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ADAPTIVE SYNCHRONIZATION OF HYPERCHAOTIC CHEN SYSTEMS WITH APPLICATION TO SECURE COMMUNICATION

Nejib Smaoui[1], Abdulrahman Karouma[1] and Mohamed Zribi[2]

[1]Department of Mathematics
[2]Department of Electrical Engineering
Kuwait University
P.O. Box 5969, Safat 13060, Kuwait
nsmaoui64@yahoo.com

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Abstract. *This paper deals with the adaptive synchronization of two identical hyperchaotic master and slave Chen systems. First, the slave system is assumed to have four inputs. Using Lyapunov theory, it is shown that the errors between the states of the master and the slave systems asymptotically converge to zero. Simulation results are presented to illustrate the ability of the adaptive controller to synchronize the hyperchaotic systems. Then, an improved adaptive control law is designed with only three control inputs in the slave system. Asymptotic convergence of the errors between the states of the master and the slave systems to zero is theoretically proven and then validated through numerical simulations. Moreover, the proposed control scheme is applied to encrypt and decrypt images where computer simulation results are provided to show the efficiency of the proposed control law.*

Keywords: Chaos, Hyperchaotic systems, Synchronization, Adaptive control, Secure communications

1. **Introduction.** The control and the synchronization of hyperchaotic systems have been studied extensively in the past few years [1-21, 26-35]. In 1999, Hsieh et al. [26] tackled the synchronization problem of two hyperchaotic Rössler systems, where a feedback control law with only two control inputs is designed to synchronize two identical hyperchaotic Rössler systems with known fixed parameters.

In 2008, Yassen [10] used the Lyapunov stability theory to synchronize hyperchaotic systems. He managed to synchronize two identical Lü systems, two identical Chen systems and a Chen system with a Lü system. The main drawback of this work is that four control inputs are needed to synchronize two hyperchaotic systems. In [21], Li et al. investigated the synchronization of two hyperchaotic Lorenz systems based on the unidirectionally linear coupled approach. Also, in [21], circuit implementation of the hyperchaotic Lorenz system is provided.

In 2009, Chen et al. [30] introduced a new hyperchaotic system by adding a nonlinear feedback controller to the Chen chaotic system. The authors study the dynamics and behaviors of the proposed system. Also, a nonlinear feedback controller was designed to achieve hybrid projective synchronization of the proposed new hyperchaotic system. In [31], Austin et al. designed an adaptive scheme to synchronize two hyperchaotic Chen unidirectionally coupling and bidirectionally coupling systems with uncertain parameters.

Adaptive control is a technique which is generally used to overcome the uncertainties of a system by modifying the controller to adapt to the system's uncertainties. Since most of the dynamical models of systems in nature contain some uncertainties, many researchers have used adaptive control in various fields and applications [2-5, 12, 13, 16-31]. A difference between an adaptive controller and a robust controller is that the adaptive controller does not require prior knowledge about the bounds on the uncertainties of the system, while a robust controller does. For instance, the change of an airplane's weight caused by the continuous consumption of fuel, where a robust control might fail because of the large bound of the change, an adaptive control would generally overcome such changes.

The adaptive control and the synchronization of hyperchaotic systems were investigated in [4, 5, 11-13, 16, 27]. In 2005, Lu and Cao [4] studied the adaptive synchronization of chaotic and hyperchaotic systems. The authors designed controllers to synchronize two identical Lorenz chaotic systems with uncertainty in the parameters. They presented three adaptive controllers to drive the three dimensional uncertain Lorenz system to be synchronized with the first three states of the four dimensional hyperchaotic Chen system. Then, the authors tackled the adaptive synchronization of the hyperchaotic Chen system with the hyperchaotic Rössler system when the parameters are totally unknown. The analytical results presented in [4] are based on the Lyapunov stability theory. The author showed the effectiveness and robustness of the proposed design against uncertainties and external disturbances by adding a random noise in the range of $[-2, 2]$ and showing through simulations that the proposed controller works. Also, Park [5] managed to design an adaptive control law for the synchronization of two hyperchaotic Chen systems; the Lyapunov stability theory was used to derive the adaptive control laws. Numerical simulation results are provided to show the effectiveness of the proposed control laws.

In 2006, Elabbasy et al. [12] discussed the adaptive synchronization of two hyperchaotic Lü systems, where adaptive control laws are derived to synchronize two hyperchaotic Lü systems when the parameters are totally unknown and when the parameters of the master and the slave systems are different.

In 2008, Wu et al. [13] tackled the problem of adaptive synchronization of two different hyperchaotic systems with parameter uncertainty, where adaptive control laws are derived to drive the hyperchaotic Henon-Heiles system to be synchronized with the hyperchaotic Chen system. The number of control inputs used (four) is considered to be the main drawback of this design. In [16], Tao and Liu proposed novel schemes to control and synchronize chaotic and hyperchaotic systems. They synchronized two hyperchaotic Chen systems whose parameters are uncertain. Also, based on the Lyapunov stability theory, Wu et al. [11] designed adaptive control laws to synchronize two non-identical uncertain hyperchaotic Chen and second-harmonic generation systems. In [27], Wang and Wang tackled the adaptive synchronization problem of the chaotic Chen system, coupled dynamos system and the hyperchaotic Rössler system.

In this paper, two adaptive control schemes are proposed. Using Lyapunov theory, it is shown that the errors between the states of the master and the slave systems converge asymptotically to zero. Numerical results are presented to reinforce the analytical results. Moreover, the proposed controllers are used for secure communications purposes; the simulation results indicate that the proposed controllers enable us the secure transmission of images.

The paper is organized as follows. A description of the hyperchaotic Chen system is presented in Section 2. Section 3 presents the design of the adaptive controller to synchronize two hyperchaotic Chen systems when the number of control inputs is four. The developed theory is validated through numerical simulations in Section 4. Section 5 presents an improved adaptive controller design based on three control inputs; the numerical simulations are given in Section 6 to validate the developed theory. Section 7 presents a secure communication scheme based on the hyperchaotic Chen system using the adaptive controller designed in Section 5. Finally, some concluding remarks are given in Section 8.

2. **System Description.** In 1999, Chen and Ueta [14] created a novel chaotic dynamical system, called the Chen system, and it is described by the following set of ordinary differential equations:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (b - a)x - xz + by \\ \dot{z} = xy - cz, \end{cases} \tag{1}$$

where x, y, z are the state variables and (a, b, c) are the system's parameters. When $a = 35$, $b = 28$ and $c = 3$, system (1) exhibits a chaotic behavior as shown in Figure 1.

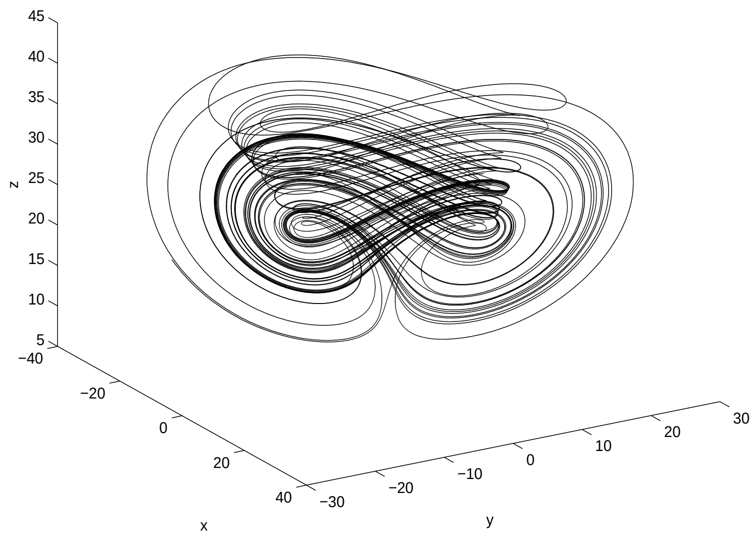

FIGURE 1. The chaotic attractor for the Chen chaotic system when $a = 35$, $b = 28$ and $c = 3$

This system differs from the chaotic Lorenz system in its dynamical behavior and topological properties. For instance, the Jacobian of the linearized system, $A = [a_{ij}]_{3\times3}$, of the Lorenz system satisfy the condition $a_{12}a_{21} > 0$ while the Jacobian of the Chen system satisfies the condition $a_{12}a_{21} < 0$. In 2002, Chen and Lü [1] created another novel dynamical system which represents the transition between the Lorenz system and the Chen system. This new system satisfy the condition $a_{12}a_{21} = 0$.

