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A DESIGN METHOD FOR SIMPLE REPETITIVE CONTROLLERS WITH SPECIFIED INPUT-OUTPUT CHARACTERISTIC

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Abstract. *The simple repetitive control system proposed by Yamada et al. is a type of servomechanism for periodic reference input. Thus, the simple repetitive control system follows a periodic reference input with small steady-state error, even if there is periodic disturbance or uncertainty in the plant. In addition, simple repetitive control systems ensure that transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles. Yamada et al. clarified the parameterization of all stabilizing simple repetitive controllers. However, when using the method of Yamada et al., it is complex to specify the low-pass filter in the internal model for the periodic reference input that specifies the input-output characteristic. To specify the input-output characteristic more easily, Murakami et al. examined the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic such that the input-output characteristic can be specified beforehand. However, they omitted complete proof on account of space limitations. This paper gives a complete proof of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic of Murakami et al. and demonstrates the effectiveness of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic. Control characteristics of a simple repetitive control system are presented, as well as a design procedure for a simple repetitive controller with the specified input-output characteristic.*

Keywords: Repetitive control, Finite number of poles, Parameterization, Low-pass filter

1. Introduction.

A repetitive control system is a type of servomechanism for a periodic reference input. In other words, the repetitive control system follows a periodic reference input without steady-state error, even if a periodic disturbance or uncertainty exists in the plant [1-13]. It is difficult to design stabilizing controllers for the strictly proper plant, because the repetitive control system that follows any periodic reference input without steady-state error is a neutral type of time-delay control system [11]. To design a repetitive control system that follows any periodic reference input without steady-state error, the plant needs to be biproper [3-11]. In practice, the plant is strictly proper. Many design methods for repetitive control systems for strictly proper plants have been given [3-11]. These systems are divided into two types. One type uses a low-pass filter [3-10] and the other type uses an attenuator [11]. The latter type of system is difficult to design because it uses a state-variable time delay in the repetitive controller [11]. The former has a simple structure and is easily designed. Therefore, the former type of repetitive control system is called the modified repetitive control system [3-10].

Using modified repetitive controllers [3-10], even if the plant does not include time delays, transfer functions from the periodic reference input to the output and from the disturbance to the output have infinite numbers of poles. This makes it difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From a practical point of view, it is desirable that these characteristics should be easy to specify. Therefore, these transfer functions should have finite numbers of poles. To overcome this problem, Yamada et al. proposed simple repetitive control systems such that the controller works as a modified repetitive controller, and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles [14]. In addition, Yamada et al. clarified the parameterization of all stabilizing simple repetitive controllers.

According to Yamada et al., the parameterization of all stabilizing simple repetitive controllers includes two free parameters. One specifies the disturbance attenuation characteristic. The other specifies the low-pass filter in the internal model for the periodic reference input that specifies the input-output characteristic. However, when employing the method of Yamada et al., it is complex to specify the low-pass filter in the internal model for the periodic reference input. When we design a simple repetitive controller, if the low-pass filter in the internal model for the periodic reference input is set beforehand, we can specify the input-output characteristic more easily than in the method employed in [14]. This is achieved by parameterizing all stabilizing simple repetitive controllers with the specified input-output characteristic, which is the parameterization when the low-pass filter is set beforehand. However, no paper has considered the problem of obtaining the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic. In addition, the parameterization is useful to design stabilizing controllers [15-18]. From this viewpoint, Murakami et al. examined the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic such that the input-output characteristic can be specified beforehand [19]. If the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic is used, it is possible to easily design a simple repetitive control system that has a desirable input-output characteristic. However, Murakami et al. omitted the complete proof of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic on account of space limitations. In addition, control characteristics were not examined using the obtained parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic in [19]. Furthermore, a design method for stabilizing a simple repetitive control system with the specified input-output characteristic was not described. Therefore, we do not know whether the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic in [19] is effective.

In this paper, we give a complete proof of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic, which was omitted in [19] and demonstrate the effectiveness of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic. First, we give a complete proof of the theorem for the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic, which was omitted in [19]. Next, we clarify control characteristics using the parameterization in [19]. In addition, a design procedure using the parameterization is presented. A numerical example is presented to illustrate the effectiveness of the proposed design method. Finally, to demonstrate the effectiveness of the parameterization for real plants, we present an application for the reduction of rotational unevenness in motors.

Notation

R	the set of real numbers.
R_+	$R \cup \{\infty\}$.
$R(s)$	the set of real rational functions of s.
RH_∞	the set of stable proper real rational functions.
H_∞	the set of stable causal functions.

2. **Simple Repetitive Controller with the Specified Input-Output Characteristic and Problem Formulation.** Consider the unity feedback control system given by

$$\left\{ \begin{array}{l} y(s) = G(s)u(s) + d(s) \\ u(s) = C(s)(r(s) - y(s)) \end{array} \right. , \tag{1}$$

where $G(s) \in R(s)$ is the strictly proper plant, $C(s)$ is the controller, $u(s) \in R(s)$ is the control input, $y(s) \in R(s)$ is the output, $d(s) \in R(s)$ is the disturbance and $r(s) \in R(s)$ is the periodic reference input with period $T > 0$ satisfying

$$r(t+T) = r(t) \quad (\forall t \geq 0). \tag{2}$$

According to [3-10], the modified repetitive controller $C(s)$ is written in the form

$$C(s) = C_1(s) + C_2(s)C_r(s), \tag{3}$$

where $C_1(s) \in R(s)$ and $C_2(s) \neq 0 \in R(s)$. $C_r(s)$ is an internal model for the periodic reference input $r(s)$ with period T and is written as

$$C_r(s) = \frac{e^{-sT}}{1 - q(s)e^{-sT}}, \tag{4}$$

where $q(s) \in R(s)$ is a proper low-pass filter satisfying $q(0) = 1$.

