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ITERATED CONSTRAINED STATE ESTIMATOR FOR NONLINEAR DISCRETE-TIME SYSTEMS WITH UNCERTAIN PARAMETERS

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Abstract. *In this paper, an estimator is developed to estimate the states of nonlinear stochastic discrete-time dynamical systems with uncertain parameters. The system model and the measurements are assumed to be corrupted by uncorrelated zero mean white Gaussian noise sequences. The parameters of the system are assumed to be uncertain. The proposed approach is based on the extended Kalman filter and the active set method, in which multiple projection approach is used to get the dynamics of the proposed estimator. Although the developed state estimator uses the nominal values of the system parameters, it shows to be more stable when compared with other existing techniques and gives satisfactory results. To illustrate the effectiveness and simplicity of the developed approach, an illustrative example is presented. Simulation results show that the developed technique leads to a stable state estimator even in many cases in which the extended Kaman filter and the extended Kalman filter with parameter estimation diverge.*
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1. **Introduction.** The well-known Kalman filter (KF) has been widely used in many areas. It has been proved that it is the optimal minimum variance state estimator for linear dynamical systems with Gaussian noise; and it is the optimal linear estimator for linear dynamical systems with non-Gaussian noise [1]. On the other hand, several algorithms have been proposed to estimate the states of nonlinear dynamical systems. Among these algorithms, the Moving Horizon estimation [2], the particle filter [3], the ensemble Kalman filter [4], the unscented Kalman filter [5] and the extended Kalman filter (EKF) [6], generally, the EKF is the most extensively used algorithm for state estimation of nonlinear systems, especially those characterized by their weak nonlinearities [7]. On the other hand, unlike the linear counter part, it is not the optimal estimator. In addition, it may quickly diverge if the system acquires strong nonlinearities [8,9], or if the initial estimate of the states is wrong, or the process is modeled incorrectly, owing to its linearization.

In many applications of state estimation problems some equality and/or inequality constraints are imposed on the states of the system. These constraints are due to physical or practical considerations. The KF and the EKF do not incorporate such information and hence they lead to a sub-optimal state estimator. In these situations it is necessary to modify the estimator resulting from KF or EKF to incorporate such additional information.

Many researchers have worked on the development of different techniques to estimate the constrained states of linear and nonlinear systems, see for example [9-43] and the references therein for some of the works on constrained state estimation. For linear systems, in one approach the equality constraints are considered as perfect (noise-free) measurements [10]. However, this technique may lead to some computational problems because of the singularity of the covariance matrix. In another approach, the model reduction technique is used to handle estimation problems with equality constraints [11]. However, this technique eliminates the physical meaning of some of the state variables. In a third approach, a quadratic optimization problem is formulated and solved at the sampling instants at which the state constraints are violated [12,14-16]. This technique can be classified as a special case of the projection approach. Recently, Teixeira et al. derived an estimator with state interval constraints [13] as well as several algorithms for linear and nonlinear state estimation with equality constraints on the states [21,22]. These estimators are based on the unscented Kalman filter. On the other hand, for nonlinear systems the moving horizon estimation technique is widely used for state estimation with equality and inequality constraints [9]. In [32,33] a procedure is proposed to include state inequality constraints in the unscented Kalman Filter. Kolas et al. [7] studied the use of the unscented Kalman filter for nonlinear state estimation. Sircoulomb et al. [34] introduced a projection approach for nonlinear state estimation when the states are constrained with nonlinear state soft inequality constraints. Dolence and Ungarala [9] derived a constrained EKF for the estimation of the states of nonlinear systems with equality and inequality constraints on the states, and the proposed estimator is computationally efficient because it does not require matrix inversion. Recently, in [44] the active set method and the multiple projection approach [45-48] are used to solve this problem. The proposed recursive technique showed to be simple and computationally efficient.

State estimation for nonlinear discrete-time dynamical systems with strong nonlinearities and uncertain parameters is a challenging problem due to the following:

1) Strong nonlinearities may lead to a quick divergence of the EKF. However, due to the fact that the estimated states resulting from the EKF are suboptimal and hence they are not orthogonal to the estimation error, such a problem can be improved by using the iterative Kalman filter (IKF) in which the projection operation is repeated several times at each sampling instant. This process increases the computational time by a factor equal to the number of repetition of the projection operations to be executed at each sampling instant.

2) Ignoring the uncertainty of the model parameters increases the tendency of divergence of the EKF. On the other hand, if the parameters of the model are estimated, the dimensionality of the system model will increase by the number of parameters to be estimated. Such a process will also increase the computational time as well as the numerical unstability of the estimator.

In this paper, we address the state estimation problem of stochastic nonlinear discrete-time dynamical systems with strong nonlinearities and uncertain parameters. Based on the fact that the EKF estimator and its associated estimation error are not orthogonal, a technique is proposed to solve this problem in a simple and recursive approach.

To clarify the idea used in our development, let us consider the estimation problem for nonlinear systems with uncertain parameters. Assume for the moment that the output measurements are noise free. In this case, the best estimator is the one that leads to an estimate of the output vector equaling the measured one. In general, this is not possible due to the input noise and the stochastic nature of the problem. However, it is still possible to get a better estimator if we can impose bounds on the differences between the actual and estimated outputs, element by element, and maintain the estimated outputs within these bounds. In other words, if a subset of the estimated output vector starts to diverge at any sampling instant of time and by some means the estimated states are modified such that this subset of the estimated output vector is maintained within the bounded region, it is expected that the stability of the estimator will be improved. In case of noisy measurements, the situation is different. If the imposed bounds are such that the estimated outputs are very close to the measured ones, we may get a worst estimator, especially for cases in which the level of the output measurement noise is high. On the other hand, it is still desired to have the estimated outputs bounded within a pre-specified region around the measured ones in order to get better performance of the estimator.

The proposed estimator is developed by extending the idea of constrained state estimation [44] to enforce constraints on the innovation. It consists of two stages. In the first stage, the standard EKF is used to estimate the states. If the imposed constraints are satisfied, we proceed to the next sampling instant of time. Otherwise, the subset of violated constraints is identified. The measurements corresponding to this subset are used to generate a new subset of equality constraints to be satisfied. This subset of equality constraints is treated as a new received vector of measurement. By applying the multiple projection approach, the assumed new received measurement vector is used to update the estimator. This procedure leads to an iterative estimator very close to IKF [49].

The rest of the paper is organized as follows. In Section 2 the constrained estimation problem for stochastic nonlinear discrete-time dynamical systems with uncertain parameters is formulated. Section 3 is devoted to the presentation of the dynamics of the developed estimator for nonlinear systems with uncertain parameters. The convergence of the update phase of the nonlinear estimator is discussed in Section 4. The algorithm used to implement the proposed estimator is presented in Section 5. In Section 6 an illustrative example is solved using the proposed estimator, the EKF and the EKF with parameter estimation. The results show good convergence behavior of the developed estimator even in the cases at which the EKF and the EKF with parameter estimation diverge. The paper is concluded in Section 7.

2. **Problem Statement.** Consider the following stochastic nonlinear discrete-time dynamical systems with uncertain parameters:

$$x_{k+1} = f_k(x_k, b_k) + w_k \tag{1}$$

$$y_{k+1} = h_{k+1}(x_{k+1}, c_{k+1}) + v_{k+1} \tag{2}$$

where $x_k \in R^n$ is the state vector, $y_k \in R^m$ is the output vector, $w_k \in R^n$ and $v_{k+1} \in R^m$ are, respectively, zero mean white Gaussian input and output noise vectors with covariance matrices $Q_k = E\{w_k w_k^T\} \in R^{n\times n}$ and $R_{k+1} = E\{v_{k+1} v_{k+1}^T\} \in R^{m\times m}$, $f_k(x_k, b_k) : R^n \to R^n$ is a vector nonlinear function of the state equation, $h_{k+1}(x_{k+1}, c_{k+1}) : R^n \to R^m$ is a vector nonlinear function of the output measurements, $x_o \in R^n$ is the initial conditions of the states assumed zero mean random Gaussian vector with covariance matrix $P_o = P_{o|o} = E\{x_o x_o^T\} \in R^{n\times n}$, $b_k \in R^l$ is the parameter vector of the system model assumed uncorrelated white Gaussian random variables with mean $\bar{b}$, equals the nominal values of the model parameters, and a covariance matrix $\Gamma_k = E\{(b_k - \bar{b})(b_k - \bar{b})^T\}$, $c_{k+1} \in R^r$ is the parameter vector of the output model assumed uncorrelated white Gaussian random variables with mean $\bar{c}$, equals the nominal values of the output model parameters, and a covariance matrix $U_{k+1} = E\{(c_{k+1} - \bar{c})(c_{k+1} - \bar{c})^T\}$. Finally $k \in \{0, 1, 2, \ldots\}$ is the discrete time instant.