The hyperchaotic Chen system is generated from the chaotic Chen system and it is described as follows:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + w_1 \\ \dot{y}_1 = dx_1 - x_1z_1 + cy_1 \\ \dot{z}_1 = x_1y_1 - bz_1 \\ \dot{w}_1 = y_1z_1 + rw_1, \end{cases} \tag{2}$$

where x_1, y_1, z_1, w_1 are the state variables and a, b, c, d, r are real constants. When $a = 35$, $b = 3$, $c = 12$, $d = 7$ and $0 \le r \le 0.085$, system (2) is chaotic. When $a = 35$, $b = 3$, $c = 12$, $d = 7$, $0.085 < r \le 0.798$, system (2) is hyperchaotic (see Figure 2). When $a = 35$, $b = 3$, $c = 12$, $d = 7$, $0.798 < r \le 0.9$, system (2) is periodic (see [15, 16]).

This paper deals with the synchronization of hyperchaotic systems. Therefore, we will consider system (2) as the master system. We will define the slave system to be the

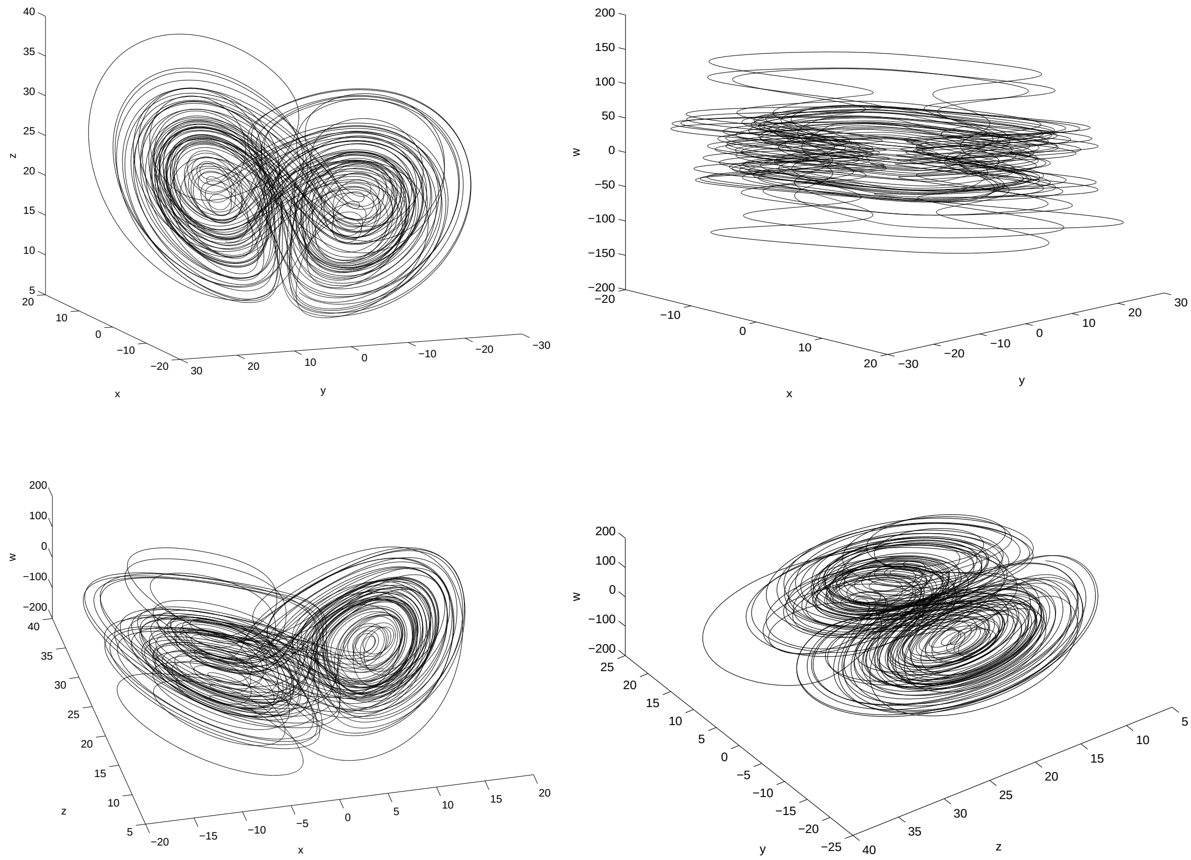

FIGURE 2. The phase portrait of the attractor of the hyperchaotic Chen system when $a = 35$, $b = 3$, $c = 12$, $d = 7$ and $r = 0.5$

hyperchaotic Chen system with the same parameter as system (2). Therefore, the slave system is defined as follows:

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + w_2 + u_1 \\ \dot{y}_2 = dx_2 - x_2z_2 + cy_2 + u_2 \\ \dot{z}_2 = x_2y_2 - bz_2 + u_3 \\ \dot{w}_2 = y_2z_2 + rw_2 + u_4, \end{cases} \tag{3}$$

where x_2, y_2, z_2, w_2 represent the states of the slave system, and u_1, u_2, u_3, u_4 represent the controllers. The rest of the paper deals with the design and implementation of control schemes to synchronize the master system with the slave system. It should be noted that the parameters a, b, c, d and r are assumed to be unknown.

3. **The Design of the First Adaptive Control Law.** Using systems (2) and (3), we can define the error system as follows:

$$\begin{cases} \dot{e}_x = a(e_y - e_x) + e_w + u_1 \\ \dot{e}_y = de_x - e_xe_z - x_1e_z - z_1e_x + ce_y + u_2 \\ \dot{e}_z = -be_z + e_xe_y + x_1e_y + y_1e_x + u_3 \\ \dot{e}_w = e_ye_z + y_1e_z + z_1e_y + re_w + u_4, \end{cases} \tag{4}$$

where $e_x = x_2 - x_1$, $e_y = y_2 - y_1$, $e_z = z_2 - z_1$ and $e_w = w_2 - w_1$. Our objective is to design the controllers u_i, $i = \overline{1,4}$ and update laws such that the controllers u_i will drive the slave system (3) to be synchronized with the master system (2) for different initial values despite the fact that the parameters are unknown. Thus forcing the origin of the error system (4) to be globally asymptotically stable. The first theorem gives the first result of the paper.

Theorem 3.1. *Let α_i, $i = \overline{1,4}$ be positive scalars, and let the controllers u_1, u_2, u_3, u_4 be such that:*

$$\begin{cases} u_1 = -e_w - \hat{a}e_y + (\hat{a} - \alpha_1)e_x \\ u_2 = -\hat{d}e_x + e_xe_z + x_1e_z + z_1e_x - (\hat{c} + \alpha_2)e_y \\ u_3 = -e_xe_y - x_1e_y - y_1e_x + (\hat{b} - \alpha_3)e_z \\ u_4 = -e_ye_z - y_1e_z - z_1e_y - (\hat{r} + \alpha_4)e_w, \end{cases} \tag{5}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{r}$ are the estimates of the parameters a, b, c, d and r, respectively, and which are updated according to the following update laws:

$$\begin{cases} \dfrac{d\hat{a}}{dt} = e_x(e_y - e_x) \\ \dfrac{d\hat{b}}{dt} = -e_z^2 \\ \dfrac{d\hat{c}}{dt} = e_y^2 \\ \dfrac{d\hat{d}}{dt} = e_xe_y \\ \dfrac{d\hat{r}}{dt} = e_w^2. \end{cases} \tag{6}$$

Then the slave Chen system is synchronized with the master Chen system for any initial conditions.

Proof: Using the controllers given in (5) into the error system (4), we get

$$\begin{cases} \dot{e}_x = -e_a(e_y - e_x) - \alpha_1e_x \\ \dot{e}_y = -e_de_x - e_ce_y - \alpha_2e_y \\ \dot{e}_z = e_be_z - \alpha_3e_z \\ \dot{e}_w = -e_re_w - \alpha_4e_w, \end{cases} \tag{7}$$

where the parameter errors are defined such that $e_a = \hat{a} - a$, $e_b = \hat{b} - b$, $e_c = \hat{c} - c$, $e_d = \hat{d} - d$ and $e_r = \hat{r} - r$.

