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STABILITY ANALYSIS OF PARAMETRIC TIME-DELAY SYSTEMS BASED ON PARAMETER PLANE METHOD

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Abstract. *The main purpose of this paper is to consider the robust stability analysis of time-delay systems based on the parameter plane method. The simple and systematic procedure proposed for solving this problem easily obtains the exact stability boundaries of time-delay systems. First, the relation between state-space representation and s-domain characteristic equations is analyzed. The effects of both the state matrix and time-delay on stability are addressed by this approach. System stabilization by state feedback gain is also derived. Performance tests of the proposed procedure confirm that it obtains more information compared with procedures reported in the literature.*
Keywords: Time delay, Parameter plane, Stability

1. **Introduction.** The use of state-space models for stability analysis of time-delay systems has received much attention in the last two decades [1-11]. Specifically, the problem of parameter-dependent and delay-dependent robust stability for time-delay systems with polytopic uncertainties was studied in [12]. The parameter plane method and parameter space method for robust stability analysis of time-delay systems was presented in [13]. A general rule has been identified for setting desirable elements in the system matrices as parameters. The exact stability boundary obtained by the parameters can also be plotted in the parameter plane or parameter space. As compared with [13], the purpose of this study is to apply the parameter plane method for evaluating the robust stability of uncertain time-delay systems reported in the literature, with an emphasis on the effects of time delay. The stability effect produced by state feedback gain and commensurate delays is also addressed.

2. **Main Results.** This section first introduces the fundamental technique for analyzing the robust stability of second order time-delay systems by the parameter plane method. The effects of scaling factors, time delay and state feedback gain are considered. Besides, the time-delay systems with third-order and commensurate delays are also analyzed.

2.1. **Effect of scaling factors.** Consider the system

$$\begin{aligned}\dot{x}(t) &= \alpha A_0 x(t) + \beta A_d x(t-T)\\ &= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \beta \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix},\end{aligned} \tag{1}$$

where A_0 and A_d are matrices, a, b, c, d, f, g, h, i are parameters, T is the time delay, and α and β are scaling factors. The characteristic equation of this system is

$$F(s,T) = \det(sI - \alpha A_0 - \beta A_d e^{-sT}) = 0 \tag{2}$$

i.e.,

$$s^2 - \alpha(a+d)s + \alpha^2(ad-bc) - \beta(f+i)se^{-sT} + \alpha\beta(fd+ai-cg-bh)e^{-sT} + \beta^2(fi-gh)e^{-2sT} = 0 \quad (3)$$

Arrange (3) as

$$X \cdot \alpha^2 + Y \cdot \beta^2 + Z \cdot \alpha + W \cdot \beta + U \cdot \alpha\beta + V = 0 \quad (4)$$

where $X = (ad-bc)$; $Y = (fi-gh)e^{-2sT}$; $Z = -(a+d)s$; $W = -(f+i)se^{-sT}$; $U = (fd+ai-cg-bh)e^{-sT}$; $V = s^2$.

Let $s = j\omega$, where ω is the frequency. Partition (4) into two stability equations including real part and imaginary part, one has

$$X_{\text{Re}} \cdot \alpha^2 + Y_{\text{Re}} \cdot \beta^2 + Z_{\text{Re}} \cdot \alpha + W_{\text{Re}} \cdot \beta + U_{\text{Re}} \cdot \alpha\beta + V_{\text{Re}} = 0 \quad (5)$$

and

$$X_{\text{Im}} \cdot \alpha^2 + Y_{\text{Im}} \cdot \beta^2 + Z_{\text{Im}} \cdot \alpha + W_{\text{Im}} \cdot \beta + U_{\text{Im}} \cdot \alpha\beta + V_{\text{Im}} = 0 \quad (6)$$

where X_{Re}, Y_{Re}, Z_{Re}, W_{Re}, U_{Re}, V_{Re}, and X_{Im}, Y_{Im}, Z_{Im}, W_{Im}, U_{Im}, V_{Im}, are the real and imaginary parts of X, Y, Z, W, U, V, respectively.

Solutions for α and β ((5) and (6), respectively) can be solved simultaneously by using the symbolic method and then plotted in the parameter plane as the stability boundary when ω is changed from 0 to ∞.

2.2. **Effect of time delay.** To determine the effect of time delay, assume that $\alpha = 1$ and $\beta = 1$ and that (3) can be expressed as

$$A_2 e^{-2sT} + A_1 e^{-sT} + A_0 = 0 \quad (7)$$

where $A_2 = fi - gh$, $A_1 = -(f+i)s + fd + ai - cg - bh$ and $A_0 = s^2 - (a+d)s + (ad-bc)$.