In the above model it is assumed that x_o, b_k, c_{k+1}, w_k, v_{k+1} are all independent. Moreover, it is assumed that the model has the following properties:

$$\begin{array}{lll} E\{x_k w_j^T\} = 0, \quad \forall\, k \le j; & E\{x_k v_j^T\} = 0, \quad \forall\, k, j; & E\{y_k v_j^T\} = 0, \quad \forall\, k < j \\ E\{x_k b_j^T\} = 0, \quad \forall\, k \le j; & E\{x_k c_j^T\} = 0, \quad \forall\, k, j; & E\{y_k c_j^T\} = 0, \quad \forall\, k < j \\ E\{b_k c_j^T\} = 0, \quad \forall\, k, j & & \end{array} \tag{3}$$

Let $Y^{k+1} = [y_1^T, y_2^T, \ldots, y_{k+1}^T]^T$ be the vector of measurement up to the instant $k+1$. Our objective is to get the estimate of the state vector $\hat{x}_{k+1|k+1}$ which:

$$\min J = E\{[x_{k+1} - \hat{x}_{k+1|k+1}]^T [x_{k+1} - \hat{x}_{k+1|k+1}] | Y^{k+1}\} \tag{4a}$$

subject to the system model (1) and (2). Moreover, the estimator has to satisfy the set of imposed inequality constraints on the differences between the actual and estimated outputs given by:

$$\underline{\zeta} \le y_{k+1} - \hat{y}_{k+1|k+1} \le \bar{\zeta} \tag{4b}$$

where $\hat{y}_{k+1|k+1} = E\{y_{k+1}|Y^{k+1}\}$ is the estimate of the output vector, $\underline{\zeta}$, $\bar{\zeta} \in R^m$ are, respectively, the lower and upper bounds of the inequality constraints element by element.

However, due to the nonlinearity of the problem, and hence the non-Gaussian nature of its probability distribution function, such an optimal estimator does not exist. Therefore, we are looking for a suboptimal state estimator which has better convergence properties than EKF and EKF with parameter estimation.

The main results to be derived in the rest of the paper are based on the following assumptions and approximations which are usually used with EKF.

1) Although nonlinear systems are non-Gaussian by their nature, they will be treated as Gaussian.

2) The covariance matrices of the predicted estimation error $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ (where $\hat{x}_{k+1|k} = E\{x_{k+1}|Y_k\}$), the filtered estimation error $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$ (where $\hat{x}_{k+1|k+1} = E\{x_{k+1}|Y_{k+1}\}$), and the Kalman gain matrix will be calculated from the linearized model around the last estimated state and the nominal values of the parameters. Higher order terms in Tailor series expansion will be neglected.

3) Although the filtered estimate $\hat{x}_{k+1|k+1}$ resulting from EKF is suboptimal, which means that $\hat{x}_{k+1|k+1}$, $\tilde{x}_{k+1|k+1}^T$ and $\hat{x}_{k+1|k+1}$, $\tilde{Y}_{k+1|k+1}^T$ are not orthogonal (i.e., $E\{\hat{x}_{k+1|k+1} \tilde{x}_{k+1|k+1}^T\} \neq 0$, $E\{\hat{x}_{k+1|k+1}\tilde{Y}_{k+1|k+1}^T\} \neq 0$), they will be approximated and treated as orthogonal (i.e., $E\{\hat{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T\} \cong 0$, $E\{\hat{x}_{k+1|k+1}\tilde{Y}_{k+1|k+1}^T\} \cong 0$).

4) Again, although $\hat{x}_{k+1|k+1}$, $\tilde{x}_{k+1|k+1}^T$ are not orthogonal which means that $P_{x_{k+1}\tilde{y}_{k+1|k}} \neq P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}}$, in gain calculation $P_{x_{k+1}\tilde{y}_{k+1|k}}$ will be approximated and treated as for linear systems in which we adopt the relation $P_{x_{k+1}\tilde{y}_{k+1|k}} \cong P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}}$.

5) It will be assumed that the conditional probability density function $p(x_k|Y^k)$ is approximated by Gaussian distribution with known conditional mean $\hat{x}_{k|k}$ and covariance matrix $P_{k|k}$ at the k^{th} sampling instant.

3. Development of the Constrained Estimator.

The main idea is to enforce an interval constraint on the innovation vector to satisfy (4b) while using a successive projection approach. This leads to an iterative estimator similar to that used by IKF [49].

3.1. The dynamics of the EKF with uncertain parameters.

Relaxing for the moment the set of imposed inequality constraints (4b), let us consider the estimation problem for the uncertain nonlinear discrete-time dynamical system described by Equations (1) and (2). For the reasons stated above, the modified dynamics to be developed for the unconstrained EKF leads to a suboptimal estimator through the linearization of the nonlinear dynamics around the last state estimate and the nominal values of the system parameters.

Let F_k, B_k, H_{k+1}, D_{k+1} be the Jacobean matrices of f_k and h_{k+1} defined by:

$$F_k = \frac{\partial f_k}{\partial x_k}\Big|_{\hat{x}_{k|k},\bar{b}} ;\; B_k = \frac{\partial f_k}{\partial b_k}\Big|_{\hat{x}_{k|k},\bar{b}} ;\; H_{k+1} = \frac{\partial h_{k+1}}{\partial x_{k+1}}\Big|_{\hat{x}_{k+1|k},\bar{c}} ;\; D_{k+1} = \frac{\partial h_{k+1}}{\partial c_{k+1}}\Big|_{\hat{x}_{k+1|k},\bar{c}} \quad (5)$$

Neglecting second and higher order terms in Tailor series expansion, the modified dynamics of the unconstrained EKF are given by the following lemma:

Lemma 3.1. *a) The predicted estimate of the state vector $\hat{x}_{k+1|k}$, its associated covariance matrix $P_{k+1|k}$, and the predicted estimate of the output vector $\hat{y}_{k+1|k}$ are given by:*

$$\hat{x}_{k+1|k} \cong f_k(\hat{x}_{k|k}, \bar{b}) \quad (6)$$

$$P_{k+1|k} \cong F_k P_{k|k} F_k^T + B_k \Gamma_k B_k^T + Q_k \quad (7)$$

$$\hat{y}_{k+1|k} \cong h_{k+1}(\hat{x}_{k+1|k},\, \bar{c}) \quad (8)$$

b) The filtered estimate of the state vector $\hat{x}_{k+1|k+1}$, its associated covariance matrix $P_{k+1|k+1}$ and the filtered estimate of the output vector $\hat{y}_{k+1|k+1}$ are such that:

$$\hat{x}_{k+1|k+1} \cong \hat{x}_{k+1|k} + K_{1\,k+1}[y_{k+1} - \hat{y}_{k+1|k}] \quad (9)$$

$$P_{k+1|k+1} \cong [I - K_{1k+1} H_{k+1}] P_{k+1|k} \quad (10)$$

with

$$K_{1k+1} \cong P_{k+1|k} H_{k+1}^T [H_{k+1} P_{k+1|k} H_{k+1}^T + D_{k+1} U_{k+1} D_{k+1}^T + R_{k+1}]^{-1} \quad (11)$$

$$\hat{y}_{k+1|k+1} \cong h_{k+1}(\hat{x}_{k+1|k+1},\, \bar{c}) \quad (12)$$

Proof: Linearizing the system dynamics $f_k(x_k, b_k)$ around $\hat{x}_{k|k}$, $\bar{b}$, while neglecting higher order terms, we have:

$$f_k(x_k, b_k) \cong f_k(\hat{x}_{k|k}, \bar{b}) + F_k[x_k - \hat{x}_{k|k}] + B_k(b_k - \bar{b}) + w_k \quad (13)$$

Substituting (13) into (1), one gets:

$$x_{k+1} \cong f_k(\hat{x}_{k|k}, \bar{b}) + F_k[x_k - \hat{x}_{k|k}] + B_k(b_k - \bar{b}) + w_k \quad (14)$$

The conditional expectation of (14) knowing Y^k is given by:

$$\hat{x}_{k+1|k} \cong E\{[f_k(\hat{x}_{k|k}, \bar{b}) + F_k[x_k - \hat{x}_{k|k}] + B_k(b_k - \bar{b}) + w_k]|Y^k\} \quad (15)$$

Since $\hat{x}_{k|k} = E\{x_k|Y^k\}$, the variables Y^k, w_k, b_k are independent, $E\{w_k\} = 0$, and $E\{b_k\} = \bar{b}$, then the predicted estimate of $\hat{x}_{k+1|k}$ takes the form:

$$\hat{x}_{k+1|k} \cong f_k(\hat{x}_{k|k}, \bar{b}) \quad (16)$$

Let $\tilde{x}_{k|k} = x_k - \hat{x}_{k|k}$; $\tilde{x}_{k+1|k} = x_k - \hat{x}_{k+1|k}$; $\tilde{b}_k = b_k - \bar{b}$, then from (14) and (16) we get:

$$\tilde{x}_{k+1|k} \cong F_k \tilde{x}_{k|k} + B_k \tilde{b}_k + w_k \quad (17)$$

The covariance matrix of the resulting error is given by:

$$P_{k+1|k} = E[\tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T] \quad (18)$$

Substituting from (17) into (18) and with the use of the model properties (3) (i.e., $\tilde{b}_k$, w_k are independent, x_k is independent of b_k, w_k and $E\{\tilde{b}_k\} = E\{b_k - \bar{b}\} = 0$) one gets, after simple mathematical manipulation, $P_{k+1|k}$ in the form:

$$P_{k+1|k} \cong F_k P_{k|k} F_k^T + B_k \Gamma_k B_k^T + Q_k \quad (19)$$

Equation (16) gives the predicted estimate of x_{k+1} denoted by $\hat{x}_{k+1|k}$, while (19) gives the covariance matrix of the prediction error $P_{k+1|k}$.