Note that since the parameters of the system are constants, then $\dot{e}_a = \dot{\hat{a}}$, $\dot{e}_b = \dot{\hat{b}}$, $\dot{e}_c = \dot{\hat{c}}$, $\dot{e}_d = \dot{\hat{d}}$ and $\dot{e}_r = \dot{\hat{r}}$.

Let the positive definite function $V_1 = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_w^2 + {e_a}^2 + {e_b}^2 + {e_c}^2 + {e_d}^2 + {e_r}^2)$ be a Lyapunov function candidate for system (7). Then, the time derivative of V_1 along the trajectories of system (7) and using the update laws given by (6) is such that:

$$\begin{aligned} \dot{V}_1 =& (e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + e_w\dot{e}_w + e_a\dot{e}_a + e_b\dot{e}_b + e_c\dot{e}_c + e_d\dot{e}_d + e_r\dot{e}_r) \\ =& e_x(-e_a(e_y - e_x) - \alpha_1e_x) + e_y(-e_de_x - e_ce_y - \alpha_2e_y) \\ &+ e_z(e_be_z - \alpha_3e_z) + e_w(-e_re_w - \alpha_4e_w) + e_a(e_x(e_y - e_x)) \\ &+ e_b(-e_z^2) + e_c(e_y^2) + e_d(e_xe_y) + e_r(e_w^2) \\ =& -(\alpha_1e_x^2 + \alpha_2e_y^2 + \alpha_3e_z^2 + \alpha_4e_w^2). \end{aligned} \tag{8}$$

Define the error vector e such that: $e = [e_x\ e_y\ e_z\ e_w]^T$. Also, let $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then, Equation (8) implies that,

$$\dot{V}_1 \le -\alpha||e||^2. \tag{9}$$

Inequality (9) implies that $\dot{V}_1$ is negative semi-definite. Therefore, it can be concluded that the state and parameters errors are bounded, i.e., $[e_x\ e_y\ e_z\ e_w\ e_a\ e_b\ e_c\ e_d\ e_r] \in L_\infty$.

Integrating the inequality $\dot{V}_1 \leq -\alpha||e||^2$ from 0 to t, we obtain the following:

$$\alpha \int_0^t ||e||^2 d\tau \leq \int_0^t -\dot{V}_1 d\tau = V_1(0) - V_1(t) \leq V_1(0). \tag{10}$$

Therefore, it can be concluded that $e \in L_2$ and using (7) it can be concluded that $\dot{e} \in L_\infty$. Hence, using Barbalat's lemma [19], we can conclude that $\lim_{t\to\infty} e = 0$. That is, the error system (4) is stabilized, and hence despite the system's uncertainties, the designed adaptive controllers given by Equation (5) with parameter estimation rules given by (6) are able to drive the slave system (3) to be synchronized with the master system (2) for any initial conditions.

4. **Numerical Simulations.** The adaptive controller given by (5)-(6) is used to synchronize systems (2) and (3). In the simulation, the fourth Runge-Kutta method with step size 0.01 is used. The initial values for the states of the master and slave systems are taken to be: $x_1(0) = 1$, $y_1(0) = 2$, $z_1(0) = 3$, $w_1(0) = 4$, $x_2(0) = 5$, $y_2(0) = 6$, $z_2(0) = 7$ and $w_2(0) = 8$, and the initial values of the parameter estimates are taken to be: $\hat{a} = 0$, $\hat{b} = 0$, $\hat{c} = 0$, $\hat{d} = 0$, $\hat{r} = 0$ and the gains α_i are chosen such as $\alpha_i = 1$, for $i = \overline{1,4}$. The adaptive control laws (5)-(6) are activated at $t = 5$. The state variables of the master and slave systems versus time are depicted in Figure 3. It is clear that the hyperchaotic slave system (3) is synchronized with the hyperchaotic master system (2). The estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{r}$ are depicted in Figures 4-8. From these figures we can notice that although the parameter estimates did not converge to their true values, the two hyperchaotic systems are synchronized. Also, note that the parameters converge to constant values.

Moreover, one can notice from the previous figures that the convergence is immediate after the activation of the four controllers.

In the next section, we improve on the results of the previous section by reducing the number of controllers used to synchronize the two hyperchaotic Chen systems.

Remark 4.1. *It is noticed that the values of the estimated parameters do not converge to the corresponding true values. This is an expected result as the persistence of excitation condition is required for convergence of the estimated parameters to their true values. However, it should be kept in mind that the objective of the paper is synchronization and not parameter identification.*

5. **An Adaptive Control Law with a Reduced Number of Control Inputs.** In this section, we reduce the number of controllers used for the synchronization of the two identical Chen systems. Let system (2) be the master system and system (3) be the slave system. Then, the error system can be defined by the error system (4). The following theorem gives the main result of this section.

Theorem 5.1. *Let α_i, $i = \overline{1,3}$ be positive scalars and let the controllers be such that:*

$$\begin{cases} u_1 = (\hat{a} - \alpha_1)e_x - e_y e_z - y_2 e_z - e_w - \hat{a}ey \\ u_2 = -(\hat{c} + \alpha_2)e_y - \hat{d}e_x + e_x e_z + z_2 e_x \\ u_3 = 0 \\ u_4 = -(\hat{r} + \alpha_3)e_w - e_y e_z - y_1 e_z - z_1 ey, \end{cases} \tag{11}$$

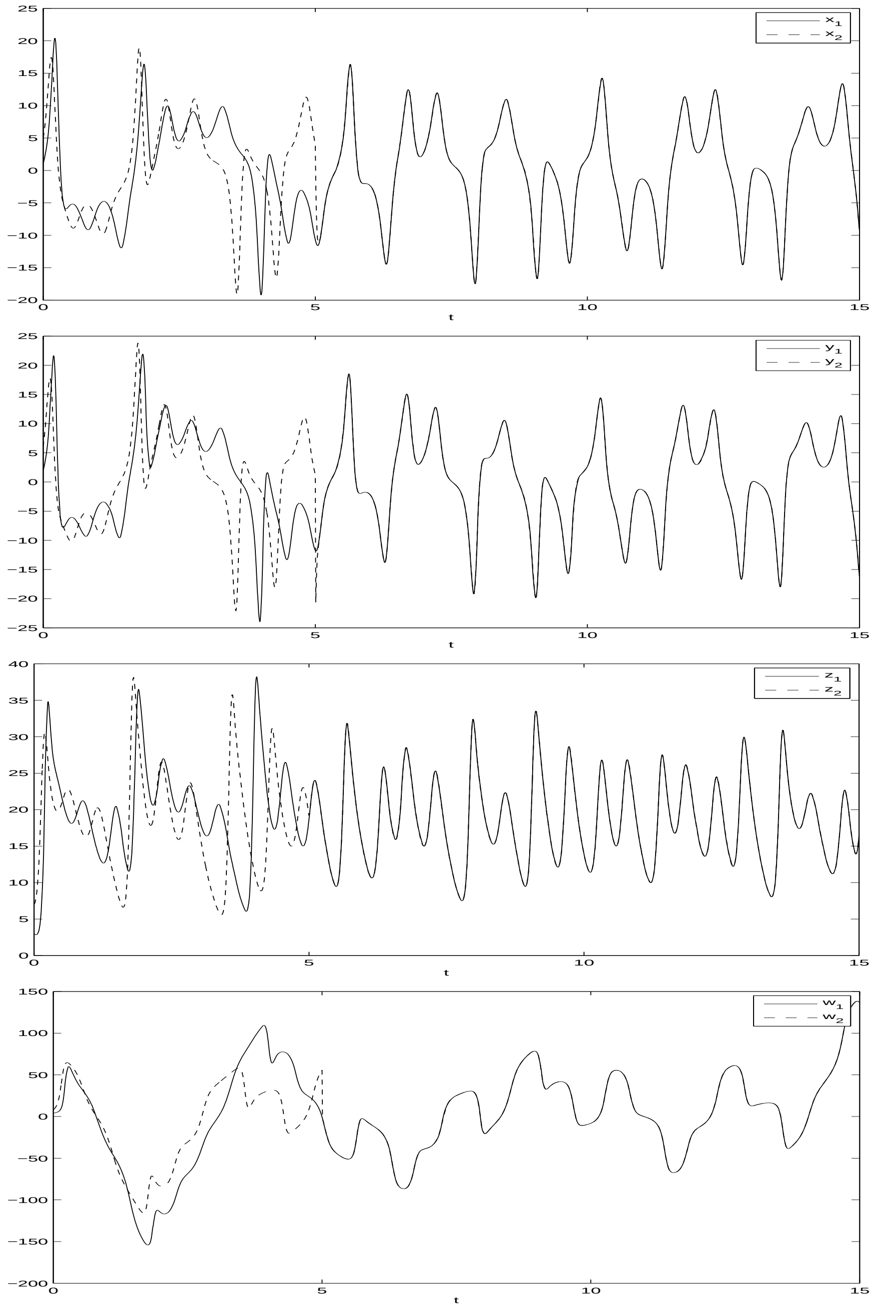

FIGURE 3. The states of the master and slave systems versus time when the first control law is used