Define

$$e^{-sT} = e^{-j\omega T} = \cos\omega T - j\sin\omega T = \delta - j\eta \quad (8)$$

where $\delta = \cos\omega T$ and $\eta = \sin\omega T$. Then

$$\begin{aligned} e^{-2j\omega T} &= \cos 2\omega T - j\sin 2\omega T \\ &= \cos^2\omega T - \sin^2\omega T - j2\cos\omega T\sin\omega T \\ &= \delta^2 - \eta^2 - j2\delta\eta \end{aligned} \quad (9)$$

After substitute (8) and (9) into (7), one has

$$A_2\delta^2 - A_2\eta^2 + A_1\delta - jA_1\eta - j2A_2\delta\eta + A_0 = 0 \quad (10)$$

To solve δ and η, let $s = j\omega$, hand partition (10) into stability equations as in (5) and (6):

$$X_{\text{Re}} \cdot \delta^2 + Y_{\text{Re}} \cdot \eta^2 + Z_{\text{Re}} \cdot \delta + W_{\text{Re}} \cdot \eta + U_{\text{Re}} \cdot \delta\eta + V_{\text{Re}} = 0 \quad (11)$$

and

$$X_{\text{Im}} \cdot \delta^2 + Y_{\text{Im}} \cdot \eta^2 + Z_{\text{Im}} \cdot \delta + W_{\text{Im}} \cdot \eta + U_{\text{Im}} \cdot \delta\eta + V_{\text{Im}} = 0 \quad (12)$$

Equations (11) and (12) can then be solved simultaneously by using the symbolic method, and the solution can be plotted in the parameter plane as the stability boundary when ω is changed from 0 to ∞.

Because solutions for δ and η must also satisfy condition (8), the boundary must intersect the unit circle in the δ-η plane. The obtained time delays are

$$T = \frac{1}{\omega}\cos^{-1}\delta, \quad (13)$$

and

$$T = \frac{1}{\omega}\sin^{-1}\eta. \quad (14)$$

Restated, (13) and (14) provide exact values for maximum time delay for asymptotic stability. If no intersection in the parameter plane is found, the system is delay-independent stable.

2.3. **Effect of state feedback gain.** The parameter plane method applied to analyze the effect of state feedback gain is presented here. Consider the state feedback gain

$$u(t) = -Kx(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{15}$$

For simplification, assume $\alpha = 1$ and $\beta = 1$. Equation (1) with state feedback gain $K = [\, k_1 \quad k_2 \,]$ can be written as the following equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_d x(t-T) + Bu(t) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} - \begin{bmatrix} m \\ n \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ &= \begin{bmatrix} a - mk_1 & b - mk_2 \\ c - nk_1 & d - nk_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} \\ &= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} \end{aligned} \tag{16}$$

where $a' = a - mk_1$, $b' = b - mk_2$, $c' = c - nk_1$, $d' = d - nk_2$ and $B = \begin{bmatrix} m & n \end{bmatrix}^T$.

The above procedures can also be used to analyze how state feedback gain and time delay affect stability in the parameter plane.

2.4. **Third order time-delay system.** This subsection addresses the stability analysis of a third order time-delay system. Consider the system

$$\dot{x}(t) = \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & v \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} k & l & m \\ n & o & p \\ q & r & u \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \\ x_3(t-T) \end{bmatrix}. \tag{17}$$

where a, b, c, d, f, g, h, i, v, k, l, m, n, o, p, q, r and u are parameters.

The characteristic equation is

$$A_3 e^{-3sT} + A_2 e^{-2sT} + A_1 e^{-sT} + A_0 = 0, \tag{18}$$

where

$$A_3 = -(uok + plq + mnr - moq - kpr - lun),$$

$$\begin{aligned} A_2 = {} & (ok + uk + uo - qm - pr - nl)s \\ & -(vok + ukf + aou + hpl + bpq + glq + mni + cnr \\ & +dmr - mho - coq - fmq - kpi - kgr \\ & -apr - dlu - bun - vnl), \\ A_1 = {} & -(k + o + u)s^2 + (fk + ao + kv + ov + au \\ & +fu - hm - cq - pi - gr - ld - bn)s \\ & -(fkv + aov + afu + bph + hgl + bgq + cni + dmi \\ & +cdr - coh - fmh - cfq - kgi - api - agr - bdu - dvl - bvn), \\ A_0 = {} & s^3 - (a + f + v)s^2 + (af + av + fv - ch - ig - bd)s \\ & -(afv + bgh + cdi - chf - agi - bvd). \end{aligned}$$