To get the filtered estimate, assume that the system measurement y_{k+1} is received at the sampling instant $k+1$. Hence, the predicted estimate $\hat{x}_{k+1|k}$ will be updated taking into consideration the new measurements. Such a filtered estimate is given by:

$$\hat{x}_{k+1|k+1} = E\{x_{k+1}|Y^k, y_{k+1}\} \tag{20}$$

where

$$y_{k+1} = h_{k+1}(x_{k+1}, c_{k+1}) + v_{k+1} \tag{21}$$

Linearizing (21) around $\hat{x}_{k+1|k}$, $\bar{c}$ while neglecting higher order terms, one gets:

$$y_{k+1} \cong h_{k+1}(\hat{x}_{k+1|k}, \bar{c}) + H_{k+1}\tilde{x}_{k+1|k} + D_{k+1}\tilde{c}_{k+1} + v_{k+1} \tag{22}$$

where $\tilde{c}_{k+1} = c_{k+1} - \bar{c}$.

Defining $\hat{y}_{k+1|k}$ as:

$$\hat{y}_{k+1|k} = E\{y_{k+1}|Y^k\} = E\{h_{k+1}(x_{k+1}, c_{k+1}) + v_{k+1}|Y^k\}$$

Since Y^k is independent of c_{k+1}, v_{k+1} while $E\{v_{k+1}\} = 0$ and $E\{c_{k+1}\} = \bar{c}$, we get:

$$\hat{y}_{k+1|k} \cong h_{k+1}(\hat{x}_{k+1|k}, \bar{c}) \tag{23}$$

Let $\tilde{y}_{k+1|k} = y_{k+1} - \hat{y}_{k+1|k}$, then from (22) and (23) $\tilde{y}_{k+1|k}$ is such that:

$$\tilde{y}_{k+1|k} \cong H_{k+1}\tilde{x}_{k+1|k} + D_{k+1}\tilde{c}_{k+1} + v_{k+1} \tag{24}$$

From (20), while using the properties of conditional expectation of Gaussian random variables, we have:

$$\begin{aligned}
\hat{x}_{k+1|k+1} &= E\{x_{k+1}|Y^k, y_{k+1}\} \cong E\{x_{k+1}|Y^k, \tilde{y}_{k+1|k}\} \\
&\cong E\{x_{k+1}|Y^k\} + E\{x_{k+1}|\tilde{y}_{k+1|k}\} \\
\hat{x}_{k+1|k+1} &\cong \hat{x}_{k+1|k} + K_{1k+1}\tilde{y}_{k+1|k}
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
K_{1k+1} &\cong P_{x_{k+1}\tilde{y}_{k+1|k}} P^{-1}_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} \cong P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}} P^{-1}_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} \\
P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}} &= E\{\tilde{x}_{k+1|k}\tilde{y}^T_{k+1|k}\} \\
K_{1k+1} &\cong E\{\tilde{x}_{k+1|k}[H_{k+1}\tilde{x}_{k+1|k} + D_{k+1}\tilde{c}_{k+1} + v_{k+1}]^T\}
\end{aligned}$$

Since $E\{\tilde{c}_{k+1}\} = 0$, $E\{v_{k+1}\} = 0$ and $\tilde{x}_{k+1|k}$ is independent of both $\tilde{c}_{k+1}$, v_{k+1}, we get:

$$P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}} = P_{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}} H^T_{k+1} \tag{26}$$

However, $P_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}}$ is such that:

$$P_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} = E\{\tilde{y}_{k+1|k}\tilde{y}^T_{k+1|k}\}$$

Using (24) and recalling that $\tilde{x}_{k+1|k}$, $\tilde{c}_{k+1}, v_{k+1}$ are mutually independent, we get:

$$P_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} \cong H_{k+1}P_{k+1|k}H^T_{k+1} + D_{k+1}U_{k+1}D^T_{k+1} + R_{k+1} \tag{27}$$

Using (26) and (27), the gain matrix K_{1k+1} takes the form:

$$K_{1k+1} \cong P_{k+1|k}H^T_{k+1}[H_{k+1}P_{k+1|k}H^T_{k+1} + D_{k+1}U_{k+1}D^T_{k+1} + R_{k+1}]^{-1} \tag{28}$$

The covariance matrix of the filtered estimate $P_{k+1|k+1}$ is given by:

$$P_{k+1|k+1} = E\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}\}$$

where the error $\tilde{x}_{k+1|k+1}$ is defined by:

$$\begin{aligned}
\tilde{x}_{k+1|k+1} &= x_{k+1} - \hat{x}_{k+1|k+1} \\
&= \tilde{x}_{k+1|k} - K_{1k+1}\tilde{y}_{k+1|k}
\end{aligned}$$

From the definitions of $P_{k+1|k+1}$, $\tilde{x}_{k+1|k+1}$ and after simple mathematical manipulation, one gets:

$$P_{k+1|k+1} \cong [I - K_{1k+1}H_{k+1}]P_{k+1|k} \tag{29}$$

By expanding $h_{k+1}(x_{k+1}, c_{k+1})$ around $\hat{x}_{k+1|k+1}$ and $\bar{c}$, it is easy to get the approximate estimate of the output vector $\hat{y}_{k+1|k+1}$ as given by (12).

3.2. **Derivation of the nonlinear update estimator.** As stated above, the different assumptions and approximations used with EKF not only lead to a suboptimal estimator $\hat{x}_{k+1|k+1}$, but may also lead to the divergence of EKF. Moreover, if the model is unprecious and/or the model parameters are random, the situation becomes worst. In this case, the probability of the divergence of EKF becomes higher, even when used with weak nonlinear systems. The stability of the estimator can be improved if the estimated output vector is enforced to be bounded, element by element, by imposing the set of inequality constraints (4b), which when violated enforces an interval constraint on the innovation vector.

In order to clarify our idea, let us assume that a subset of s-elements of the estimated output vector $\hat{y}_{k+1|k+1}$, as given by (12), violates the constraints (4b) at the sampling instant $k+1$. It is, therefore, necessary to saturate the violated constraints to the upper or lower levels of the violated bounds. Let:

$$z_{k+1} = y'_{k+1} - \theta_{k+1}; \quad \hat{x}_{1k+1|k+1} = \hat{x}_{k+1|k+1}; \quad P_{1k+1|k+1} = P_{k+1|k+1} \tag{30}$$

where $y'_{k+1} \in R^s$ is the output measurement sub-vector corresponding to the violated constraints, $\theta_{k+1} \in R^s$ is a vector that contains the corresponding violated elements of the upper bound vector $\bar{\zeta}$ or the lower bound vector $\underline{\zeta}$ and $\hat{x}_{k+1|k+1}$, $P_{k+1|k+1}$ are as given by (25) and (29) respectively.

Our objective is to get an update estimator $\hat{x}_{k+1|k+1}$ such that the following equality constraint is satisfied:

$$z_{k+1} = g_{k+1}(\hat{x}_{k+1|k+1}, \bar{c}') \tag{31}$$

where $g_{k+1}(\hat{x}_{k+1|k+1}, \bar{c}') : R^n \to R^s$ is a vector nonlinear function that contains the s-elements of $h_{k+1}(x_{k+1}, c_{k+1})$ corresponding to the s-violated constraints from (4b) after replacing x_{k+1} by its desired estimated value $\hat{x}_{k+1|k+1}$ and c_{k+1} by the nominal values of the subset of parameters $\bar{c}'$ included in $g_{k+1}(x_{k+1}, c'_{k+1})$.

Let $\Delta x_{k+1|k+1} = \hat{x}_{k+1|k+1} - \hat{x}_{1k+1|k+1}$, then the first two terms of Tailor series expansion of (31) takes the form:

$$z_{k+1} = g_{k+1}(\hat{x}_{1k+1|k+1}, \bar{c}') + \left.\frac{\partial g_{k+1}(x, \bar{c}')}{\partial x}\right|_{x=\hat{x}_{1k+1|k+1}} \Delta\hat{x}_{k+1|k+1} \tag{32}$$

Since $g_{k+1}(\hat{x}_{1k+1|k+1}, \bar{c}') = \hat{y}'_{k+1|k+1}$, then we have:

$$\tilde{z}_{k+1|k+1} = z_{k+1} - \hat{y}'_{k+1|k+1} = \left.\frac{\partial g_{k+1}(x, \bar{c}')}{\partial x}\right|_{x=\hat{x}_{1k+1|k+1}} \Delta x_{k+1|k+1} \tag{33}$$

In (33), $\tilde{z}_{k+1|k+1}$ can be calculated by using z_{k+1} as defined by (30) and the estimate of the subset of the violated output vector $\hat{y}'_{k+1|k+1}$. Our objective is to get an estimate of $\Delta x_{k+1|k+1}$ to satisfy (33). One of the procedures to solve this problem is to formulate an optimization problem in which we minimize a quadratic cost function of $\Delta x_{k+1|k+1}$ subject to the equality constraint (33) [14-16]. Although this approach is feasible, it leads to an unstable estimator, at least in the illustrative example given in Section 6.