Using the modified repetitive controller $C(s)$ in (3), transfer functions from the periodic reference input $r(s)$ to the output $y(s)$ and from the disturbance $d(s)$ to the output $y(s)$ in (1) are written as

$$\begin{aligned} \frac{y(s)}{r(s)} &= \frac{C(s)G(s)}{1 + C(s)G(s)} \\ &= \frac{\left\{C_1(s) - (C_1(s)q(s) - C_2(s))e^{-sT}\right\} G(s)}{1 + C_1(s)G(s) - \left\{(1 + C_1(s)G(s))q(s) - C_2(s)G(s)\right\} e^{-sT}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{y(s)}{d(s)} &= \frac{1}{1 + C(s)G(s)} \\ &= \frac{1 - q(s)e^{-sT}}{1 + C_1(s)G(s) - \left\{(1 + C_1(s)G(s))q(s) - C_2(s)G(s)\right\} e^{-sT}}, \end{aligned} \tag{6}$$

respectively. Generally, transfer functions from the periodic reference input $r(s)$ to the output $y(s)$ in (5) and from the disturbance $d(s)$ to the output $y(s)$ in (6) have infinite numbers of poles. When transfer functions from the periodic reference input $r(s)$ to the output $y(s)$ and from the disturbance $d(s)$ to the output $y(s)$ have infinite numbers of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From a practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. To specify the input-output characteristic and the disturbance attenuation characteristic easily, it is desirable for transfer functions from the periodic reference input $r(s)$ to the output

$y(s)$ and from the disturbance $d(s)$ to the output $y(s)$ to have finite numbers of poles. To achieve this, Yamada et al. proposed simple repetitive control systems such that the controller works as a modified repetitive controller, and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles [14]. In addition, Yamada et al. clarified the parameterization of all stabilizing simple repetitive controllers.

On the other hand, according to [3-10], if the low-pass filter $q(s)$ satisfies

$$1 - q(j\omega_i) \simeq 0 \quad (\forall i = 0, \ldots, N_{\max}), \tag{7}$$

where ω_i is the frequency component of the periodic reference input $r(s)$ written by

$$\omega_i = \frac{2\pi}{T} i \quad (i = 0, \ldots, N_{\max}) \tag{8}$$

and $\omega_{N_{\max}}$ is the maximum frequency component of the periodic reference input $r(s)$, then the output $y(s)$ in (1) follows the periodic reference input $r(s)$ with small steady-state error. Using the result in [14], for $q(s)$ to satisfy (7) in a wide frequency range, we must design $q(s)$ to be stable and of minimum phase. If we obtain the parameterization of all stabilizing simple repetitive controllers such that $q(s)$ in (4) is set beforehand, we can design the simple repetitive controller satisfying (7) more easily than in the method in [14].

From the above practical requirement, Murakami et al. proposed the concept of the simple repetitive controller with the specified input-output characteristic as follows [19].

Definition 2.1. *(Simple repetitive controller with the specified input-output characteristic)* [19] *We call the controller $C(s)$ a "simple repetitive controller with the specified input-output characteristic" if the following expressions hold true.*

1. *The low-pass filter $q(s) \in RH_\infty$ in (4) is set beforehand. That is, the input-output characteristic is set beforehand.*
2. *The controller $C(s)$ works as a modified repetitive controller. That is, the controller $C(s)$ is written as (3), where $C_1(s) \in R(s)$, $C_2(s) \neq 0 \in R(s)$ and $C_r(s)$ is written as (4).*
3. *The controller $C(s)$ ensures transfer functions from the periodic reference input $r(s)$ to the output $y(s)$ in (1) and from the disturbance $d(s)$ to the output $y(s)$ in (1) have finite numbers of poles.*

In addition, Murakami et al. examined the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic such that the input-output characteristic can be specified beforehand [19]. However, they omitted a complete proof on account of space limitations. The problem considered in this paper is to give the complete proof of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic in [19] and to propose a design method for a control system using the parameterization in [19].

3. **Parameterization of All Stabilizing Simple Repetitive Controllers with the Specified Input-Output Characteristic.** According to [19], the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic is summarized in the following theorem.

Theorem 3.1. *There exists a stabilizing simple repetitive controller with the specified input-output characteristic if and only if the low-pass filter $q(s) \in RH_\infty$ in (4) takes the form:*

$$q(s) = N(s)\bar{q}(s). \tag{9}$$

Here, $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = \frac{N(s)}{D(s)} \tag{10}$$

and $\bar{q}(s) \neq 0 \in RH_\infty$ is any function. When the low-pass filter $q(s) \in RH_\infty$ in (4) satisfies (9), the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic is given by

$$C(s) = \frac{X(s) + D(s)Q(s) + D(s)\left(Y(s) - N(s)Q(s)\right)\bar{q}(s)e^{-sT}}{Y(s) - N(s)Q(s) - N(s)\left(Y(s) - N(s)Q(s)\right)\bar{q}(s)e^{-sT}}. \tag{11}$$

Here, $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are functions satisfying

$$X(s)N(s) + Y(s)D(s) = 1 \tag{12}$$

and $Q(s) \in RH_\infty$ is any function [19].

Proof of this theorem requires the following lemma.

Lemma 3.1. *The unity feedback control system in (1) is internally stable if and only if $C(s)$ is written as*

$$C(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)}, \tag{13}$$

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ on RH_∞ satisfying (10), $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are functions satisfying (12) and $Q(s) \in RH_\infty$ is any function [18].

Using Lemma 3.1, we present the proof of Theorem 3.1.