Let $s = j\omega$ and solve (8) as follows:

$$\begin{aligned} e^{-3j\omega T} &= \cos 3\omega T - j \sin 3\omega T \\ &= (4\cos^3 \omega T - 3\cos \omega T) - j(3\sin \omega T - 4\sin^3 \omega T) \\ &= (4\delta^3 - 3\delta) - j(3\eta - 4\eta^3) \end{aligned} \tag{19}$$

Substitute (8), (9) and (19) into (18) as follows:

$$4A_3\delta^3 + j4A_3\eta^3 + A_2\delta^2 - A_2\eta^2 + (-3A_3 + A_1)\delta + (-j3A_3 - jA_1)\eta - j2A_2\delta\eta + A_0 = 0 \quad (20)$$

By adopting the same procedure applied in the second-order case in Subsection 2.2, the solution for δ and η can be depicted in the parameter plane as ω is changed from 0 to ∞. Moreover, the bound of time delay can be obtained by using (13) and (14).

2.5. **Effect of commensurate time delays.** Consider the following linear system with commensurate time delays.

$$\dot{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix} \begin{bmatrix} x_1(t-2T) \\ x_2(t-2T) \end{bmatrix}, \quad (21)$$

The characteristic equation is

$$A_4 e^{-4sT} + A_3 e^{-3sT} + A_2 e^{-2sT} + A_1 e^{-sT} + A_0 = 0, \quad (22)$$

where $A_4 = kn - lm$, $A_3 = fn + ki - gm - hl$, $A_2 = -(n+k)s + (an + fi + kd - bm - gh - cl)$, $A_1 = -(i+f)s + (ai + fd - bh - cg)$ and $A_0 = s^2 - (a+d)s + ad - bc$.

Let $s = j\omega$, Equation (8) gets

$$\begin{aligned} e^{-4j\omega T} &= \cos 4\omega T - j \sin 4\omega T \\ &= (8\cos^4 \omega T - 8\cos^2 \omega T + 1) - j(4\cos \omega T \sin \omega T - 8 \cos \omega T \sin^3 \omega T) \\ &= (8\delta^4 - 8\delta^2 + 1) - j(4\delta\eta - 4\delta\eta^3) \end{aligned} \quad (23)$$

Substitute (8), (9), (19) and (23) into (22). This gives

$$\begin{aligned} &8A_4\delta^4 + 4A_3\delta^3 + j4A_3\eta^3 + j8A_4\delta\eta^3 + (A_2 - 8A_4)\delta^2 - A_2\eta^2 \\ &\quad + (-3A_3 + A_1)\delta + (-j3A_3 - jA_1)\eta - j2A_2\delta\eta + A_0 + A_4 = 0 \end{aligned} \quad (24)$$

Applying a similar procedure obtains the solutions for δ and η in the parameter plane as ω is changed from 0 to ∞. Moreover, the time delay bound can be obtained by using (13) and (14).

3. Numerical Examples.

This section gives examples of time delay systems reported in the literature to verify the design procedure.

Example 3.1. *Consider the time-delay system*

$$\dot{x}(t) = \alpha \begin{bmatrix} -0.6 & 0.2 \\ 0.2 & -0.9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \beta \begin{bmatrix} -2.1 & -1 \\ -1 & -0.6 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}. \quad (25)$$

In [2], the asymptotic stability is obtained when $T = 0.5$, $\alpha = 1$ and $\beta = [0, 1.333)$. Figure 1 plots the solutions for α and β obtained by (5) and (6), respectively. Point Q_1 is the obtained result in [2]. The accuracy of Figure 1 is tested at two operating points: (Q_2 : $\alpha = 2$, $\beta = 1$, stable) and (Q_3: $\alpha = 2$, $\beta = 2$, unstable). Figure 2 shows the time responses of x_1. Compared with [2], the proposed method obtains more stability information with scaling factors α and β in the parameter plane.

Example 3.2. *Consider the time-delay system*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}. \quad (26)$$

The minimum time delay to destabilize (26) is 6.1726 [7]. In Figure 3, the solutions obtained by (11) and (12) for δ and η when ω is changed are plotted as the dotted line. The unit circle is also plotted as a dashed line in Figure 3. The curves clearly intersect at Q_4, and ω and δ are 0.436 and -0.9, respectively. According to (13), $T = 6.1726$.