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{r}$ are the estimates of the parameters a, b, c, d and r, respectively, and which are updated according to the following update laws:

$$\begin{cases} \dfrac{d\hat{a}}{dt} = e_x(e_y - e_x) \\ \dfrac{d\hat{b}}{dt} = 0 \\ \dfrac{d\hat{c}}{dt} = e_y^2 \\ \dfrac{d\hat{d}}{dt} = e_x e_y \\ \dfrac{d\hat{r}}{dt} = e_w^2. \end{cases} \tag{12}$$

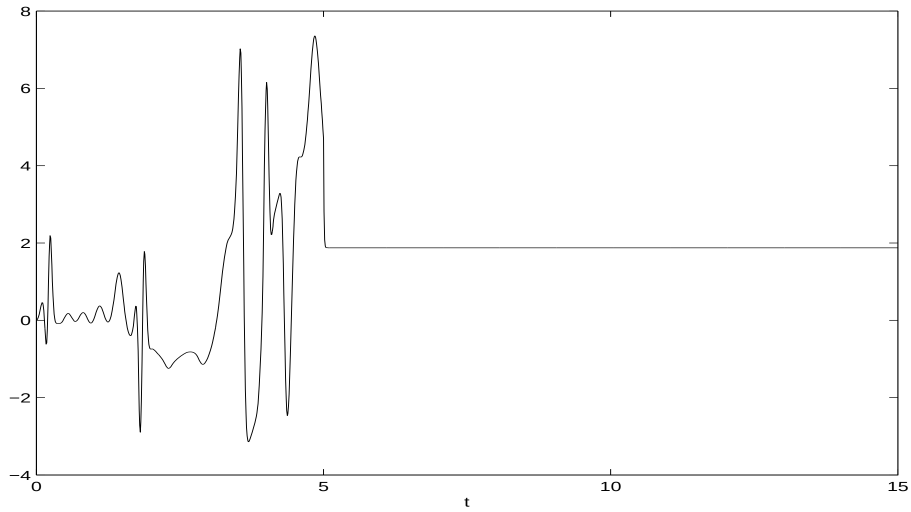

FIGURE 4. The estimate $\hat{a}$ versus time when the first control law is used

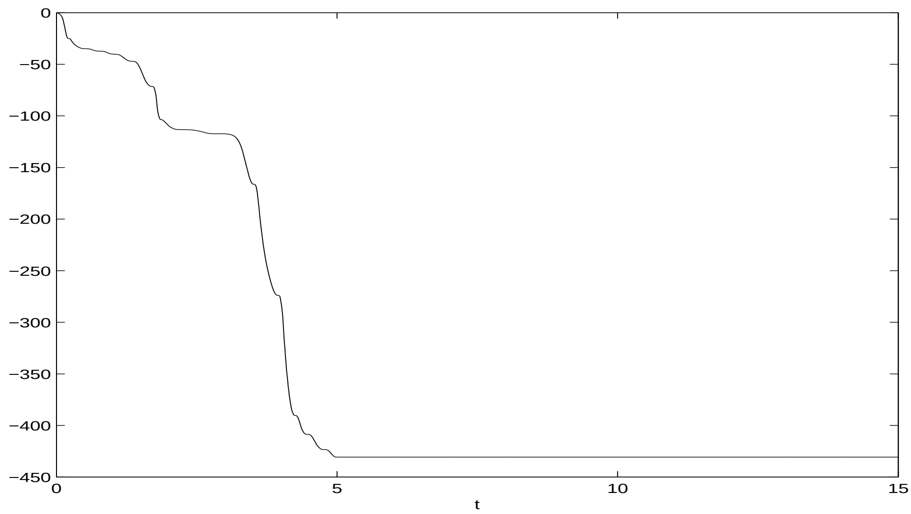

FIGURE 5. The estimate $\hat{b}$ versus time when the first control law is used

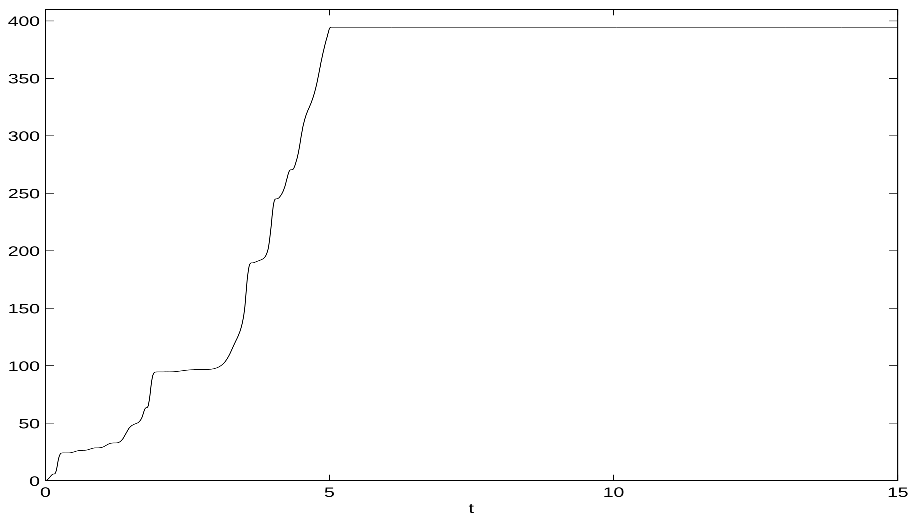

FIGURE 6. The estimate $\hat{c}$ versus time when the first control law is used

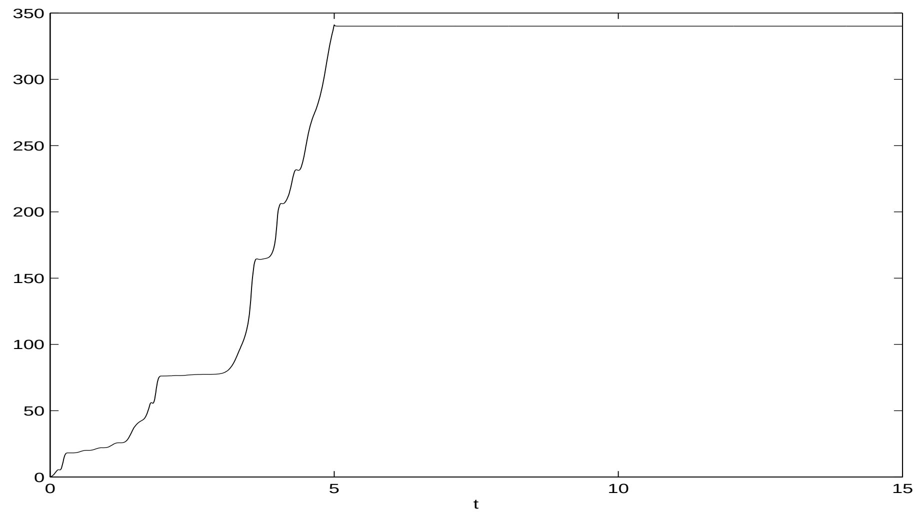

FIGURE 7. The estimate $\hat{d}$ versus time when the first control law is used

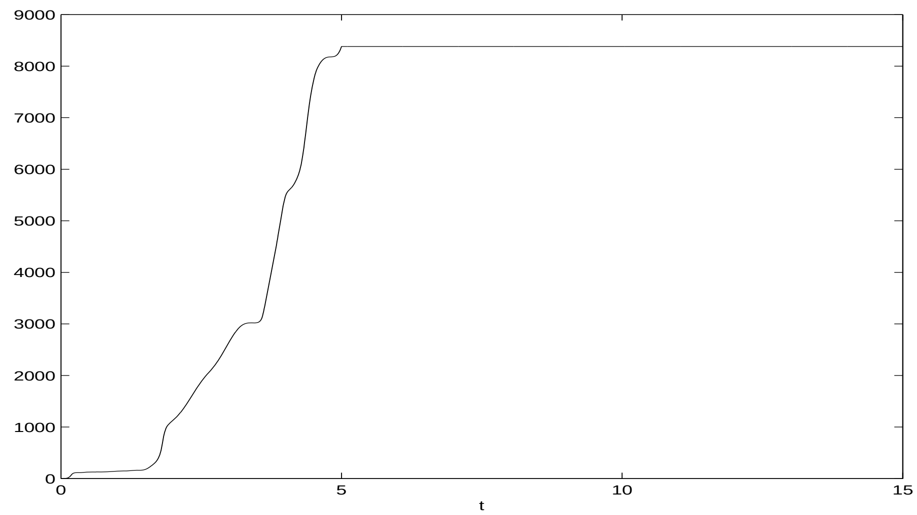

FIGURE 8. The estimate $\hat{r}$ versus time when the first control law is used

Then, the slave uncertain Chen system is synchronized with the master Chen system for any initial conditions.

Proof: Using the controllers (11) into the error system (4), we get

$$\begin{cases} \dot{e}_x = -e_a e_y + e_a e_x - \alpha_1 e_x - y_1 e_z - e_y e_z \\ \dot{e}_y = -e_c e_y - \alpha_2 e_y - e_d e_x - x_1 e_z \\ \dot{e}_z = -b e_z + e_x e_y + x_1 e_y + y_1 e_x \\ \dot{e}_w = -e_r e_w - \alpha_3 e_w, \end{cases} \tag{13}$$

where the parameter errors are defined such that $e_a = \hat{a} - a$, $e_b = \hat{b} - b$, $e_c = \hat{c} - c$, $e_d = \hat{d} - d$ and $e_r = \hat{r} - r$. Note that since the parameters of the system are constants, then $\dot{e}_a = \dot{\hat{a}}$, $\dot{e}_b = \dot{\hat{b}}$, $\dot{e}_c = \dot{\hat{c}}$, $\dot{e}_d = \dot{\hat{d}}$ and $\dot{e}_r = \dot{\hat{r}}$.

Let the positive definite function $V_2 = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_w^2 + {e_a}^2 + {e_b}^2 + {e_c}^2 + {e_d}^2 + {e_r}^2)$ be a Lyapunov function candidate for system (13). Then, the time derivative of V_2 along the trajectories of system (13) while using the update laws (12) is given by:

$$\begin{aligned}\dot{V}_2 &= (e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + e_w\dot{e}_w + e_a\dot{e}_a + e_b\dot{e}_b + e_c\dot{e}_c + e_d\dot{e}_d + e_r\dot{e}_r)\\ &= e_x(-e_ae_y + e_ae_x - \alpha_1e_x - y_1e_z - e_ye_z)\\ &\quad +e_y(-e_ce_y - \alpha_2e_y - e_de_x - x_1e_z)\\ &\quad +e_z(-be_z + e_xe_y + x_1e_y + y_1e_x) + e_w(-e_re_w - \alpha_3e_w)\\ &\quad +e_a(e_x(e_y - e_x)) + e_c(e_y^2) + e_d(e_xe_y) + e_r(e_w^2)\\ &= -(\alpha_1e_x^2 + \alpha_2e_y^2 + be_z^2 + \alpha_3e_w^2) \end{aligned} \tag{14}$$

Let the error vector e be such: $e = [e_x\ e_y\ e_z\ e_w]$. Also, let $\gamma = \min\{\alpha_1, \alpha_2, \alpha_3, b\}$. Then, Equation (14) implies that