Proof: First, the necessity is shown. That is, we show that if the controller $C(s)$ in (3) stabilizes the control system in (1) and ensures that the transfer function from the periodic reference input $r(s)$ to the output $y(s)$ of the control system in (1) has a finite number of poles, then the low-pass filter $q(s)$ must take the form (9). From the assumption that the controller $C(s)$ in (3) ensures that the transfer function from the periodic reference input $r(s)$ to the output $y(s)$ of the control system in (1) has a finite number of poles, we know that

$$\frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{\left\{C_1(s) - (C_1(s)q(s) - C_2(s))e^{-sT}\right\}G(s)}{1 + G(s)C_1(s) - \left\{(1 + G(s)C_1(s))q(s) - C_2(s)G(s)\right\}e^{-sT}} \tag{14}$$

has a finite number of poles. This implies that

$$C_2(s) = \frac{(1 + G(s)C_1(s))q(s)}{G(s)} \tag{15}$$

is satisfied; that is, $C(s)$ is necessarily

$$C(s) = \frac{G(s)C_1(s) + q(s)e^{-sT}}{G(s)\left(1 - q(s)e^{-sT}\right)}. \tag{16}$$

From the assumption that $C(s)$ in (3) stabilizes the control system in (1), we know that $G(s)C(s)/(1+G(s)C(s))$, $C(s)/(1+G(s)C(s))$, $G(s)/(1+G(s)C(s))$ and $1/(1+G(s)C(s))$ are stable. From simple manipulation and (16), we have

$$\frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{G(s)C_1(s) + q(s)e^{-sT}}{1 + G(s)C_1(s)}, \tag{17}$$

$$\frac{C(s)}{1 + G(s)C(s)} = \frac{G(s)C_1(s) + q(s)e^{-sT}}{(1 + G(s)C_1(s))G(s)}, \tag{18}$$

$$\frac{G(s)}{1+G(s)C(s)} = \frac{(1-q(s)e^{-sT})G(s)}{1+G(s)C_1(s)} \tag{19}$$

and

$$\frac{1}{1+G(s)C(s)} = \frac{1-q(s)e^{-sT}}{1+G(s)C_1(s)}. \tag{20}$$

From the assumption that all transfer functions in (17), (18), (19) and (20) are stable, we know that $G(s)C_1(s)/(1+G(s)C_1(s))$, $C_1(s)/(1+G(s)C_1(s))$, $G(s)/(1+G(s)C_1(s))$ and $1/(1+G(s)C_1(s))$ are stable. This means that $C_1(s)$ is an internally stabilizing controller for $G(s)$. From Lemma 3.1, $C_1(s)$ must take the form:

$$C_1(s) = \frac{X(s)+D(s)Q(s)}{Y(s)-N(s)Q(s)}, \tag{21}$$

where $Q(s) \in RH_\infty$. From the assumption that the transfer function in (18) is stable, we know that

$$\frac{q(s)}{G(s)\left(1+G(s)C_1(s)\right)} = \frac{\left(Y(s)-N(s)Q(s)\right)D^2(s)q(s)}{N(s)} \tag{22}$$

is stable. This implies that $q(s)$ must take the form:

$$q(s) = N(s)\bar{q}(s), \tag{23}$$

where $\bar{q}(s) \neq 0 \in RH_\infty$ is any function. In this way, it is shown that if there exists a stabilizing simple repetitive controller with the specified input-output characteristic, then the low-pass filter $q(s)$ must take the form (9).

Next, we show that if (9) holds true, then $C(s)$ is written as (11). Substituting (15), (21) and (23) into (3), we have (11). Thus, the necessity has been shown.

Next, the sufficiency is shown. That is, it is shown that if $q(s)$ and $C(s)$ take the form (9) and (11), respectively, then the controller $C(s)$ stabilizes the control system in (1), ensures that the transfer functions from $r(s)$ and $d(s)$ to $y(s)$ of the control system in (1) have finite numbers of poles and works as a stabilizing modified repetitive controller. After simple manipulation, we have

$$\begin{aligned}&\frac{G(s)C(s)}{1+G(s)C(s)}\\ &= \left\{X(s)+D(s)Q(s)+D(s)\left(Y(s)-N(s)Q(s)\right)\bar{q}(s)e^{-sT}\right\}N(s),\end{aligned} \tag{24}$$

$$\begin{aligned}&\frac{C(s)}{1+G(s)C(s)}\\ &= \left\{X(s)+D(s)Q(s)+D(s)\left(Y(s)-N(s)Q(s)\right)\bar{q}(s)e^{-sT}\right\}D(s),\end{aligned} \tag{25}$$

$$\begin{aligned}&\frac{G(s)}{1+G(s)C(s)}\\ &= \left\{Y(s)-N(s)Q(s)-N(s)\left(Y(s)-N(s)Q(s)\right)\bar{q}(s)e^{-sT}\right)N(s)\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\frac{1}{1+G(s)C(s)}\\ &= \left\{Y(s)-N(s)Q(s)-N(s)\left(Y(s)-N(s)Q(s)\right)\bar{q}(s)e^{-sT}\right\}D(s).\end{aligned} \tag{27}$$

Since $X(s) \in RH_\infty$, $Y(s) \in RH_\infty$, $N(s) \in RH_\infty$, $D(s) \in RH_\infty$, $Q(s) \in RH_\infty$ and $\bar{q}(s) \in RH_\infty$, the transfer functions in (24), (25), (26) and (27) are stable. In addition, for the same reason, transfer functions from $r(s)$ and $d(s)$ to $y(s)$ of the control system in (1) have finite numbers of poles.