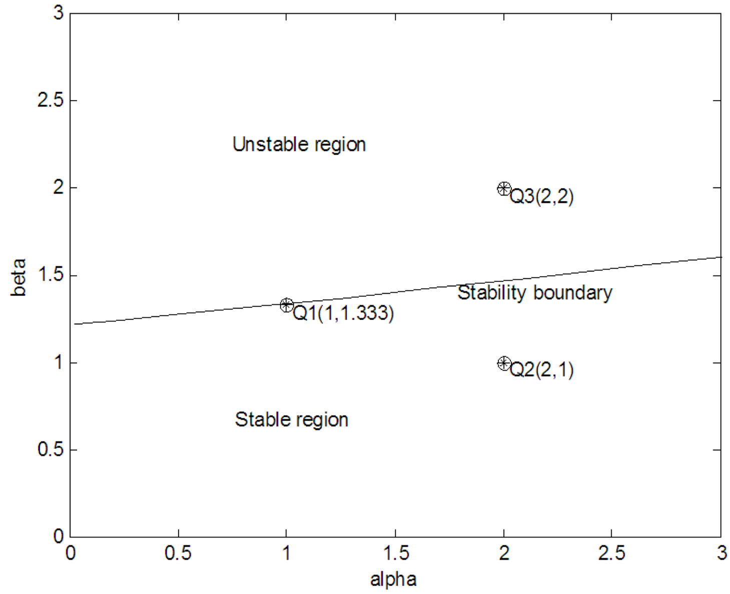
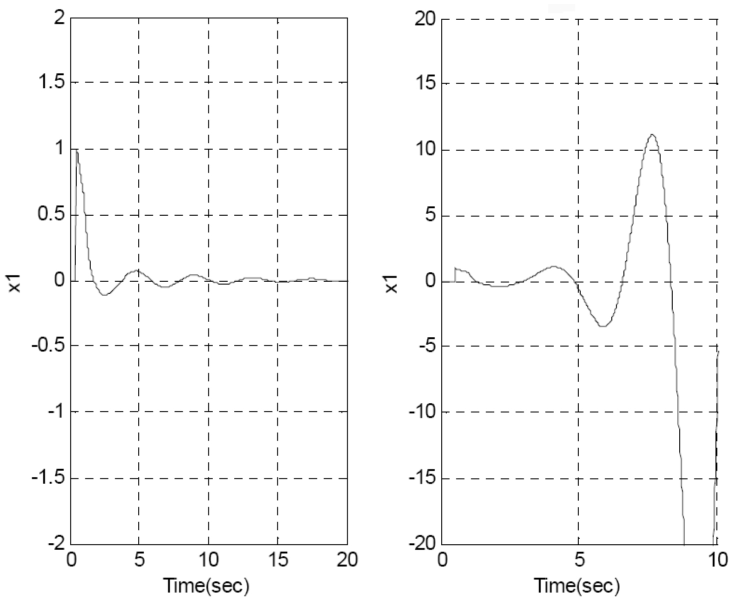


FIGURE 1. Stability boundary



(a) Q2 (b) Q3

FIGURE 2. Time responses

Example 3.3. *Consider the time-delay system*

$$\dot{x}(t) = \begin{bmatrix} -6 & 0 \\ 0.2 & -5.8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ -8 & -8 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}. \tag{27}$$

The system proposed in (27) is stable for arbitrary delay [7]. In Figure 4, the solutions for δ and η obtained by the proposed approach are plotted as dotted lines. The results show no intersection in the parameter plane, and the system is delay-independent stable.

Example 3.4. *Consider the time-delay system*

$$\dot{x}(t) = \begin{bmatrix} -3 & -2.5 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}. \tag{28}$$