The idea of our approach is to get an estimate of $\Delta x_{k+1|k+1}$ as a linear function of $\tilde{z}_{k+1|k+1}$. Since the last estimate $\hat{x}_{1k+1|k+1}$, the statistical information of the output model uncertainties and the measurement noise have direct impact on the results, they have to be included in the projection operator to be applied on $\tilde{z}_{k+1|k+1}$. Moreover, since the system is nonlinear, and due to the different approximations used to get (9)-(12), such an estimate is calculated iteratively through successive application of the projection approach till the satisfaction of the constraints (4b) within a pre-specified accuracy.

Since $\Delta x_{k+1|k+1}$ is not a state variable, it is desired to define z_{k+1} as function of x_{k+1} such that:

a) $\hat{z}_{k+1|k+1} = \hat{y}'_{k+1|k+1}$, and hence $\tilde{z}_{k+1|k+1}$ is as given by (33).

b) The projection operator to be used to update the estimator incorporates the statistical information of the output model uncertainties and the measurement noise.

c) The violated upper or lower bounds are used to enforce an interval constraint on the innovation vector.

Therefore, we define z_{k+1} as:

$$z_{k+1} = g_{k+1}(x_{k+1}, c'_{k+1}) + v'_{k+1} \tag{34}$$

where z_{k+1}, y'_{k+1} and θ_{k+1} are as defined above, $g_{k+1}(x_{k+1}, c'_{k+1}) : R^n \to R^s$ is a vector nonlinear function that contains the elements of $h_{k+1}(x_{k+1}, c_{k+1})$ corresponding to the violated constraints in (4b), c'_{k+1} is the subset of parameters included in $g_{k+1}(x_{k+1}, c'_{k+1})$, $v'_{k+1} \in R^s$ is a vector that contains the corresponding elements of the output noise vector v_{k+1}.

Since z_{k+1} can be calculated, Equation (34) will be treated as a new received subset of measurements. Therefore, the new estimation problem is to get an estimator for the system (1) using the measurements (2) and (34) while satisfying the constraints (4b).

Again, let us denote by $\hat{x}_{1k+1|k+1}$, $P_{1k+1|k+1}$, $\hat{y}_{1k+1|k+1}$ as, respectively, the filtered estimate (9), the covariance matrix (10), and the estimated output vector (12) resulting from the unconstrained EKF. The following lemma gives the dynamics of the constrained estimator (update estimator) for the nonlinear discrete-time system:

Lemma 3.2. *The updated filtered estimate $\hat{x}_{k+1|k+1}$, the covariance matrix of the constrained estimator $P_{k+1|k+1}$, and the estimate of the assumed new subset of measurements $\hat{z}_{k+1|k+1}$, are computed such as:*

$$\hat{x}_{k+1|k+1} \cong \hat{x}_{1k+1|k+1} + K_{2\,k+1}[z_{k+1} - \hat{z}_{k+1|k+1}] \tag{35}$$

$$P_{k+1|k+1} \cong [I - K_{2k+1}G_{k+1}]P_{1k+1|k+1} \tag{36}$$

$$K_{2k+1} \cong P_{1k+1|k+1}G^T_{k+1}[G_{k+1}P_{1k+1|k+1}G^T_{k+1} + M_{k+1}U'_{k+1}M^T_{k+1} + R'_{k+1}]^{-1} \tag{37}$$

$$\hat{z}_{k+1|k+1} \cong E\{z_{k+1}|Y^k, y_{k+1}\} \cong g_{k+1}(\hat{x}_{1k+1|k+1}, \bar{c}') \tag{38}$$

where

$$G_{k+1} = \frac{\partial g_{k+1}}{\partial x_{k+1}}\Big|_{\hat{x}_{1k+1|k+1},\bar{c}'}\ ; \quad M_{k+1} = \frac{\partial g_{k+1}}{\partial c_{k+1}}\Big|_{\hat{x}_{1k+1|k+1},\bar{c}'} \tag{39}$$

$$R'_{k+1} = E\{v'_{k+1}v'^T_{k+1}\}; \quad U'_{k+1} = E\{(c'_{k+1} - \bar{c})(c'_{k+1} - \bar{c})^T\} \tag{40}$$

Proof: Due to system nonlinearities, the estimator $\hat{x}_{k+1|k+1}$ is approximated by:

$$\begin{aligned}\hat{x}_{k+1|k+1} &\cong E\{x_{k+1}|Y^k, y_{k+1}, \tilde{z}_{k+1|k+1}\} \\ &\cong E\{x_{k+1}|Y^k, y_{k+1}\} + E\{x_{k+1}|\tilde{z}_{k+1|k+1}\} \\ \hat{x}_{k+1|k+1} &\cong \hat{x}_{1k+1|k+1} + K_{2k+1}\,\tilde{z}_{k+1|k+1}\end{aligned} \tag{41}$$

where

$$\tilde{z}_{k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{42}$$

and

$$K_{2k+1} \cong P_{x_{k+1}\tilde{z}_{k+1|k+1}}P^{-1}_{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}} \cong P_{\tilde{x}_{1k+1|k+1}\tilde{z}_{k+1|k+1}}P^{-1}_{\tilde{z}_{k+1|k+1}\tilde{z}_{k+1|k+1}} \tag{43}$$

Equation (34) can be rewritten in the form:

$$z_{k+1} = g_{k+1}(x_{k+1}, c'_{k+1}) + v'_{k+1}$$
$$z_{k+1} = g_{k+1}(\hat{x}_{1k+1|k+1} + \tilde{x}_{1k+1|k+1}, \bar{c}' + \tilde{c}'_{k+1}) + v'_{k+1} \tag{44}$$

Linearizing (44) around $\hat{x}_{1k+1|k+1}$, $\bar{c}'$ while neglecting higher order terms, we have:

$$z_{k+1} \cong g_{k+1}\left(\hat{x}_{1k+1|k+1}, \bar{c}'\right) + G_{k+1}\tilde{x}_{1k+1|k+1} + M_{k+1}\tilde{c}'_{k+1} + v'_{k+1} \tag{45}$$

Substituting from (38) and (45) into (42), one gets:

$$\tilde{z}_{k+1|k+1} \cong G_{k+1}\tilde{x}_{1k+1|k+1} + M_{k+1}\tilde{c}_{k+1} + v'_{k+1} \tag{46}$$

It is clear that $\tilde{x}_{1k+1|k+1}$ depends on y_{k+1} which in turn depends on c_{k+1} and v_{k+1}. However, in order to get expressions for $P_{\tilde{x}_{1k+1|k+1}\tilde{z}_{k+1|k+1}}$, $P_{\tilde{z}_{k+1|k+1}\tilde{z}_{k+1|k+1}}$, an approximation is made through which it is assumed that $\tilde{x}_{1k+1|k+1}$ is independent on c_{k+1} and v_{k+1}. Therefore, using (46), the covariance matrices $P_{\tilde{x}_{1k+1|k+1}\tilde{z}_{k+1|k+1}}$, $P_{\tilde{z}_{k+1|k+1}\tilde{z}_{k+1|k+1}}$ are such that:

$$P_{\tilde{x}_{1k+1|k+1}\tilde{z}_{k+1|k+1}} \cong P_{\tilde{x}_{1k+1|k+1}\tilde{x}_{1k+1|k+1}} G^T_{k+1} \tag{47}$$
$$P_{\tilde{z}_{k+1|k+1}\tilde{z}_{k+1|k+1}} \cong G_{k+1}P_{1k+1|k+1}G^T_{k+1} + M_{k+1}U'_{k+1}M^T_{k+1} + R'_{k+1} \tag{48}$$

With direct substitution from (47) and (48) into (43), one gets the equation for the gain matrix K_{2k+1}, as given by (37).

To get an expression for $P_{k+1|k+1}$, we have:

$$P_{k+1|k+1} = E\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}\} \tag{49}$$

Using (41) and the definition of $\tilde{x}_{k+1|k+1}$ as $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$, we get:

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{1k+1|k+1} - K_{2k+1}\,\tilde{z}_{k+1|k+1} \tag{50}$$

Substituting from (50) into (49), $P_{k+1|k+1}$ takes the form:

$$P_{k+1|k+1} = [I - K_{2k+1}G_{k+1}]P_{1k+1|k+1} \tag{51}$$

which proves the assertion.