Next, we show that the controller in (11) works as a modified repetitive controller. The controller in (11) is rewritten in the form in (3), where

$$C_1(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)} \tag{28}$$

and

$$C_2(s) = \frac{\bar{q}(s)}{(Y(s) - N(s)Q(s))}. \tag{29}$$

From the assumption of $\bar{q}(s) \neq 0$, $C_2(s) \neq 0$ holds true. These expressions imply that the controller $C(s)$ in (11) works as a modified repetitive controller. Thus, the sufficiency has been shown.

We have thus proved Theorem 3.1.

Remark 3.1. *Note that from Theorem 3.1, when the plant $G(s)$ is of non-minimum phase, the low-pass filter $q(s)$ cannot be set to be of minimum phase.*

4. Control Characteristics.

In this section, we describe control characteristics of the control system in (1) using the stabilizing simple repetitive controller in (11).

First, we mention the input-output characteristic. The transfer function $S(s)$ from the periodic reference input $r(s)$ to the error $e(s) = r(s) - y(s)$ is written as

$$S(s) = \frac{1}{1 + G(s)C(s)} = D(s)\left(Y(s) - N(s)Q(s)\right)\left(1 - q(s)e^{-sT}\right). \tag{30}$$

From (30), since $q(s)$ is set beforehand to satisfy (7), the output $y(s)$ follows the periodic reference input $r(s)$ with small steady-state error. That is, we find that by using the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic in [19], the input-output characteristic can be specified beforehand.

Next, we mention the disturbance attenuation characteristic. The transfer function from the disturbance $d(s)$ to the output $y(s)$ is written as (30). From (30), for the frequency component ω_i ($i = 0, \ldots, N_{\max}$) in (8) of the disturbance $d(s)$ that is the same as that of the periodic reference input $r(s)$, since $S(s)$ satisfies $S(j\omega_i) \simeq 0$ ($\forall i = 0, \ldots, N_{\max}$), the disturbance $d(s)$ is attenuated effectively. For the frequency component ω_d of the disturbance $d(s)$ that is different from that of the periodic reference input $r(s)$ (that is, $\omega_d \neq \omega_i$), even if

$$1 - q(j\omega_d) \simeq 0, \tag{31}$$

the disturbance $d(s)$ cannot be attenuated because

$$e^{-j\omega_d T} \neq 1 \tag{32}$$

and

$$1 - q(j\omega_d)e^{-j\omega_d T} \neq 0. \tag{33}$$

To attenuate the frequency component ω_d of the disturbance $d(s)$ that is different from that of the periodic reference input $r(s)$, we need to set $Q(s)$ satisfying

$$Y(j\omega_d) - N(j\omega_d)Q(j\omega_d) \simeq 0. \tag{34}$$

From the above discussion, the role of $q(s)$ is to specify the input-output characteristic for the periodic reference input $r(s)$ and it can be specified beforehand. The role of $Q(s)$ is to specify the disturbance attenuation characteristic for the frequency component of the disturbance $d(s)$ that is different from that of the periodic reference input $r(s)$.

5. **Design Procedure.** In this section, a design procedure for stabilizing the simple repetitive controller with the specified input-output characteristic is presented.

A design procedure for stabilizing simple repetitive controllers satisfying Theorem 3.1 is summarized as follows.

Procedure

Step 1) Obtain coprime factors $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ of $G(s)$ satisfying (10).

Step 2) $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are set satisfying (12).

Step 3) $\bar{q}(s) \in RH_\infty$ in (9) is set so that for the frequency component ω_i ($i = 0, \ldots, N_{\max}$) of the periodic reference input $r(s)$,

$$1 - q(j\omega_i) = 1 - N(j\omega_i)\bar{q}(j\omega_i) \simeq 0. \tag{35}$$

To satisfy $1 - N(j\omega_i)\bar{q}(j\omega_i) \simeq 0$, $\bar{q}(s) \in RH_\infty$ is set according to

$$\bar{q}(s) = \frac{1}{N_o(s)}\bar{q}_r(s), \tag{36}$$

where $N_o(s) \in RH_\infty$ is an outer function of $N(s)$ satisfying

$$N(s) = N_i(s)N_o(s), \tag{37}$$

$N_i(s) \in RH_\infty$ is an inner function satisfying $N_i(0) = 1$ and $|N_i(j\omega)| = 1$ ($\forall\omega \in R_+$), $\bar{q}_r(s)$ is a low-pass filter satisfying $\bar{q}_r(0) = 1$, as

$$\bar{q}_r(s) = \frac{1}{(1 + s\tau_r)^{\alpha_r}} \tag{38}$$

is valid, α_r is an arbitrary positive integer that ensures $\bar{q}_r(s)/N_o(s)$ is proper and $\tau_r \in R$ is any positive real number satisfying

$$1 - N_i(j\omega_i)\frac{1}{(1 + j\omega_i\tau_r)^{\alpha_r}} \simeq 0 \quad (\forall i = 0, \ldots, N_{\max}). \tag{39}$$

Step 4) $Q(s) \in RH_\infty$ is set so that for the frequency component ω_d of the disturbance $d(s)$, $Y(j\omega_d) - N(j\omega_d)Q(j\omega_d) \simeq 0$ is satisfied. To design $Q(s)$ to hold $Y(j\omega_d) - N(j\omega_d)Q(j\omega_d) \simeq 0$, $Q(s)$ is set according to

$$Q(s) = \frac{Y(s)}{N_o(s)}\bar{q}_d(s), \tag{40}$$

where $\bar{q}_d(s)$ is a low-pass filter satisfying $\bar{q}_d(0) = 1$, as

$$\bar{q}_d(s) = \frac{1}{(1 + s\tau_d)^{\alpha_d}} \tag{41}$$

is valid, α_d is an arbitrary positive integer that ensures $\bar{q}_d(s)/N_o(s)$ is proper and $\tau_d \in R$ is any positive real number satisfying

$$1 - N_i(j\omega_d)\frac{1}{(1 + j\omega_d\tau_d)^{\alpha_d}} \simeq 0. \tag{42}$$

6. **Numerical Example.** In this section, a numerical example is presented to illustrate the effectiveness of the proposed method.