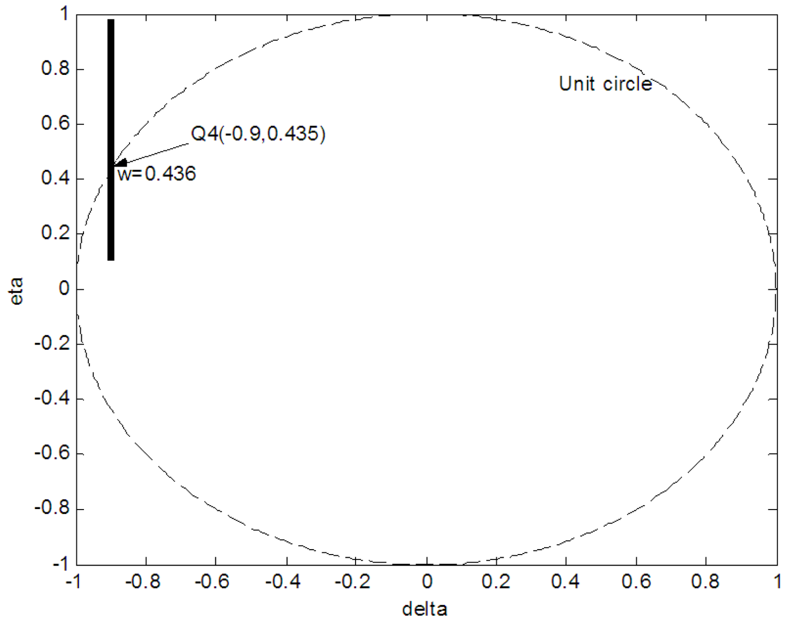


FIGURE 3. Parameter plane

In Figure 5, the solutions obtained for δ *and* η *when* ω *is changed are plotted as the dotted line. Notably, the curve passes through the unit circle at* Q_5*. Therefore,* $\omega = 0.866$*,* $\delta = -0.5$*, and* $T = 2.4184$*, which is consistent with [7].*

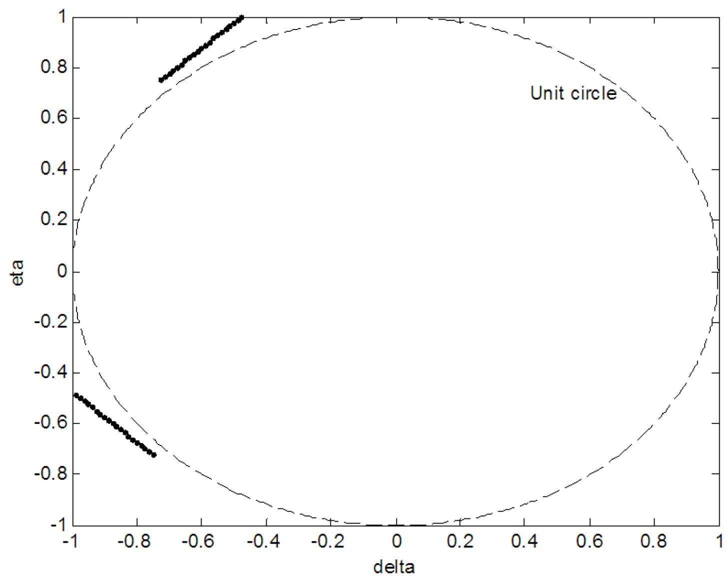


FIGURE 4. Parameter plane

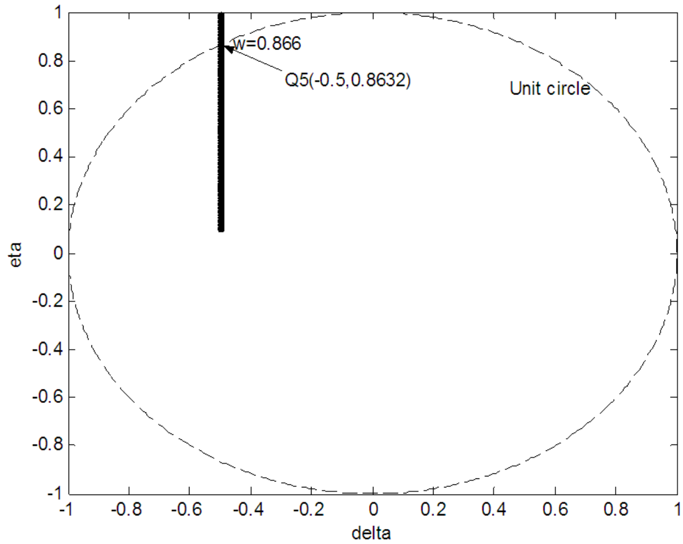


FIGURE 5. Parameter plane

Example 3.5. *Consider the uncertain time-delay system*

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1+g_1 & -0.5 \end{bmatrix}\begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} \\ &\quad + \begin{bmatrix} -1+g_2 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} (-1+g_2)k_1 & (-1+g_2)k_2 \\ k_1 & k_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 1 \\ -1+g_1 & -0.5 \end{bmatrix}\begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}\end{aligned} \tag{29}$$

where $|g_1| \leq 0.53$ *and* $|g_2| \leq 1.7$. *The uncertain boundaries of* g_1 *and* g_2 *are indicted by solid lines in Figure 6. If time-delay* $T = 0.2$ *and state feedback gain matrix* $\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0.0329 & -0.1016 \end{bmatrix}$ *are selected [9], the stability boundary of* g_1 *and* g_2 *can be plotted in the parameter plane as shown by the dotted line in Figure 6 when using (5), (6) and (16). The regions are classified as stable or unstable. The uncertain system is clearly stabilized by the state feedback gain. Figure 7 shows the time responses of* x_1 *at four vertices.*

Example 3.6. *Consider the following uncertain time-delay system*

$$\dot{x}(t) = \begin{bmatrix} 0 & -0.12+12\rho \\ 1 & -0.465-\rho \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}\begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix}, \tag{30}$$

where $|\rho| \leq 0.035$. *The asymptotic stability is guaranteed for all delays that are less than or equal to* $T \leq 0.863$ *[9]. In Figure 8, the solutions obtained for* δ *and* η *when* ω *is changed are plotted as the dotted line. The results are consistent with [9] when* $\omega = 0.23$ *and* $\delta = 0.981$ *at intersection* Q_6.