Now, the estimated output is such that:

$$\hat{y}_{k+1|k+1} = h_{k+1}(\hat{x}_{k+1|k+1}, \bar{c}) \tag{52}$$

If the constraints are satisfied, we proceed to the next sampling instant of time. Otherwise, we let $\hat{x}_{1k+1|k+1} = \hat{x}_{k+1|k+1}$; $P_{1k+1|k+1} = P_{k+1|k+1}$ and then repeat the same procedure till the satisfaction of the imposed constraints.

Therefore, the proposed estimation algorithm consists of the following two phases:

***Phase* I:** In this phase, the filtered estimate of the state vector and its covariance matrix are calculated using the modified dynamics of the unconstrained EKF. If all the constraints are satisfied within a pre-specified accuracy, proceed to the next sampling instant. Otherwise go to Phase II.

***Phase* II:** Formulate the subset of equality constraints to be satisfied, and then apply the dynamics of the update estimator. Check the constraints, if satisfied within a pre-specified accuracy, proceed to the next sampling instant, otherwise calculate the norm of the error vector between the actual and estimated subset of outputs used in this phase. If decreasing repeat Phase II, else terminate the program.

Remark 3.1. *The proposed estimator can be considered as a modified form of the IKF. The differences between the two estimators can be summarized in the following:*

a) The developed estimator is applied only at sampling instants at which a subset of the constraints is violated while the IKF is applied at each sampling instant.

b) *The developed estimator is used with the assumed new output measurement vector* z_{k+1} *rather than the output vector* y_{k+1}*, and the results of the* i^{th} *iteration are used to get the state estimator and its covariance matrix at the* $(i+1)^{\text{th}}$ *iteration.*

c) *In case of linear output model with certain parameters, the IKF will not have any updating effect on the estimator except if the initial estimator and its covariance matrix are replaced by the results of the* i^{th} *iteration. However, the proposed technique can always be used to adjust the estimated states.*

d) *The IKF is used to improve the estimator, resulting from the EKF, for nonlinear stochastic dynamical systems with certain parameters using the fact that the estimated states and their estimation errors are not orthogonal. On the other hand, the developed estimator is applicable to nonlinear stochastic dynamical systems with uncertain parameters. As it is made clear by Lemmas 3.1 and 3.2, the dynamics of the proposed filter are different from those used with the EKF.*

Remark 3.2. *In some cases, the assumed new received measurement vector (34) leads to an unobservable system. Hence, the updating process will not affect all the state variables. This problem can be avoided by defining* $g_{k+1}(x_{k+1}, c'_{k+1}) = h_{k+1}(x_{k+1}, c_{k+1})$*, i.e., the original m-dimensional vector nonlinear function of output measurements,* $v'_{k+1} = v_{k+1}$*, and* $z_{k+1} = y_{k+1} - \theta_{k+1}$*, where* $z_{k+1} \in R^m$ *and* $\theta_{k+1} \in R^m$ *is a vector contains, element by element, zeros at positions corresponding to output elements satisfying the constraints, and the values of the violated upper or lower bounds in (4b) at positions corresponding to output elements violating the constraints. In this case, if the algorithm converges, we insure the boundedness of the estimated outputs and also gain the other stabilizing effects of IKF.*

Remark 3.3. *Since the nonlinear updating algorithm is an application of Gauss-Newton method for approximating a maximum likelihood estimate (as will be shown in Section 4), it is controlled by the rate of convergence of the this method. It shows correct convergence behavior as the observation becomes more accurate. In other words, it may diverge as the level of the measurement noise and/or the uncertainties of the output model parameters increase. This fact will be justified in the illustrative example.*

Remark 3.4. *Assume that the output measurements are noise free and the output model parameters are certain. Then, the best estimate of the output vector is that it matches the measured one. In case of noisy measurements and uncertain parameters, each measured element of the output vector is disturbed – in the average – from its ideal value (noise free and certain parameters) by the standard deviations of its measurement noise and the set of parameters related to this output. Therefore, a good choice of the upper and lower bounds of the estimation error,* $\bar{\zeta}$*,* $\underline{\zeta}$ *– which have to be symmetric – is found to be (element by element) within the range* $\sigma \leq \left\{ \left|\underline{\zeta}\right|, \left|\bar{\zeta}\right| \right\} \leq 2\sigma$ *(e.g., if* $\underline{\zeta} = -1.5\sigma$ *then* $\bar{\zeta} = 1.5\sigma$*) where* σ *is a vector of elements each of which represents the sum of the standard deviations of the output noise and the uncertain parameters affecting the corresponding element of the output vector. This means that, such a range allows for instantaneous disturbance levels up to the value of these limits before applying the algorithm. However, this range can be updated depending on the sensitivity of the problem and design requirements.*

Remark 3.5. *The filter dynamics and its stability are highly affected by the chosen sampling period which depends on the system dynamics. Bad choice of the sampling period may lead to an unstable estimator. However, the proposed estimator shows to be less sensitive to sampling period. This feature will be verified through simulation. It will be shown that if the sampling period is chosen within certain limits, the proposed estimator will be stable while the EKF may diverge. Beyond these limits both estimators will diverge.*

4. **Conversion of the Nonlinear Update Estimator.** In this section, the maximum likelihood/least square technique is formulated to the nonlinear update estimator. It will be shown that the proposed update procedure is an application of Gauss-Newton method to approximate a solution.

Since the update procedure is applied only at the sampling instants at which some of the constraints are violated, it is independent of the dynamics of the system [50]. Therefore, the update estimator is equivalent to the static estimation problem in which we correct the current state estimator using the assumed new received subset of measurements in which the interval constraint is enforced. To simplify our notation, and without introducing any ambiguity, the subscript indicates the time is omitted from the variables.

Linearizing (34) around $\bar{c}$ we get:

$$z = g(x, \bar{c}') + M\tilde{c}' + v' \tag{53}$$

where $M = \frac{\partial g}{\partial c}|_{x,\bar{c}'}$.

Regarding z and $\hat{x}_i$ as realization of independent multivariate random vectors with normal distribution at the i^{th} iteration:

$$z : N\left(g(x, \bar{c}'), \bar{R}_i\right), \quad \hat{x}_i : N(x, P_i) \tag{54}$$

where $\bar{R}_i = R' + M_i U' M_i^T$ and $M_i = \frac{\partial g}{\partial c}|_{\hat{x}_i,\bar{c}'}$.

Values for z, $\hat{x}_i$, $\bar{R}_i$, P_i are given. The update problem is to find a better estimate $\hat{x}_{i+1}$ and its covariance matrix P_{i+1} using the available information.

Lumping the current observation and the state estimate into a single vector, the augmented vectors are such as:

$$\eta_i = \left[z^T \ \hat{x}_i^T\right]^T, \quad \gamma(x) = \left[g^T(x, \bar{c}') \ x^T\right]^T \tag{55}$$

Using the independent assumptions, we get:

$$\eta_i : N(\gamma(x), \bar{Q}_i)$$

where

$$\bar{Q} = \begin{bmatrix} \bar{R}_i & 0 \\ 0 & P_i \end{bmatrix} \tag{56}$$

Replacing x by $\hat{x}_{i+1}$, where $\hat{x}_{i+1}$ is the estimate we are looking for, and let $\hat{x}_{i+1} = \hat{x}_i + \Delta x_i$. Assume that $\hat{x}_{i+1}$ is close to $\hat{x}_i$ that we can replace γ by its first order approximation, i.e., we assume γ affine on a neighborhood of $\hat{x}_{i+1}$, $\hat{x}_i$. Thus, $\gamma(\hat{x}_{i+1}) = \gamma(\hat{x}_i) + \Pi(\hat{x}_i)\Delta x_i$ where

$$\Pi(\hat{x}_i) = \left.\frac{\partial \gamma(x)}{\partial x}\right|_{x=\hat{x}_i} = \begin{bmatrix} G(\hat{x}_i) \\ I \end{bmatrix}, \quad G(\hat{x}_{i)} = \left.\frac{\partial g(x, \bar{c}')}{\partial x}\right|_{x=\hat{x}_i} \tag{57}$$

and $I \in R^{n\times n}$ is a unity matrix.

Re-arranging (55) while using (57), the approximate representation of $\gamma(\hat{x}_{i+1})$ and substituting for x by $\hat{x}_{i+1} = \hat{x}_i + \Delta x_i$, we get:

$$\tilde{\eta}_i = \left[\tilde{z}_i^T \quad O^T\right]^T, \quad \tilde{\gamma}(\Delta x_i) = \Pi(\hat{x}_i)\Delta x_i = \left[(G(\hat{x}_i)\Delta x_i)^T \quad \Delta x_i^T\right]^T \tag{58}$$

where $\tilde{z}_i = z - g(\hat{x}_i, \bar{c}')$; $O \in R^n$ is a vector of zero elements.