We consider the problem of obtaining the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic for the plant $G(s)$ written as

$$G(s) = \frac{s - 50}{(s + 1)(s - 1)} \tag{43}$$

that follows the periodic reference input $r(t)$ with period $T = 2$[s].

A pair of coprime factors $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ of $G(s)$ in (43) satisfying (10) is given by

$$N(s) = \frac{s-50}{(s+30)(s+40)} \tag{44}$$

and

$$D(s) = \frac{(s+1)(s-1)}{(s+30)(s+40)}. \tag{45}$$

$q(s)$ is set according to

$$\begin{aligned} q(s) &= N_i(s)\bar{q}_r(s) \\ &= \frac{-s+50}{s+50} \cdot \frac{1}{0.001s+1}, \end{aligned} \tag{46}$$

where

$$N_i(s) = \frac{-s+50}{s+50} \tag{47}$$

and

$$\bar{q}_r(s) = \frac{1}{0.001s+1}. \tag{48}$$

$X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ satisfying (12) are derived as

$$X(s) = -\frac{3943s+29024}{(s+30)(s+40)} \tag{49}$$

and

$$Y(s) = \frac{s^2+140s+11244}{(s+30)(s+40)}. \tag{50}$$

From Theorem 3.1, the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic for $G(s)$ in (43) is given by (11), where $Q(s) \in RH_\infty$ in (11) is any function. So that the disturbance

$$d(t) = \sin\left(\frac{\pi t}{2}\right) \tag{51}$$

can be attenuated effectively, $Q(s)$ is set by (40), where

$$\bar{q}_d(s) = \frac{1}{0.001s+1} \tag{52}$$

and

$$N_o(s) = \frac{-s-50}{(s+30)(s+40)}. \tag{53}$$

Using the above mentioned parameters, we have a stabilizing simple repetitive controller with the specified input-output characteristic.

Using the designed stabilizing simple repetitive controller with the specified input-output characteristic, the response of the error $e(t) = r(t) - y(t)$ in (1) for the periodic reference input $r(t) = \sin(\pi t)$ is shown in Figure 1. Here, the dotted line shows the response of the periodic reference input $r(t) = \sin(\pi t)$ and the solid line shows that of the error $e(t) = r(t) - y(t)$. Figure 1 shows that the output $y(t)$ follows the periodic reference input $r(t)$ with a small steady-state error.

Next, using the designed simple repetitive controller with the specified input-output characteristic $C(s)$, the disturbance attenuation characteristic is shown. The response of the output $y(t)$ for the disturbance $d(t) = \sin(2\pi t)$ of which the frequency component

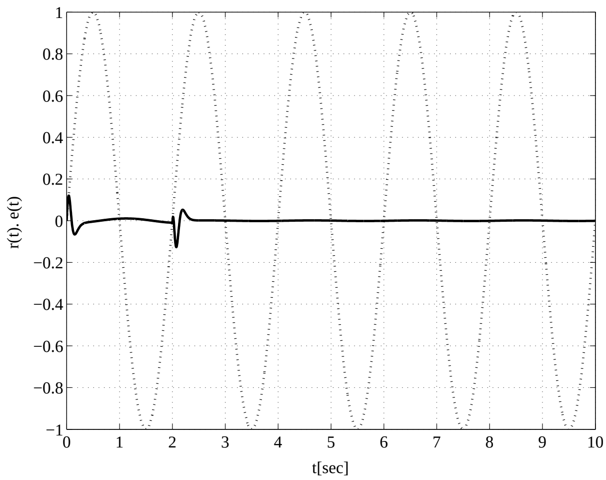

FIGURE 1. Response of the error $e(t) = r(t) - y(t)$ for the periodic reference input $r(t) = \sin(\pi t)$

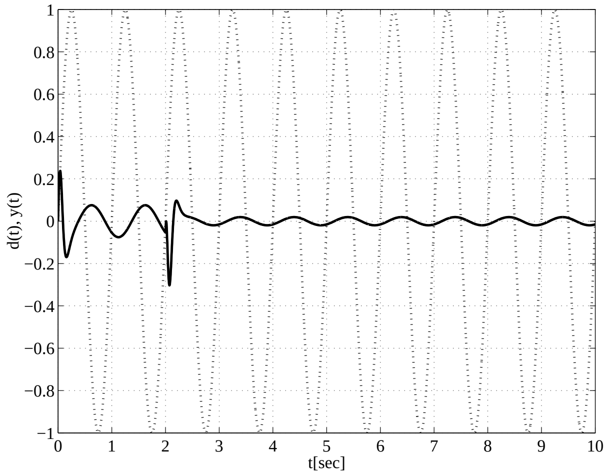

FIGURE 2. Response of the output $y(t)$ for the disturbance $d(t) = \sin(2\pi t)$

is equivalent to that of the periodic reference input $r(t)$ is shown in Figure 2. Here, the dotted line shows the response of the disturbance $d(t) = \sin(2\pi t)$ and the solid line shows that of the output $y(t)$. Figure 2 shows that the disturbance $d(t) = \sin(2\pi t)$ is attenuated effectively. Finally, the response of the output $y(t)$ for the disturbance $d(t)$ in (51) of which the frequency component is different from that of the periodic reference input $r(t)$ is shown in Figure 3. Here, the dotted line shows the response of the disturbance $d(t)$ in (51) and the solid line shows that of the output $y(t)$. Figure 3 shows that the disturbance $d(t)$ in (51) is attenuated effectively.