Example 3.7. *Consider the third order time-delay system*

$$\dot{x}(t) = \begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1(t-T) \\ x_2(t-T) \\ x_3(t-T) \end{bmatrix}. \tag{31}$$

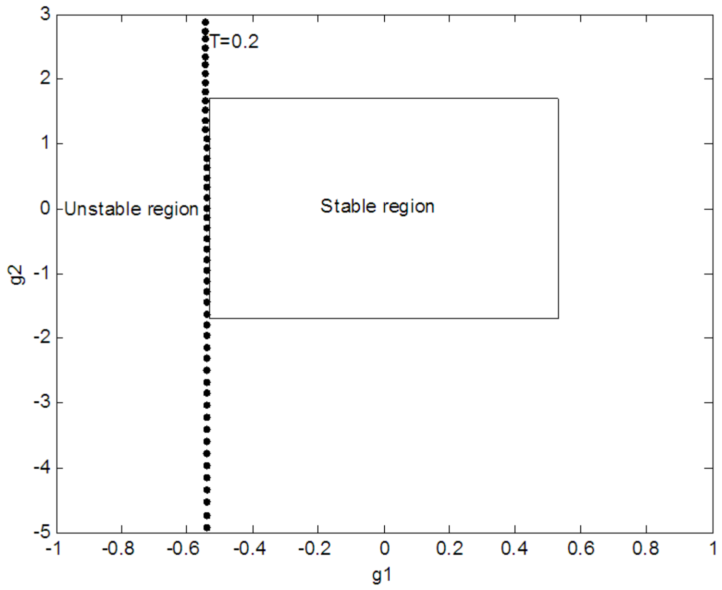


FIGURE 6. Parameter plane

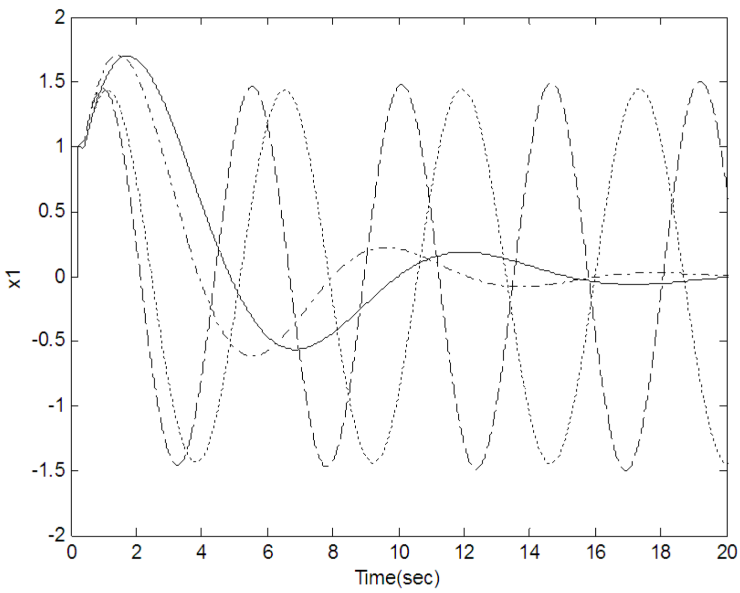


FIGURE 7. State responses

In Figure 9, the solutions obtained by (20) for δ and η when ω is changed are plotted as dotted lines. The curves pass through the unit circle at three points (Q_7, Q_8, Q_9), where $T = 0.1624$, $T = 0.1859$ and $T = 0.222$, respectively. The same results were reported in [8].

