W_{lf} & W_{ll} \end{bmatrix} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

The graph G is uncontrollable because (W_{ff}, W_{fl}) is uncontrollable. (W_{ff}, W_{fl}) will be transformed into generalized controllability canonical form by the following steps.

Step 1. Decompose it into controllable/uncontrollable subsystems by P_1:

$$(P_1 W_{ff} P_1^{-1}, P_1 W_{fl}) = \left(\left[\begin{array}{cc|ccc} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2/3 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{array}\right], \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \tag{23}$$

Step 2. Only transform the controllable subsystem in (23) into controllability canonical form by P_2:

$$(P_2 P_1 W_{ff} P_1^{-1} P_2^{-1}, P_2 P_1 W_{fl}) = \left(\left[\begin{array}{cc|ccc} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2/3 & 0 & 1 & 0 \\ 0 & -1/3 & 0 & 0 & 1 \\ 1 & 2/3 & 0 & 3 & 1 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right] \right)$$

Step 3. Compute the maximum controllability index of the upper left block in above matrix $P_2 P_1 W_{ff} P_1^{-1} P_2^{-1}$, and the value is 2. So a virtual matrix pair can be constructed as follows with an auxiliary input vector, which is already controllable and need not be decomposed anymore:

$$\left(\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Step 4. Only transform the virtual controllable subsystem into controllability canonical form by similar transformation P_3, and we obtain the generalized controllability canonical form below

$$\begin{aligned} (M, h) &= (P_3 P_2 P_1 W_{ff} P_1^{-1} P_2^{-1} P_3^{-1}, P_3 P_2 P_1 W_{fl}) \\ &= \left(\left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & \eta_1 = 0 & 0 & 0 \\ \hline 2/3 & 2/3 & 0 & 1 & 0 \\ -1/3 & -1/3 & 0 & 0 & 1 \\ 5/3 & 2/3 & 0 & 3 & 1 \end{array}\right], \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Note that the entire similar transformation is $P = P_3 P_2 P_1$.

The index of η_1 in M is (2,3). To improve the controllability of the system, let the value of η_1 be altered from 0 to 1: $\Delta M = E_{(2,3)}$. Evidently, $(M + \Delta M, h)$ is controllable.

It follows that,

$$\Delta W_{ff} = P^{-1}\Delta M P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & -1/3 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System $(W_{ff} + \Delta W_{ff}, W_{fl})$ is also controllable in accordance with $(M + \Delta M, h)$. The result is illustrated by Figure 1.

7. **Conclusions.** This paper presented approaches to improving controllability of two types of LTI dynamical multi-agent system models by adjusting the configurations of graphs. The difference between the two models mainly lies in the form of interactive dynamics among neighboring agents. As theoretical foundation, the concept of maximum controllability index for square matrices was introduced. It was shown that the maximum controllability index of a matrix is equal to the degree of its minimal polynomial. Approaches to adjusting graph topology were described respectively for two specific scenarios: fixed follower-follower topology and fixed leader-follower topology. For dynamical multi-agent system model I, if the follower-follower topology is fixed and the corresponding fixed adjacency matrix is derogatory, the system cannot be adjusted to be controllable with only one leader. If the leader-follower topology is fixed, the corresponding uncontrollable dynamic system should be transformed into a generalized controllability canonical form, which can be regarded as a series of subsystems. The adjustment can be made by rebuilding connections for the information to pass through these subsystems. For dynamical multi-agent system model II, if the follower-follower topology is fixed, in contrast, the system can always be adjusted to be controllable with only one leader. If the leader-follower topology is fixed, controllability can be improved by the same technique for model I, with considering some more restrictions. The current results are mainly limited in two aspects: 1) The controllability improvement problem should be handled in scenarios with either fixed follower-follower topology or leader-follower topology; 2) If the leader-follower topology is fixed, only systems with a single leader can be dealt with. These limitations will be the emphasis of our future work. For example, the limitation of a single leader may be removed by developing a generalized controllability canonical form for the multi-input case.

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