A stabilizing simple repetitive controller with the specified input-output characteristic can be easily designed in the way shown here.

7. **Application of Reducing Rotational Unevenness in Motors.** In this section, to demonstrate the effectiveness of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic for real plants, we present an application of reducing rotational unevenness in motors.

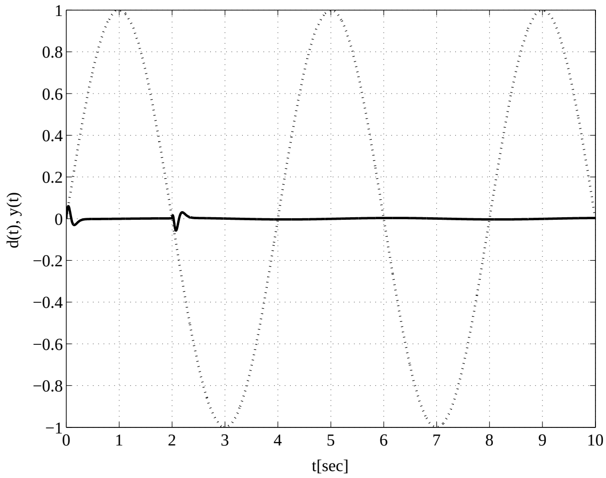

FIGURE 3. Response of the output $y(t)$ for the disturbance $d(t) = \sin\left(\frac{\pi t}{2}\right)$

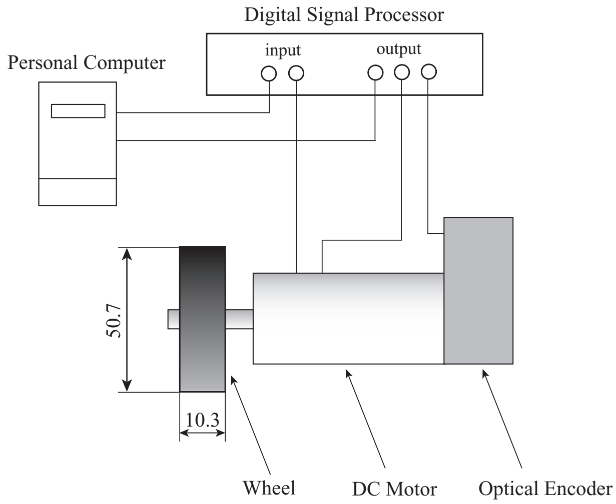

FIGURE 4. Illustrated motor control experiment

7.1. **Motor control experiment and problem description.** A motor control experiment is illustrated in Figure 4. The motor control experiment consists of a direct-current motor with an optical encoder of 1000[counts/revolution] and a wheel that has a diameter of 50.7[mm], a width of 10.3[mm] and mass of 72.5[g] attached to the motor. We denote with T_v[rad/s] the estimated value of the angular velocity of the wheel calculated from the measurement of the angle of the wheel. V_m denotes a control input for the direct-current motor, and the available voltage of V_m is $-24[\mathrm{V}] \le V_m \le 24[\mathrm{V}]$. When we set $V_m = 2.1[\mathrm{V}]$, the response of T_v, which is the angular velocity of the wheel, is shown in Figure 5 and Figure 6. Figure 5 and Figure 6 show disturbances including rotational unevenness in the motor. Since the rotational unevenness in the motor depends on the angle of the motor, the disturbance is considered a periodic disturbance.

The problem considered in this experiment is to design a control system to attenuate periodic disturbances including the rotational unevenness in the motor by parameterizing all stabilizing simple repetitive controllers with the specified input-output characteristic in [19], which is an effective compensator for attenuating periodic disturbances effectively.

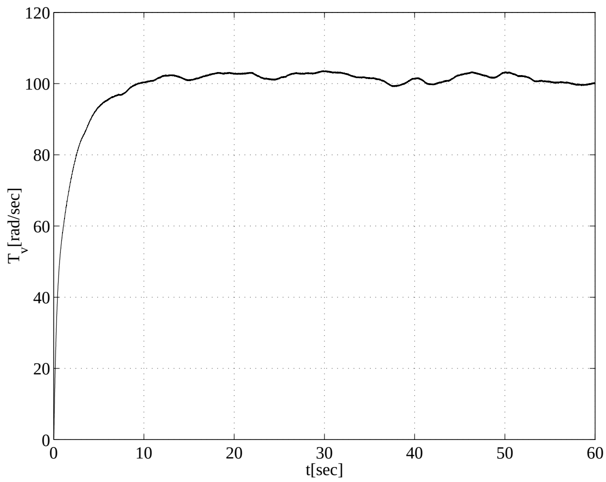

FIGURE 5. Response of T_v when $V_m = 2.1$[V]

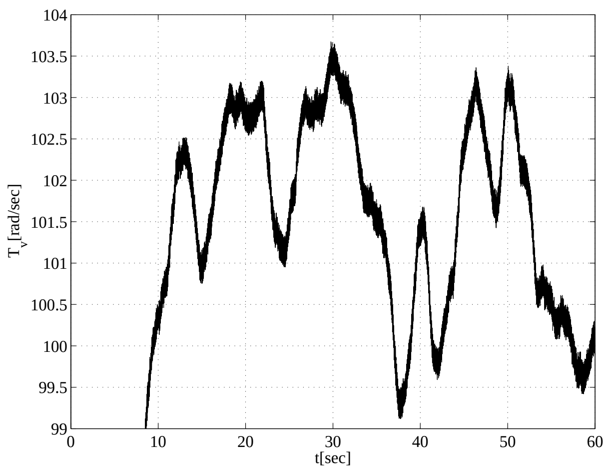

FIGURE 6. Magnified plot of Figure 5 between 99[rad/s] and 104[rad/s]

7.2. **Experimental result.** In this subsection, we present experimental results of controlling the angular velocity in the motor control experiment in Figure 4 using the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic.