Example 3.8. *Consider the commensurate delay system*

$$\dot{x}(t) = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \end{bmatrix} + \begin{bmatrix} -0.8 & 0 \\ 0.8 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(t-2T) \\ x_2(t-2T) \end{bmatrix}. \tag{32}$$

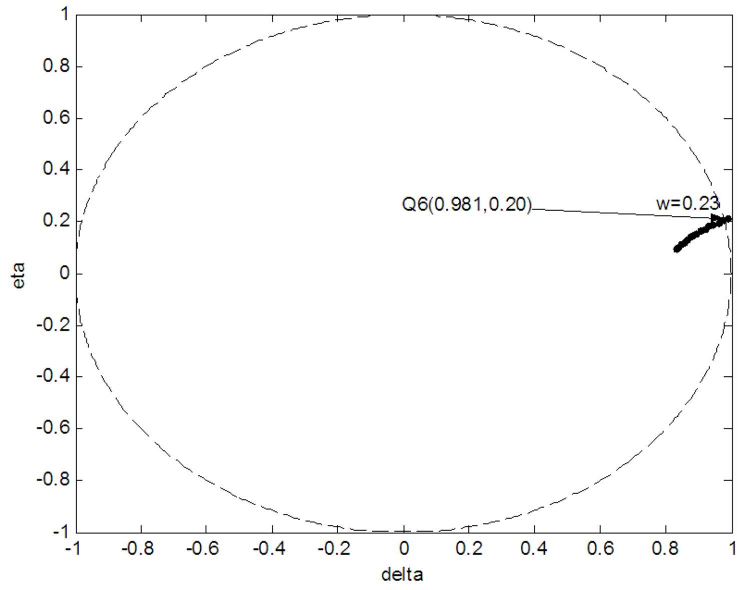


FIGURE 8. Parameter plane

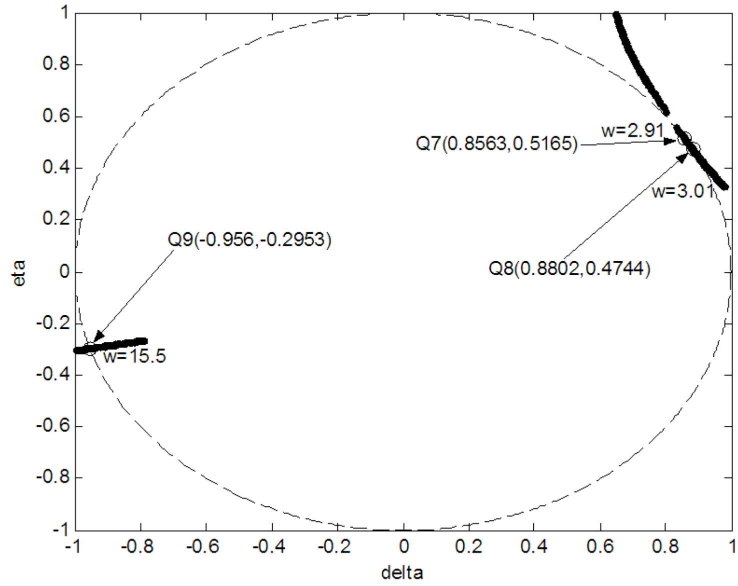


FIGURE 9. Parameter plane

In Figure 10, the solutions obtained by (24) for δ *and* η *when* ω *is changed are plotted as the dotted line. The curve passes through the unit circle at* Q_{10}*. Additionally,* $\omega = 1.5$ *and* $\delta = 0.9532$*. Therefore,* $T = 0.2048$*, which is consistent with [5].*

Example 3.9. *The characteristic equation for a third order system with delayed state feedback is*

$$hs^3 + (6h+1)s^2 + (13.75h + 6 + 1.82he^{-sT} + 0.42he^{-2sT})s + 13.75 + 1.82e^{-sT} + (0.42 - 1305k)e^{-2sT} = 0 \quad (33)$$