$$\dot{V}_2 \leq -\gamma||e||^2. \tag{15}$$

Inequality (15) implies that $\dot{V}_2$ is negative semi definite. Therefore, it can be concluded that the state and parameters errors are bounded, i.e., $[e_x\ e_y\ e_z\ e_w\ e_a\ e_b\ e_c\ e_d\ e_r] \in L_\infty$.

Integrating the inequality $\dot{V}_2 \leq -\gamma||e||^2$ from 0 to t, we obtain the following:

$$\gamma\int_0^t ||e||^2 d\tau \leq \int_0^t -\dot{V}_2 d\tau = V_2(0) - V_2(t) \leq V_2(0). \tag{16}$$

Therefore, it can be concluded that $e \in L_2$ and using (13) we have $\dot{e} \in L_\infty$. Hence, using Barbalat's lemma [19], we can conclude that $\lim_{t\to\infty} e = 0$. That is, the error system (4) is stabilized and hence despite the system's uncertainties, the designed adaptive controllers given by Equation (11) with parameter estimation rules given by (12) are able to drive the slave system (3) to be synchronized with the master system (2) for any initial conditions.

6. **Numerical Simulations for the Improved Controllers.** The adaptive control law given by (11)-(12) is used to synchronize the hyperchaotic Chen systems (2) and (3). The fourth Runge-Kutta method with step size 0.01 is used. The initial conditions are the same as the ones in Section 4. The initial values of the parameter estimates are chosen as $\hat{a} = 0$, $\hat{b} = 0$, $\hat{c} = 0$, $\hat{d} = 0$, $\hat{r} = 0$ and the gains α_i are chosen such as $\alpha_i = 1$, for $i = \overline{1, 3}$. The control law is activated at $t = 5$ sec. The state variables of the master and slave systems versus time are depicted in Figure 9. The estimates of the parameters versus time are depicted in Figures 10-14. The simulation results indicate that the adaptive control given by (11)-(12) is able to synchronize the hyperchaotic slave system (3) with the hyperchaotic master system (2). Moreover, the figures indicate that the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{r}$ converge to constant values. Note that these constant values are different from the true values of the parameters. Also, note that even though the number of controllers has changed from four to three, the performance of the controlled system hardly changed. Hence, it can be concluded that the adaptive controller (11)-(12) is more useful than the controller (5)-(6).

7. **A Secure Communication Scheme Based on the Hyperchaotic Chen System.** In this section, we introduce a secure communication scheme based on the hyperchaotic Chen system using an adaptive synchronization controller. This secure communication scheme is used to send text messages and to encrypt images.

Recall that the hyperchaotic Chen system is defined as follows:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + w_1 \\ \dot{y}_1 = dx_1 - x_1z_1 + cy_1 \\ \dot{z}_1 = x_1y_1 - bz_1 \\ \dot{w}_1 = y_1z_1 + rw_1. \end{cases} \tag{17}$$