4.1. **The maximum likelihood update.** It is well known that the likelihood function $L(\xi)$ is the probability density of $\tilde{\eta}$ with Δx_i replaced by the free variable ξ. Therefore,

$$L(\xi) = \left(1\Big/\sqrt{(2\pi)^{n+s}\left|\bar{Q}_i\right|}\right)\exp\left(-\frac{1}{2}(\tilde{\eta}_i - \tilde{\gamma}(\xi))^T\bar{Q}_i^{-1}(\tilde{\eta}_i - \tilde{\gamma}(\xi))\right) \tag{59}$$

Maximizing $L(\xi)$ with respect to ξ is equivalent to minimizing the exponent of (59) with respect to ξ. Such a minimization leads to the following:

$$0 = \Pi(\hat{x}_i)^T Q_i^{-1}(\tilde{\eta}_i - \tilde{\gamma}(\xi))$$
$$\Delta x_i = (\Pi(\hat{x}_i)^T Q_i^{-1}\Pi(\hat{x}_i))^{-1}\Pi(\hat{x}_i)^T Q_i^{-1}\tilde{\eta}_i \tag{60}$$

Since $(\Pi^T(\hat{x}_i)\bar{Q}_i^{-1}\Pi(\hat{x}_i))^{-1}\Pi^T(\hat{x}_i)\bar{Q}_i^{-1}\tilde{\eta}_i = (G^T(\hat{x}_i)\bar{R}_i^{-1}G(\hat{x}_i) + P_i^{-1})^{-1}G^T(\hat{x}_i)\bar{R}_i^{-1}\tilde{z}_i$ and by using the matrix inversion lemma [48] we get:

$$(G^T(\hat{x}_i)\bar{R}_i^{-1}G(\hat{x}_i) + P_i^{-1})^{-1}G^T(\hat{x}_i)\bar{R}_i^{-1} = K_i = P_i G^T(\hat{x}_i)(G(\hat{x}_i)P_i G^T(\hat{x}_i) + \bar{R}_i)^{-1} \tag{61}$$

Therefore,

$$\hat{x}_{i+1} = \hat{x}_i + K_i\tilde{z}_i \tag{62}$$

The covariance matrix P_{i+1} is such that:

$$P_{i+1} \triangleq E\{(\hat{x}_{i+1} - \hat{x}_i)(\hat{x}_{i+1} - \hat{x}_i)^T\} = E\{\Delta x_i \Delta x_i^T\}$$
$$P_{i+1} = (\Pi^T(\hat{x}_i)\bar{Q}_i^{-1}\Pi(\hat{x}_i))^{-1} \tag{63}$$

Using (57) we get:

$$P_{i+1} = (G(\hat{x}_i)^T \bar{R}_i^{-1} G(\hat{x}_i) + P_i^{-1})^{-1} \tag{64}$$

Hence, by using (61) and (64), the covariance matrix is given by:

$$P_{i+1} = (I - K_i G(\hat{x}_i))P_i \tag{65}$$

From (61), (62) and (65), it is clear that the proposed estimator given by (35)-(37) is equivalent to the maximum likelihood update.

4.2. **The nonlinear update estimator is a Gauss-Newton method.** Consider the following nonlinear least square problem [46]:

$$\min f(\alpha) = \frac{1}{2} r(\alpha)^T r(\alpha) \tag{66}$$

where $r(\alpha) : R^s \to R^s$ is twice differentiable. The Gauss-Newton method for solving this problem is based on the successive application of Newton method to find the roots of $f_\alpha(\alpha) = 0$, where $*_\alpha(\alpha)$ is the first derivative of $*(\alpha)$ with respect to (w.r.t.)α. Let $F_{\alpha\alpha}$ be the second derivative of $f(\alpha)$ w.r.t.α, then given an initial approximation α_o, we inductively define:

$$\alpha_{i+1} = \alpha_i - F_{\alpha\alpha}^{-1}(\alpha_i) f_\alpha(\alpha_i) \tag{67}$$

where

$$F_{\alpha\alpha} = r_\alpha(\alpha)^T r_\alpha(\alpha) + \Im(\alpha) \tag{68}$$

The second term in the right hand side (R.H.S.) of (68) is given by $\Im(\alpha) = \sum_{j=1}^{s} r_j(\alpha)\chi_j(\alpha)$ where $\chi_j(\alpha)$ is the Hessian of the j^{th} component of $r(\alpha)$.

Dropping the term $\Im(\alpha)$ from (68), the sequence (67) is approximated by:

$$\alpha_{i+1} = \alpha_i - (r_\alpha(\alpha_i)^T r_\alpha(\alpha_i))^{-1} r_\alpha(\alpha_i)^T r(\alpha_i) \tag{69}$$

It is worth mentioning that the same results can be obtained if we let $\alpha_{i+1} = \alpha_i + \Delta\alpha_i$ and approximate $r(\alpha_{i+1})$ by its first order approximation, i.e., $r(\alpha_{i+1}) = r(\alpha_i) + r_\alpha(\alpha_i)\Delta\alpha_i$. Then by minimizing $f(\alpha_i + \Delta\alpha_i)$ w.r.t. $\Delta\alpha_i$ we get the optimal value of $\Delta\alpha_i$ as given by the second term in the R.H.S. of (69).

Define $r : R^n \to R^{n+s}$ as:

$$r(\alpha) = S(\tilde{\eta} - \tilde{\gamma}(\alpha)) \tag{70}$$

where $S^T S = \bar{Q}^{-1}$.

Using (57), (70) and $r_\alpha(\alpha) = -S\Pi(\alpha)$, while substituting for α_i, α_{i+1} by $\hat{x}_i, \hat{x}_{i+1}$ respectively, the Gauss-Newton iteration (69) takes the form:

$$\hat{x}_{i+1} = \hat{x}_i + ((\Pi(\hat{x}_i)^T \bar{Q}^{-1}\Pi(\hat{x}_i))^{-1}\Pi(\hat{x}_i)^T \bar{Q}^{-1}(\tilde{\eta} - \tilde{\gamma}(\hat{x}_i)) \tag{71}$$

From (56)-(58), (61), (64) and by using the matrix inversion lemma, we get after simple mathematical manipulation:

$$\hat{x}_{i+1} = \hat{x}_i + K_i \tilde{z}_i \tag{72}$$

Therefore, the sequence of iterates generated by Gauss-Newton method is identical to the proposed update procedure. Hence, the rate of convergence of the proposed nonlinear update estimator is as stated for Gauss-Newton method [51].

5. **The Developed Recursive Algorithm.** The algorithm used to solve the estimation problem of nonlinear systems with uncertain parameters is as follows:

Step 1: *Initialize the value of* $\hat{x}_{o|o}$, $P_{o|o} = P_o$, Q_o, R_o, Γ_o, U_1, *the pre-specified accuracy* ε *and set* $\upsilon_o =$ *large number*, $k = 0$.

Step 2: *Calculate* $\hat{x}_{k+1|k}$, F_k, B_k, H_{k+1}, D_{k+1} , $P_{k+1|k}$, $\hat{y}_{k+1|k}$, *using (6), (5), (7) and (8) respectively.*

Step 3: *Calculate* $K_{1k+1|k+1}$, $\hat{x}_{k+1|k+1}$, $\hat{y}_{k+1|k+1}$ *and* $P_{k+1|k+1}$ *using (11), (9), (12) and (10) respectively.*

Step 4: *Check the satisfaction of the constraints.*

If *all the constraints are satisfied within* ε
Then
Go to Step 5
Else
Calculate $\upsilon_{k+1} = \left\| \hat{y}_{k+1|k+1} - y_{k+1} \right\|$
If $\upsilon_{k+1} > \upsilon_k$
Then
Stop
Else
Assign: $\hat{x}_{1k+1|k+1} = \hat{x}_{k+1|k+1}$, $P_{1k+1|k+1} = P_{k+1|k+1}$

Identify the violated constraints and formulate Equation (34). Calculate G_{k+1}, M_{k+1}, K_{2k+1}, $\hat{x}_{k+1|k+1}$, $P_{k+1|k+1}$ *using (39), (37), (35) and (36) respectively.*

Go to the beginning of ***Step 4.***

Step 5: ***Set*** $k = k + 1$;
If $k = k_f$ (k_f *is the last sampling instant),*
Then
Stop
Else
Go to Step 2

6. **Simulation Results and Discussion.** For the completeness of our presentation, we start this section by introducing Monte Carlo simulation technique, then we present our illustrative practical problem representing a synchronous machine.

6.1. **Monte Carlo simulation.** For deterministic systems with certain inputs, no matter how many times the calculations are repeated to get the set of variables, one gets the same results. On the other hand, for stochastic systems, since random variables and inputs are involved, this is not the case. Therefore, to complement the theoretical derivations and to test the applicability and efficiency of any new approach related to stochastic processes, it has to pass firstly through a statistical analysis phase. In general, Monte Carlo simulation [52] is commonly used for stochastic systems. It is a sampling technique that was invented by scientists long time ago. Such an approach is a problem solving technique used to approximate the probability of certain outcomes by running multiple trial runs using the random variables.