From Figure 5, we find that the transfer function from V_m to T_v, which is the angular velocity of the wheel, is

$$T_v = \frac{48}{1 + 1.31s} V_m. \tag{54}$$

T_v and V_m are considered as the output $y(s)$ and the control input $u(s)$ in the control system. $G(s)$ is then written as

$$G(s) = \frac{48}{1 + 1.31s} \in RH_\infty. \tag{55}$$

The reference input $r(s)$ is set as $r(t) = v_r = 100$[rad/s]. The period T of the disturbance $d(t)$ is

$$T = \frac{2\pi}{v_r} = \frac{2\pi}{100}. \tag{56}$$

To attenuate the periodic disturbance $d(t)$ with period T, we design a simple repetitive controller with the specified input-output characteristic $C(s)$ in (11). Coprime factors $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ of the plant $G(s)$ in (55) on RH_∞ are given by

$$N(s) = \frac{114.2857}{s+1} \tag{57}$$

and

$$D(s) = \frac{s+2.381}{s+1}. \tag{58}$$

A pair of $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ satisfying $N(s)X(s) + D(s)Y(s) = 1$ is written as

$$X(s) = \frac{0.0167}{s+1} \tag{59}$$

and

$$Y(s) = \frac{s-0.381}{s+1}. \tag{60}$$

$q(s)$ is set according to

$$q(s) = N_i(s)\bar{q}_r(s) = \frac{1}{0.2s+1}, \tag{61}$$

where

$$N_i(s) = 1 \tag{62}$$

and

$$\bar{q}_r(s) = \frac{1}{0.2s+1}. \tag{63}$$

Using the abovementioned parameters, the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic for $G(s)$ in (55) is given by (11), where $Q(s) \in RH_\infty$ in (11) is any function.

$Q(s)$ is set by (40), where

$$\bar{q}_d(s) = \frac{1}{0.03s+1} \tag{64}$$

and

$$N_o(s) = N(s). \tag{65}$$

Substitution of $Q(s)$ into (11) gives a stabilizing simple repetitive controller with the specified input-output characteristic $C(s)$.

Using the designed simple repetitive controller with the specified input-output characteristic $C(s)$, the response of the output $y(t)$, which is the angular velocity of the wheel T_v, for the reference input $r(t) = 100$[rad/s], is shown in Figure 7 and Figure 8. Figure 7 and Figure 8 show that the output $y(t)$, which is the angular velocity of the wheel T_v, follows the reference input $r(t) = 100$[rad/s] with small steady-state error. In addition, the disturbance $d(t)$ that includes the rotational unevenness in the motor is attenuated effectively.

To demonstrate the effectiveness of the simple repetitive controller with the specified input-output characteristic, a comparison was made with the response when using the

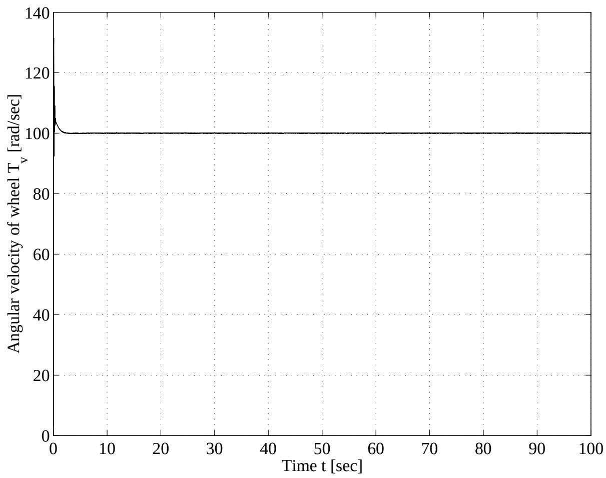

FIGURE 7. Response of the output $y(t)$, which is the angular velocity of the wheel T_v, for the reference input $r(t) = 100$[rad/s] using the simple repetitive controller with the specified input-output characteristic

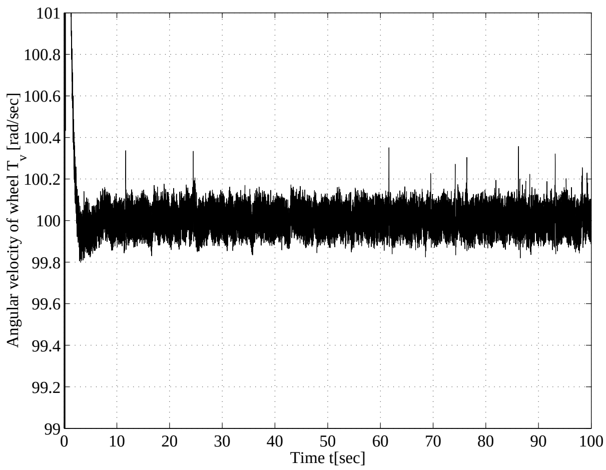

FIGURE 8. Magnified plot of Figure 7 between 99[rad/s] and 101[rad/s]

parameterization of all stabilizing modified repetitive controllers with the specified input-output characteristic in [20] written as

$$C(s) = \frac{X(s) + D(s)\hat{Q}(s)}{Y(s) - N(s)\hat{Q}(s)}, \tag{66}$$

where

$$\hat{Q}(s) = \frac{Q_n(s) + \left(Y(s)\bar{Q}(s) - Q_n(s)\right) q(s)e^{-sT}}{Q_d(s) + \left(N(s)\bar{Q}(s) - Q_d(s)\right) q(s)e^{-sT}} \in H_\infty. \tag{67}$$