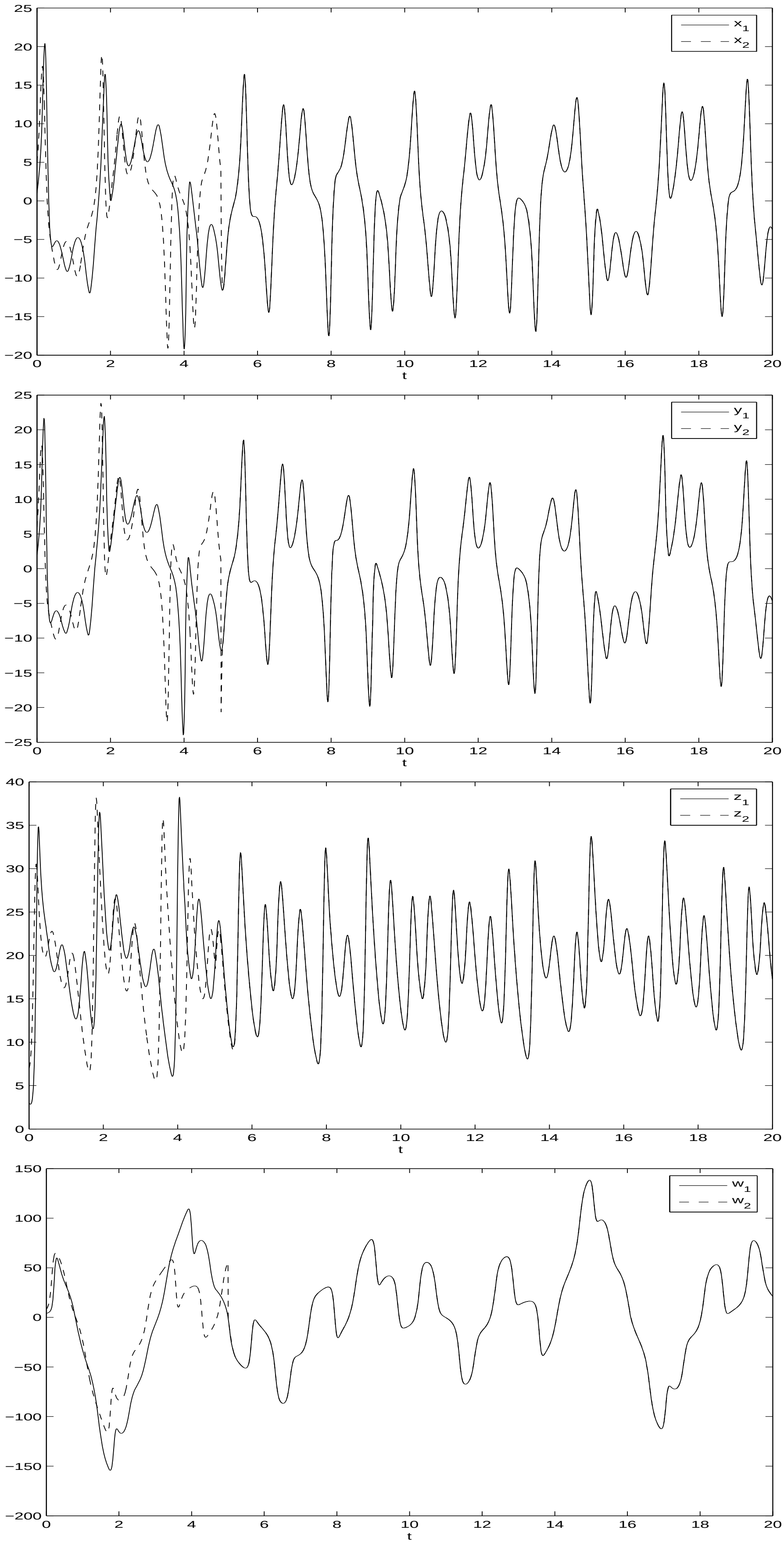

FIGURE 9. The states of the master and slave systems versus time when the second control law is used

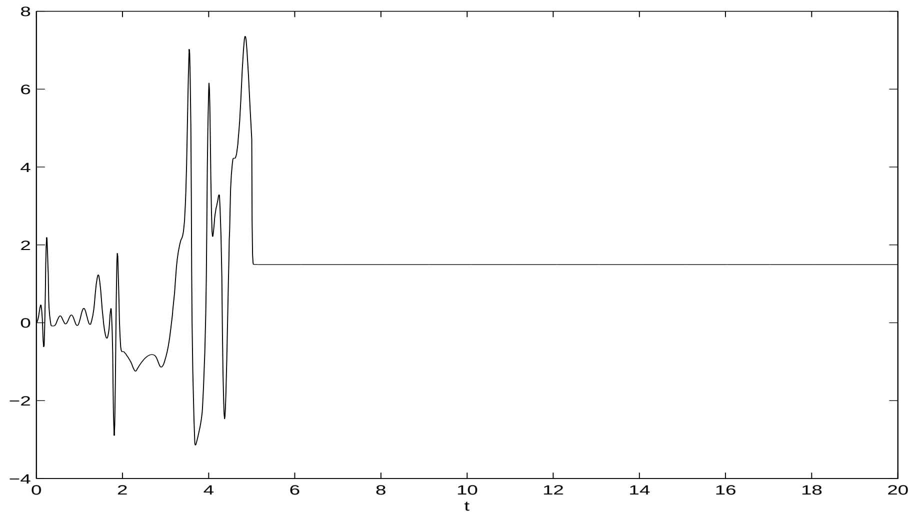

FIGURE 10. The estimate $\hat{a}$ versus time when the second control law is used

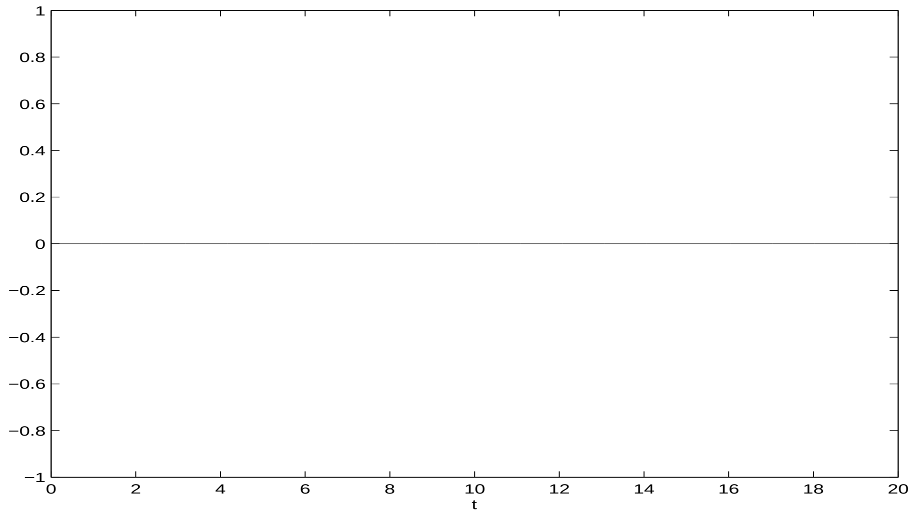

FIGURE 11. The estimate $\hat{b}$ versus time when the second control law is used

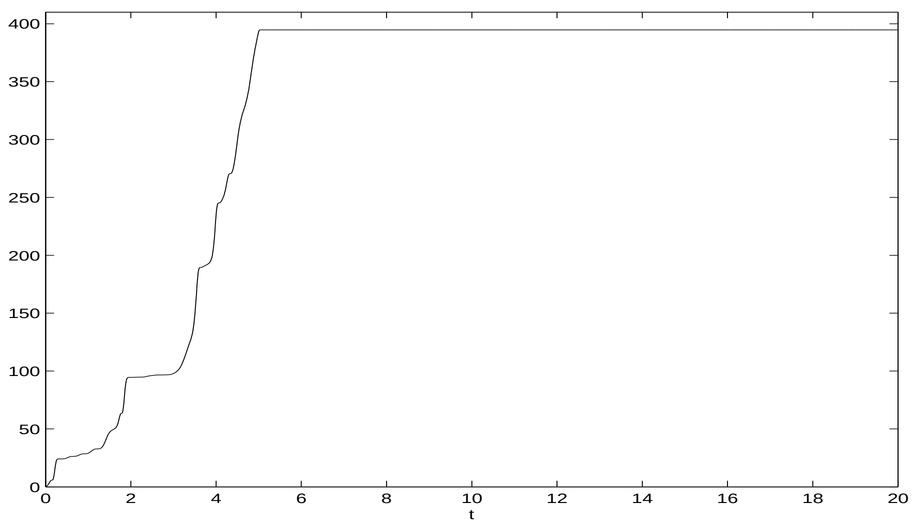

FIGURE 12. The estimate $\hat{c}$ versus time when the second control law is used