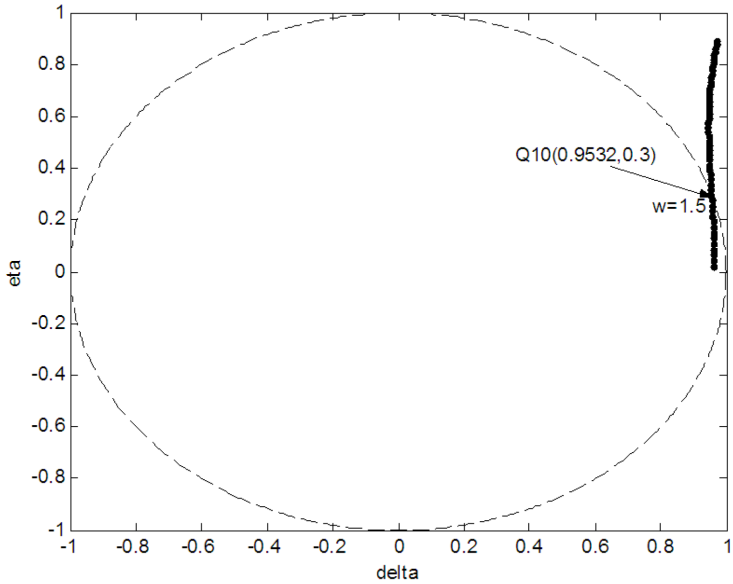


FIGURE 10. Parameter plane

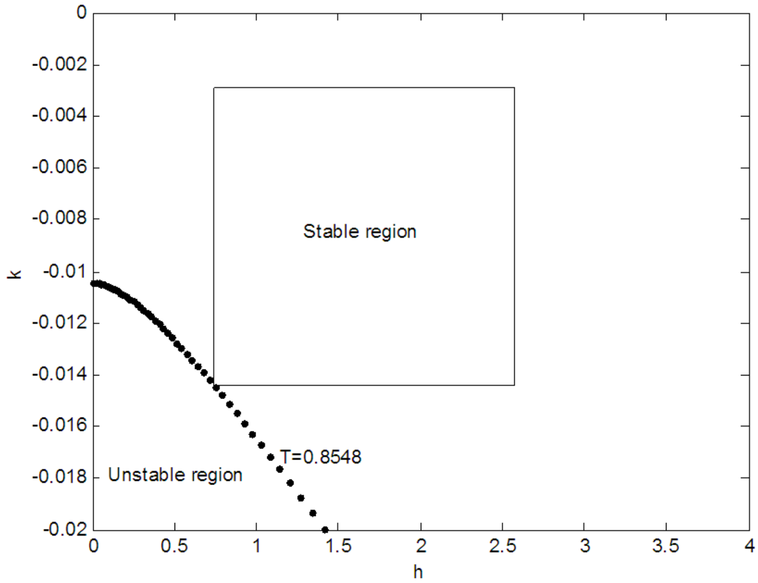


FIGURE 11. Parameter plane

where $k \in [-0.0144, -0.0029]$ *and* $h \in [0.739, 2.58]$. *In order to check the stability bound of time-delay* $T = 0.8548$ *[10], the solutions for* k *and* h *are plotted in the parameter plane. Figure 11 shows the results. The uncertain boundaries are also drawn as solid lines in Figure 11. The stability boundary intersects the vertex (*$k = -0.0144$, $h = 0.739$*) as* $T = 0.8548$, *and the uncertain system is stable with state feedback.*

Example 3.10. *Consider the following specific three-variable biochemical system [14]*

$$\begin{aligned} \dot{x}_1 &= 1 - x_1^{0.5} \\ \dot{x}_2 &= x_1^{0.5} - x_2^{0.5}(t-T) \\ \dot{x}_3 &= x_2^{0.5}(t-T) - x_3 \end{aligned} \quad . \tag{34}$$

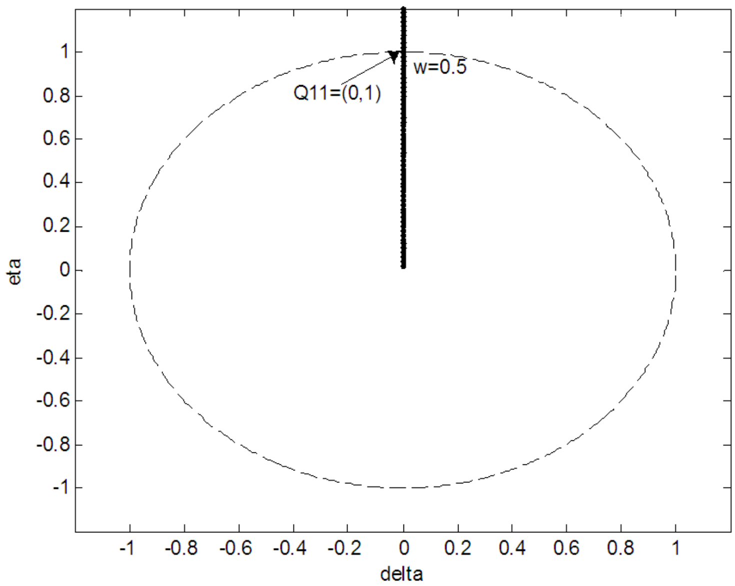


FIGURE 12. Parameter plane

Linearizing this system gives

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1(t-T) \\ x_2(t-T) \\ x_3(t-T) \end{bmatrix} \tag{35}$$

In Figure 12, the solutions obtained by (20) for δ and η when ω is changed are plotted as the dotted line. The curve intersects the unit circle at point Q_{11}. Using (14) obtains $T = \pi$, which is consistent with [14].

4. **Conclusion.** Parameter plane method was used for robust stability analysis of different time-delay systems. A simple systematic design procedure is also proposed. Examples in the literature are given to verify the approach. The approach can easily be extended to high order time-delay systems to acquire system element information in the parameter plane.

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