This approach is used in our numerical analysis not only to show the effectiveness of the developed procedure, but also to make a quantitative comparison with other techniques usually used to handle similar problems. In order to achieve this objective, two indicators will be calculated through our simulation and their results will be used as a base for our analysis. These indicators are:

1) The Root Mean Square Index (RMSI). This index is an overall index which calculates the root mean square estimation error for each state variable over the whole horizon of estimation and for the total number of Monte Carlo simulation. It is given by:

$$RMSI_i = \sqrt{\frac{\sum\limits_{j=1}^{NOMI} \sum\limits_{k_o}^{k_f} [x_{i_j}(k) - \hat{x}_{i_j}(k|k)]^2}{NOMI * (k_f - k_o + 1)}} \tag{73}$$

where $NOMI$ is the number of trial runs or Monte Carlo simulation, $RMSI_i$ is the root mean square index for x_i, k_o is the initial time and k_f is the final time.

It is obvious that, the lower the value of this index, the better the performance of the estimator.

2) The Root Mean Square Estimation Error (RMS). This index calculates the root mean square estimation error of each state variable at each sampling instant of time. It is given by:

$$RMS_i(k) = \sqrt{\frac{\sum\limits_{j=1}^{NOMI} [x_{i_j}(k) - \hat{x}_{i_j}(k|k)]^2}{NOMI}} \tag{74}$$

where $RMS_i(k)$ is the root mean square estimation error for x_i at the sampling instant k.

The progress of this indicator with time gives an indication of the behavior of the estimation procedure. In other words, if the indicator increases with time, this means that the chosen technique has the tendency to diverge.

6.2. **Illustrative problem.** To illustrate the effectiveness of the developed approach, let us consider the following model for a synchronous machine [48]:

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= \alpha_1 - \alpha_2 x_2 - \alpha_3 x_3 \sin x_1 - 0.5\alpha_4 \sin 2x_1 + w_2 \\ \dot{x}_3 &= u_{ref} - \alpha_5 x_3 + \alpha_6 \cos x_1 + w_3 \end{aligned} \tag{75}$$

$$y = cx_1 + v \tag{76}$$

where $x_1 = \theta$ is the angular position of the rotor (rad.), $x_2 = \Delta\omega$ is the change in the angular speed ω (rad./sec.), x_3 is the flux variation (web.), u_{ref} is the reference input, w_i; $i \in \{1, 2, 3\}$ is the input noise, y is the output measurement and v is the output measurement noise.

The following assumptions and data are used within our simulation:

(1) w, v are assumed to be uncorrelated zero mean white Gaussian input and output noise vectors with covariance matrices $Q = diag[q\ q\ 0.1q]$ and $R = [r]$ respectively. The values of q, r will be specified within our case studies.

(2) $P_o = diag[1.0\ \ 1.0\ \ 1.0]$. With $\hat{x}_{o|o} = [0.0\ \ 0.0\ \ 0.0]$ the estimated states were stable and the results were very satisfactory. However, the graphical representation of the states and their estimated trajectories were not very clear due to the scale used for plotting the results. To avoid this problem, the obtained trajectories will be plotted for $\hat{x}_{o|o} = [0.7461\ \ 0.0\ \ 7.7438]$.

(3) The set of model parameters, α_i; $i \in \{1, 2, \ldots, 6\}$, and the output measurement parameter c are assumed to be uncertain and represented by uncorrelated white Gaussian random variables with the following statistics: α_1: $N(38.18, 36.4428)$; α_2: $N(0.27, 0.001\ 82)$; α_3: $N(12.01, 3.606)$; α_4: $N(-48.04, 57.696)$; α_5: $N(0.32, 0.00255)$; α_6: $N(1.9, 0.9024)$; c: $N(1, 0.1)$. Again, the mean values of these parameters are their physical nominal values.

(4) Simulations are carried out over a time horizon $t \in [0,\ 150]$sec. and measurements are taken at a sampling rate $\Delta T = 0.01$sec.

(5) To illustrate the effectiveness of the proposed approach, different case studies are simulated and the results are compared with those achieved from the different techniques used for the same purpose. For each case, 100 Monte Carlo simulation runs are performed to estimate the states using the following estimation procedures:

a) Extended Kalman filter with the nominal values of the parameters (denoted by EKF).
b) Extended Kalman filter with parameter estimation (denoted by EKF.P.E). In this approach the parameters are treated as static states with the prior at $k = 0$ being $N(\bar{b}, \Gamma)$ as specified in the assumption (3) above.
c) The proposed stabilized estimator (denoted by St. EKF).

The $RMSI_i$ ($i = 1, 2, 3$), averaged over 100 Monte Carlo simulations and 15001 sampling points, are calculated for each state variable and for each case study considered in our simulation using the three different estimation procedures. The results are given in Table 1.

TABLE 1. *RMSI* of the estimation errors

Case No.	Filter Type	q	R	X1	X2	X3
1	EKF	0.001	1.0	0.687	4.847	0.584
	EKF P.E.			0.459	3.009	0.676
	St. EKF			0.124	0.978	0.087
2	EKF	0.01	1.0	0.662	4.720	0.550
	EKF P.E.			0.348	2.654	0.528
	St. EKF			0.124	0.979	0.087
3	EKF	0.1	1.0	0.254	1.979	0.125
	EKF P.E.			0.203	1.578	0.187
	St. EKF			0.125	0.985	0.088
4	EKF	1.0	1.0	0.138	1.080	0.097
	EKF P.E.			0.135	1.051	0.116
	St. EKF			0.133	1.039	0.095
5	EKF	7.0	1.0	0.588	2.994	0.256
	EKF P.E.			0.363	2.690	0.251
	St. EKF			0.366	2.714	0.231

Moreover, the $RMS_i(k)$ ($i = 1, 2, 3$) averaged over the 100 Monte Carlo simulations, are calculated for each state variable and at each sampling instant using the three approaches.

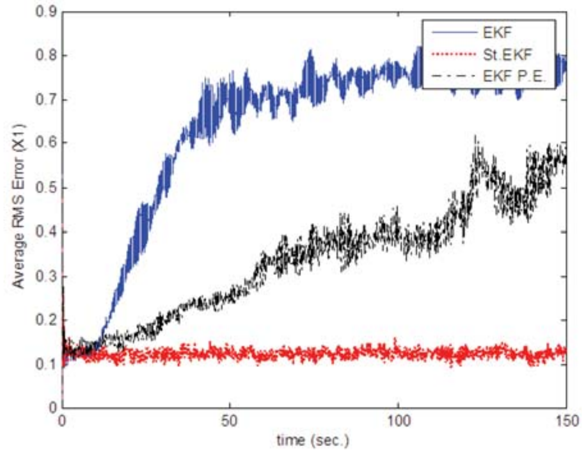

Figure 1. $RMS_1(k)$ for x_1 (Case 2)

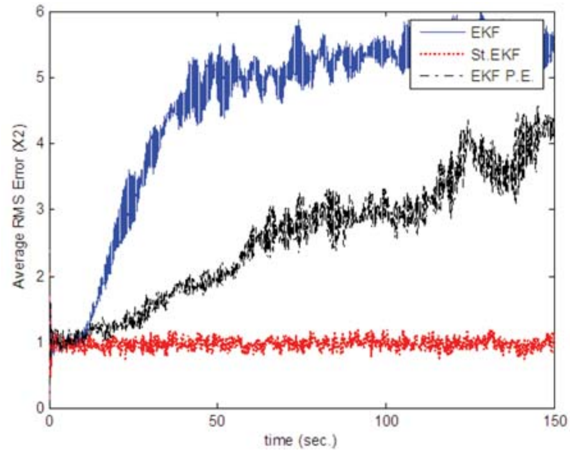

Figure 2. $RMS_2(k)$ for x_2 (Case 2)

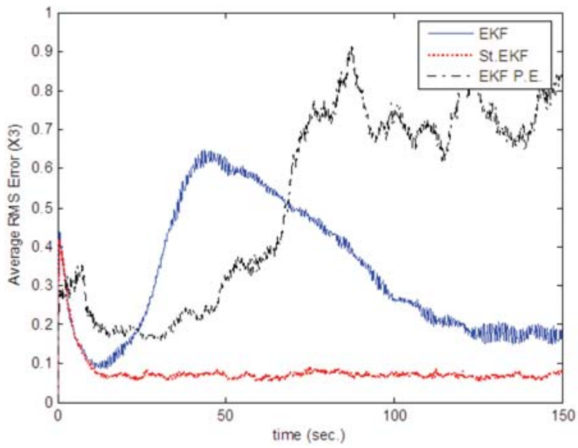

Figure 3. $RMS_3(k)$ for x_3 (Case 2)

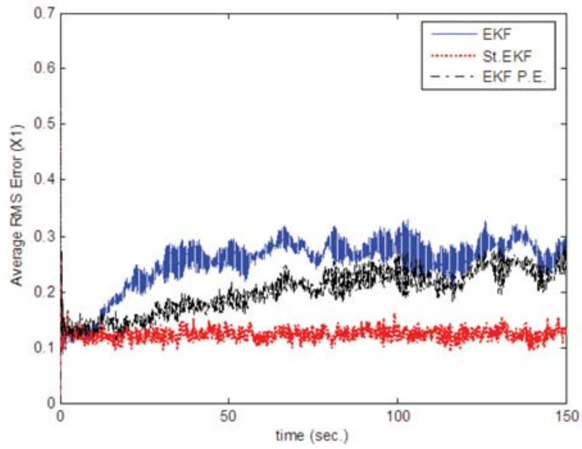

Figure 4. $RMS_1(k)$ for x_1 (Case 3)