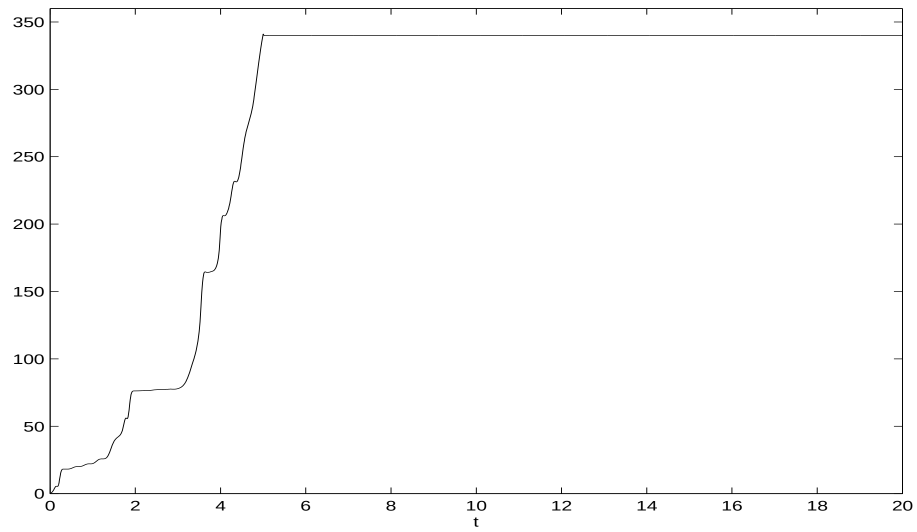

FIGURE 13. The estimate $\hat{d}$ versus time when the second control law is used

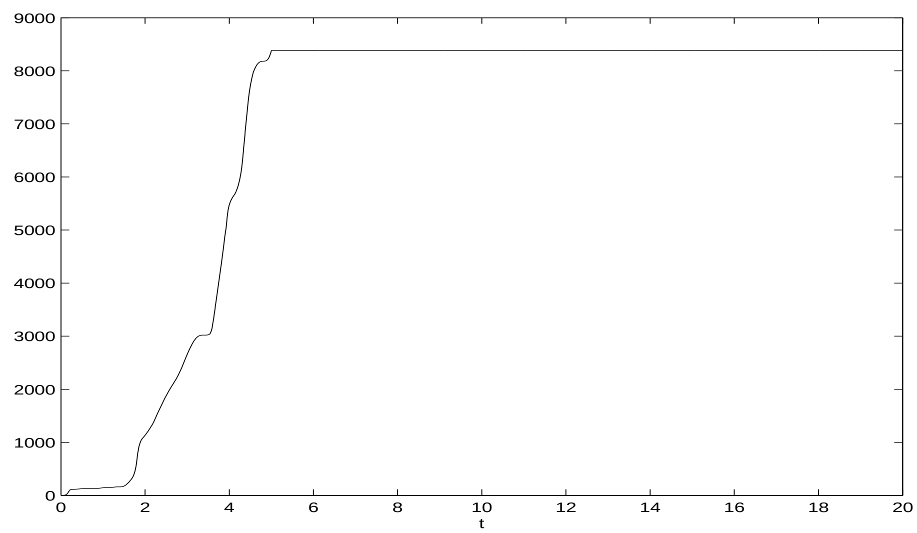

FIGURE 14. The estimate $\hat{r}$ versus time when the second control law is used

Figure 15 depicts the block diagram of a secure communication scheme based on the hyperchaotic Chen system using an adaptive synchronization control method. The message $\beta m(t)$, where β is a scaling factor, is sent by inserting it to the $y_1(t)$ o.d.e. of the hyperchaotic master Chen system denoted as system (A). The states $x_1(t)$, $y_1(t)$, $z_1(t)$ and $w_1(t)$ are sent through a public channel. At the receiver side, we use an adaptive synchronization controller to synchronize the master and the slave system and then recover the message. In the following scheme analysis, we consider a noise-free channel, i.e., $\tilde{x}_1 = x_1$, $\tilde{y}_1 = y_1$, $\tilde{z}_1 = z_1$ and $\tilde{w}_1 = w_1$.

7.1. **Scheme analysis.** In this section, we will show that the signal sent from the transmitter and the signal received by the receiver will be synchronized. Therefore, the master hyperchaotic Chen system (A) will be synchronized with the slave hyperchaotic Chen system (B) by using the designed adaptive control laws.

The hyperchaotic Chen system (A) is defined as follows:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + w_1 \\ \dot{y}_1 = dx_1 - x_1 z_1 + cy_1 - D \\ \dot{z}_1 = x_1 y_1 - bz_1 \\ \dot{w}_1 = y_1 z_1 + rw_1, \end{cases} \tag{18}$$

where x_1, y_1, z_1 and w_1 are the state variables of the hyperchaotic Chen system (A) with the parameters a, b, c, d and r. The message $D = \beta m(t)$ is considered to be an external disturbance and β is a scaling factor used to reduce signal to noise ration (SNR). Similarly, the hyperchaotic Chen system (B) at the receiver side is defined as follows:

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + w_2 + u_1 \\ \dot{y}_2 = dx_2 - x_2 z_2 + cy_2 + u_2 \\ \dot{z}_2 = x_2 y_2 - bz_2 + u_3 \\ \dot{w}_2 = y_2 z_2 + rw_2 + u_4, \end{cases} \tag{19}$$

where $U = [u_1 \quad u_2 \quad u_3 \quad u_4]$ is the controller.

Define the error system by subtracting system (18) from system (19). We obtain

$$\begin{cases} \dot{e}_x = a(e_y - e_x) + e_w + u_1 \\ \dot{e}_y = de_x - z_2 e_x - x_2 e_z + e_x e_z + ce_y + D + u_2 \\ \dot{e}_z = x_2 e_y + y_2 e_x - e_y e_x - be_z + u_3 \\ \dot{e}_w = z_2 e_y + y_2 e_z - e_y e_Z + re_w + u_4, \end{cases} \tag{20}$$

where $e_x = x_2 - x_1$, $e_y = y_2 - y_1$, $e_z = z_2 - z_1$ and $e_w = w_2 - w_1$.

The hyperchaotic Chen system (A) will be synchronized with the hyperchaotic Chen system (B) if the zero fixed point of the error system (20) is globally asymptotically stable.

An adaptive control law which is motivated by the control scheme presented in Theorem 5.1 is used to synchronize the two hyperchaotic Chen systems for secure communication purposes.

Theorem 7.1. *Let the control laws be such that:*

$$\begin{cases} u_1 = (\hat{a} - K_1)e_x - e_y e_z - y_2 e_z - e_w - \hat{a} e_y \\ u_2 = -(\hat{c} + K_2)e_y - \hat{d} e_x + e_x e_z + z_2 e_x \\ u_3 = -K_3 e_z \\ u_4 = -(\hat{r} + K_4)e_w - e_y e_z - y_1 e_z - z_1 e_y, \end{cases} \tag{21}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{r}$ are the estimates of the parameters a, b, c, d and r, respectively, and which are updated according to the following rules:

$$\begin{cases} \dfrac{d\hat{a}}{dt} = e_x(e_y - e_x) \\ \dfrac{d\hat{b}}{dt} = 0 \\ \dfrac{d\hat{c}}{dt} = e_y^2 \\ \dfrac{d\hat{d}}{dt} = e_x e_y \\ \dfrac{d\hat{r}}{dt} = e_w^2, \end{cases} \tag{22}$$

with K_i $(i = \overline{1,4})$ being positive gain parameters. Then, the slave hyperchaotic Chen system (19) will be synchronized with the master hyperchaotic Chen system (18) for any initial conditions.

Proof: (Similar to the proof of Theorem 5.1).

Remark 7.1. *It should be noted that the controllers used in (22) reduce to the one in (11) when* $K_3 = 0$.

Since we are using adaptive synchronization, the designed controllers can overcome the external disturbance $D = \beta m(t)$ and synchronize the two hyperchaotic systems. Then, a noisy version of the message can be recovered from the error system $e_y = y_2 - y_1$. Using a filter and a threshold detector, the transmitted message can be completely retrieved.