Here, $Q_n(s) \in RH_\infty$, $\bar{Q}(s) \neq 0 \in RH_\infty$ and $Q_d(s) \neq 0 \in RH_\infty$ are any functions. $N(s) \in RH_\infty$, $D(s) \in RH_\infty$, $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are given by (57), (58), (59) and (60), respectively. $q(s)$ is a low-pass filter that satisfies $q(0) = 1$ and specifies the input-output characteristic for the periodic reference input $r(s)$ and the disturbance attenuation characteristic for the frequency component of the disturbance $d(s)$ that is the same as that of the periodic reference input $r(s)$. To compare the simple repetitive controller and the modified repetitive controller fairly, $q(s)$ in (67) is set as that of the simple repetitive controller; that is, $q(s)$ is set by (61). Using the above mentioned parameters, the parameterization of all stabilizing modified repetitive controllers with the specified input-output characteristic $C(s)$ is written as (66) with (67).

For $\hat{Q}(s)$ to satisfy $\hat{Q}(s) \in H_\infty$, $Q_d \in RH_\infty$ and $\bar{Q}(s) \in RH_\infty$ are set according to

$$Q_d(s) = \frac{2s + 100}{s + 0.1} \tag{68}$$

and

$$\bar{Q}(s) = \frac{5(s^2 + s + 1)}{3\left(10s^2 + s + 2\right)}, \tag{69}$$

respectively. $Q_n(s)$ in (67) is set according to

$$Q_n(s) = \frac{Y(s)Q_d(s)}{N(s)}\bar{q}_d(s), \tag{70}$$

where $\bar{q}_d(s)$ is given by (64). Substitution of $Q_n(s)$, $Q_d(s)$ and $\bar{Q}(s)$ into (67) gives a stabilizing modified repetitive controller $C(s)$.

Using the obtained modified repetitive controller $C(s)$, the response of the output $y(t)$, which is the angular velocity of the wheel T_v, for the reference input $r(t) = 100$[rad/s] is shown in Figure 9 and Figure 10. Figure 9 and Figure 10 show that the output $y(t)$, which

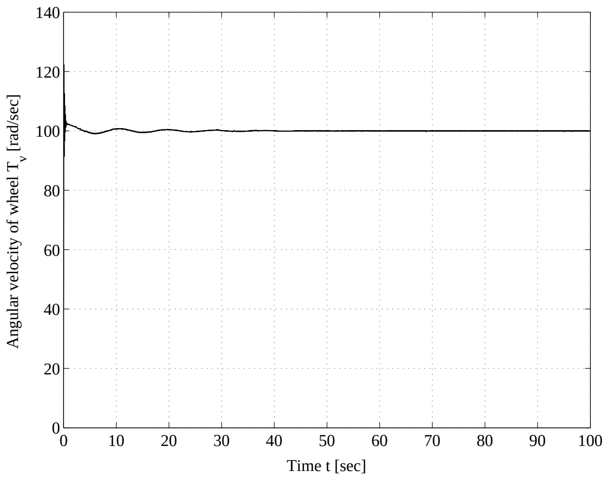

FIGURE 9. Response of the output $y(t)$, which is the angular velocity of the wheel T_v, for the reference input $r(t) = 100$[rad/s] using the modified repetitive controller with the specified input-output characteristic

is the angular velocity of the wheel T_v, follows the reference input $r(t) = 100$[rad/s] with small steady-state error. In addition, the disturbance $d(t)$ that includes the rotational unevenness of the motor is attenuated effectively.

The comparison of Figure 8 with Figure 10 shows that the convergence of the simple repetitive control system is faster than that of the modified repetitive control system. In addition, the simple repetitive control system attenuates the disturbance that includes the rotational unevenness in the motor more effectively than the modified repetitive control system. The simple repetitive control system has merits such as the transfer functions from the periodic reference input to the output having finite numbers of poles and the system being easy to design. This result illustrates that the simple repetitive control system is more effective for the reduction of rotational unevenness in motors than the modified repetitive control system.

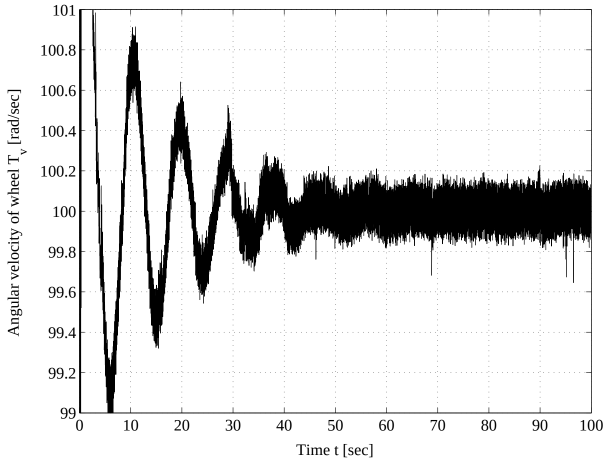

Figure 10. Magnified plot of Figure 9 between 99[rad/s] and 101[rad/s]

In this way, the effectiveness of the control system employing the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic in (11) for real plants has been shown.

8. **Conclusions.** In this paper, we gave a complete proof of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic such that the low-pass filter in the internal model for the periodic reference input is set beforehand, the controller works as a stabilizing modified repetitive controller, and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles. In addition, we demonstrated the effectiveness of the parameterization of all stabilizing simple repetitive controllers with the specified input-output characteristic. Control characteristics of a simple repetitive control system were presented, as well as a design procedure for a simple repetitive controller with the specified input-output characteristic. Finally, a numerical example and an application for the reduction of rotational unevenness in motors were presented to illustrate the effectiveness of the proposed method.

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