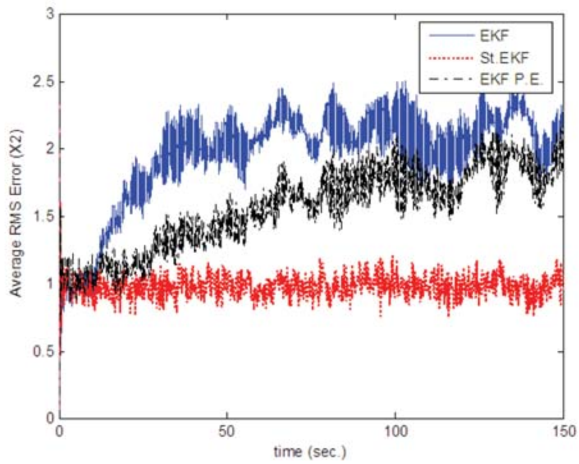

Figure 5. $RMS_2(k)$ for x_2 (Case 3)

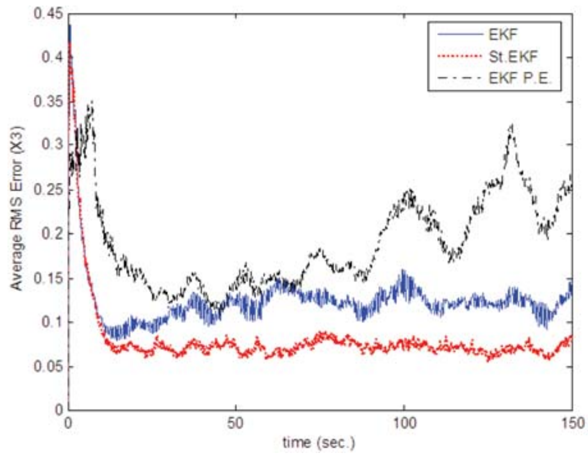

Figure 6. $RMS_3(k)$ for x_3 (Case 3)

Sample of the results are plotted in Figures 1-6. Figures 1-3 present the results of Case 2, while Figures 4-6 demonstrate the results of Case 3.

6.3. **Discussion of the results and remarks.** Based on simulation results, one can conclude the following:

1) From Table 1 it is clear that the least $RMSI^s$ are achieved with the application of the developed estimator.

2) The estimated states using the developed approach are stable either in Cases 1, 2, 3 and 5 in which the EKF and the EKF with parameter estimation are unstable (see for example Figure 7 for x_3 which has been found to be the most sensitive state variable).

3) The best results are always achieved through the application of the proposed estimator, either in Case 4 in which the three estimators are stable (see for example Figure 8 for the most sensitive state variable x_3). Moreover, the developed approach is faster than the EKF with parameter estimation.

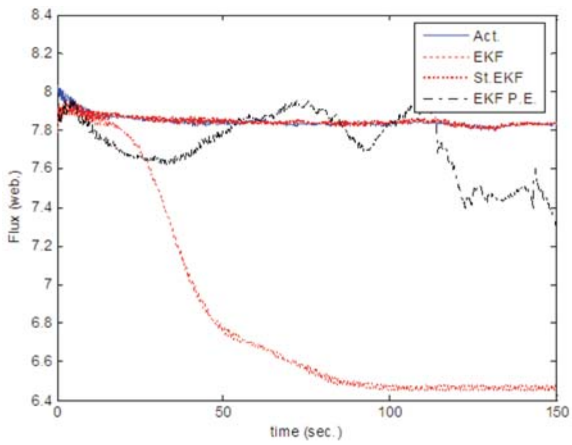

FIGURE 7. Actual and estimated state x_3 (Case 1)

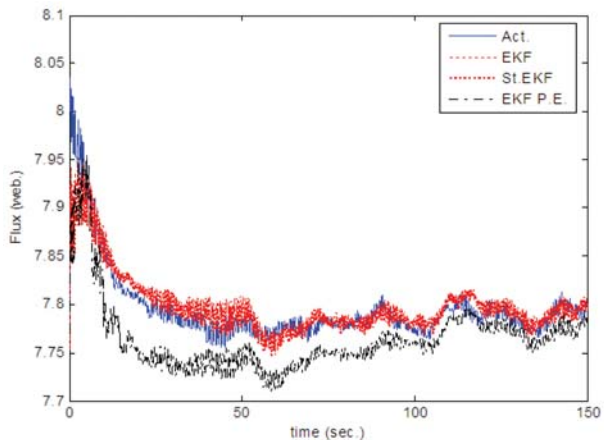

FIGURE 8. Actual and estimated state x_3 (Case 4)

4) Since the main objective is to insure that the estimated outputs are bounded and not tracking the actual measurements, the upper and lower bounds are chosen as stated in Remark 3.4 at the end of Section 3.

5) Since the developed approach can be imbedded in the class of techniques dealing with constrained estimation problems, it is worth comparing with others dealing with the same problem. One of the important techniques used in this domain is that presented in [14-16]. Such a technique is adopted to enforce constraints on the innovation, and then applied to our problem with the data set given in Case 5. The weighting matrix of the quadratic cost function is chosen as a unity matrix and as the covariance matrix $P_{k+1|k+1}$ at the same sampling instant of time at which the constraints are violated. For each choice of the weighting matrices, the $RMSI^s$ averaged over 100 Monte Carlo simulations and 15001 sampling points, are calculated for each state variable. The obtained results are 0.367, 3.3266 and 0.2120 for x_1, x_2, x_3 with the use of unity matrix as a weighting matrix, and 0.366, 2.752 and 0.2352 for x_1, x_2, x_3 with the use of the covariance matrix $P_{k+1|k+1}$ as a weighting matrix. Although the above numbers indicate that we may have good results, the calculated averages at each sampling instant $RMS_i(k)$ as well as the actual and estimated states are unstable.

6) As stated in Section 3 (Remark 3.3), we can have better convergence as the measurement becomes more accurate. In other words, the estimator diverges as the level of the measurement noise and/or the output parameter uncertainty increases. In order to justify this fact, the problem is simulated using different values of $R = \{15, 20, 30\}$ with $q = 1$ and different values of the variance of the output parameter $c = \{0.1, 0.5, 1.0\}$. For each combination, 100 Monte Carlo simulation runs are performed and the different

indicators are calculated. For example, the $RMSI^s$ for the states in case of $R = 20$, and the variance of $c = 0.5$ are 0.5428, 3.972, 0.396 for x_1, x_2, x_3. Comparing this result with those obtained from the application of EKF (7.0248, 6.11, 0.5188) and EKF with parameter estimation (0.570 4.207 0.434), it is clear that although the proposed approach still performing much better than the others, it starts to give unacceptable results specially for x_3 as shown in Figure 9. The situation becomes worst by increasing R and the variance of the output parameter.

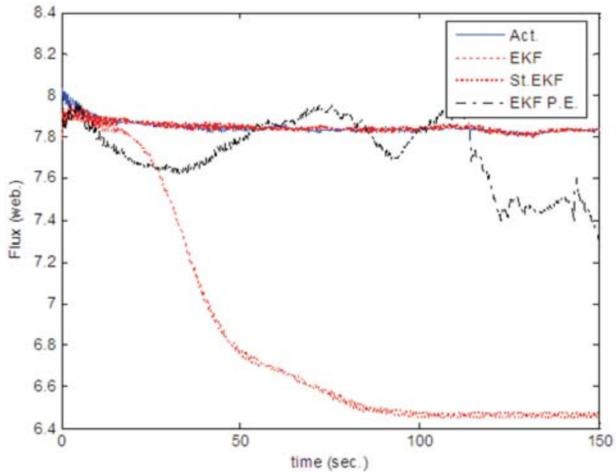

FIGURE 9. Actual and estimated state x_3 (R = 20)

7) Although this approach leads to better results when compared with others dealing with the same problem, it is still an open area for future research. In this direction we are looking for an improvement of this technique or developing others leading to:

a) Faster convergence behavior than the proposed multiple projection approach.
b) Better control on the behavior of the estimators of the unmeasured state variables.

7. **Conclusion.** In this paper, a state estimator is developed for nonlinear stochastic discrete-time dynamical systems with uncertain parameters. The states of the system as well as the measured output vector are assumed to be corrupted by zero mean white Gaussian noise. The statistical data of the parameters, the input and the output noise vectors are assumed to be known. By imposing a set of inequality constraints on the error between the actual and estimated outputs, an estimator is developed based on the EKF and the active set method. This approach can be imbedded in the class of constrained estimation algorithms. An illustrative example is presented to show the effectiveness of the developed approach. Simulations results show that the developed technique improves the stability of the estimator and leads to stable state estimates even in many cases in which the EKF and the EKF with parameter estimation are unstable. Research activities are now going on to use this approach in generating state dependent control strategies to control the performance of stochastic nonlinear discrete-time control systems with uncertain parameters.

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