7.2. **Numerical results.** The above mentioned secure communication scheme is used to transmit an encrypted image. Let the message to be transmitted be an image of the Eiffel tower as shown in Figure 16. Black and white images are $n \times m$ matrices of binary bits

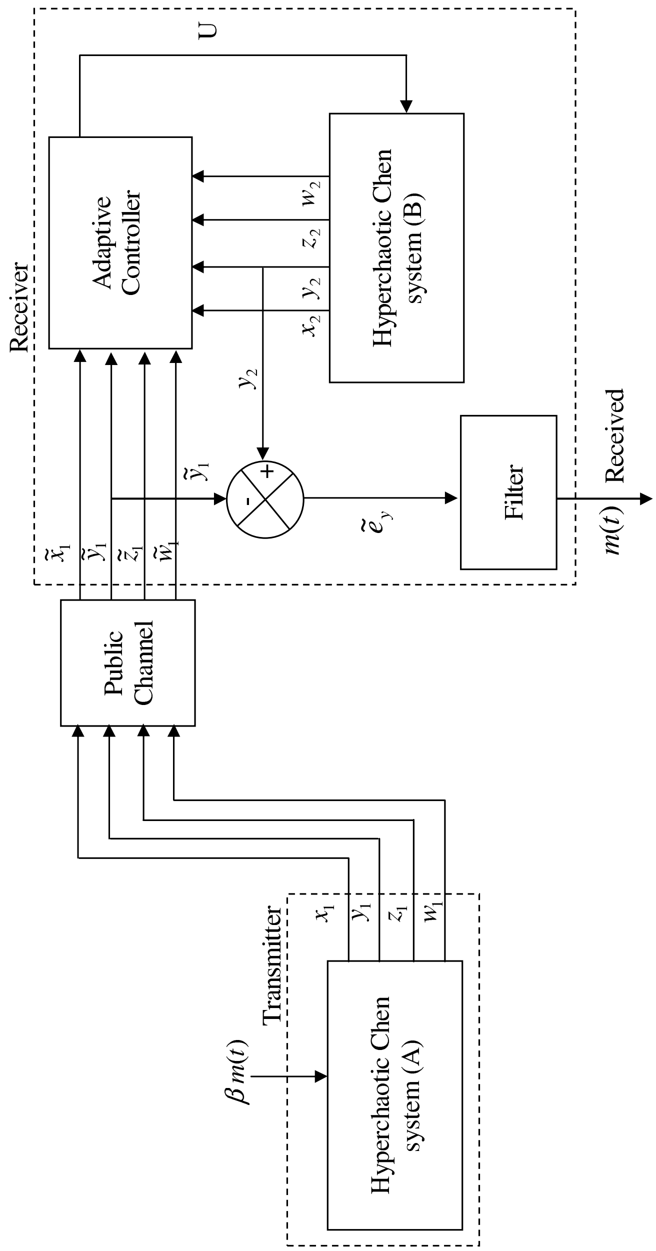

FIGURE 15. The block diagram of a secure communication scheme based on the hyperchaotic Chen system and the adaptive synchronization control method

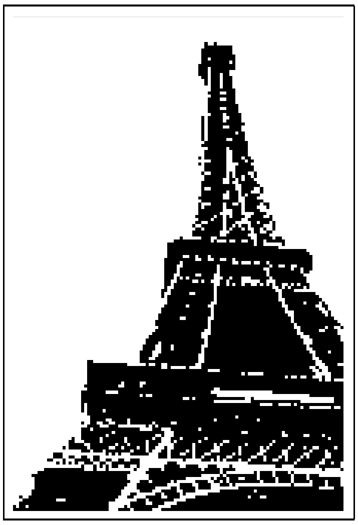

Figure 16. The transmitted Eiffel image

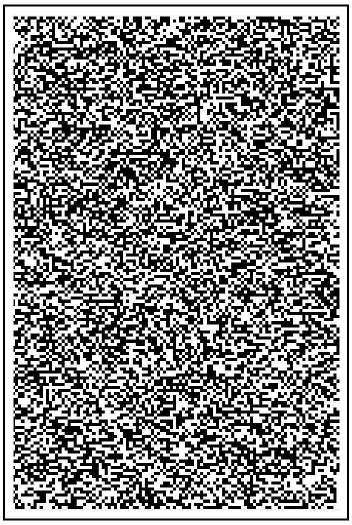

Figure 17. The encrypted Eiffel image taken from the state $y_1(t)$ of the public channel

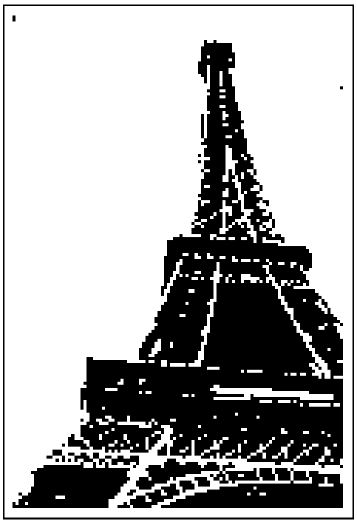

Figure 18. The retrieved Eiffel image

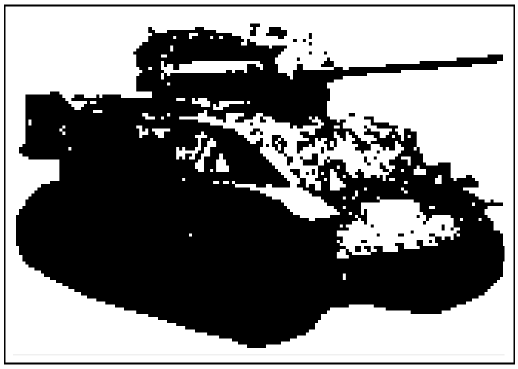

Figure 19. The transmitted tank image

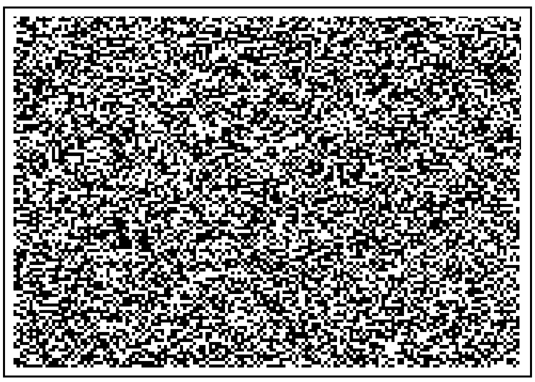

Figure 20. The encrypted image of the tank taken from the state $y_1(t)$ of the public channel

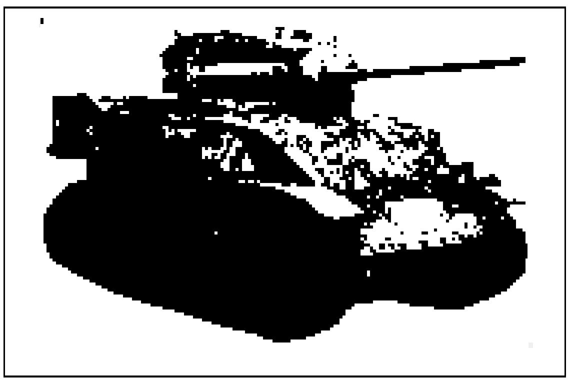

Figure 21. The retrieved tank image

and colored images are three dimensional arrays [28]. Hence, to send a black and white image using the proposed transmission scheme, we need to re-dimension the image to become a binary sequence and re-scale the binary sequence to become pulses with period $T = 1$. Hence,

$$\text{the pulse} = \begin{cases} 1 & \text{if the bit is 1} \quad \forall t \in [t_0, t_0 + T] \\ 0 & \text{if the bit is 0} \quad \forall t \in [t_0, t_0 + T]. \end{cases}$$

Next, the binary sequence is transmitted by considering it as a disturbance acting on the second o.d.e. of the hyperchaotic master Chen system (18). Figure 17 depicts the encrypted image where the state $y_1(t)$ is taken from the output of the public channel and re-dimensioned to be an $n \times m$ array. Figure 17 clearly shows that the image has been completely hidden by the encryption process.

At the receiver side, we can recover the message by using the adaptive synchronization control scheme (21)-(22). Then, we retrieve the image by re-dimensioning the recovered binary sequence back to become an $n \times m$ matrix. Figure 18 shows the retrieved Eiffel image.

Figures 19-21 present another image transmitted and retrieved using the proposed scheme. Therefore, it can be concluded that the proposed synchronization scheme works well as it is successfully used to securely transmit and retrieve different images.

8. **Conclusions.** The adaptive synchronization of two hyperchaotic master and slave Chen systems is tackled in this paper. Two adaptive control schemes are proposed. Using these control schemes, it is shown that the errors between the states of the master and the slave systems asymptotically converge to zero. Simulation results are presented to illustrate and reinforce the ability of the adaptive controller to synchronize the hyperchaotic systems. In addition, the proposed control scheme is applied to securely transmit images where computer simulation results were provided to show the efficiency of the proposed controller.

Future research will address the problem of using other control techniques to synchronize hyperchaotic systems.

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