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CONTROLLER DESIGN FOR TAKAGI-SUGENO SYSTEMS IN CONTINUOUS-TIME

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Abstract. *The stabilization of continuous-time Takagi-Sugeno systems is studied and solved by using a non-PDC controller. First, stability conditions are derived in terms of Linear Matrix Inequalities, using a multiple Lyapunov function; then, algorithms to calculate controllers that ensure closed-loop stability are derived. To show the applicability of the proposed approach, some examples are also provided.*
Keywords: Takagi-Sugeno systems, State-feedback controller, Multiple Lyapunov function

1. Introduction.

It is well known that many continuous-time nonlinear systems can be represented by equivalent Takagi-Sugeno (T-S) models [1, 2], which simplifies developing nonlinear controllers that guarantee stability. This representation as T-S systems is precise when the so-called nonlinearity sector approach is used [3]. Nonetheless, controller design for this kind of systems is frequently carried out after transformation to discrete-time, as stability analysis of T-S models is easier in discrete-time, by using Lyapunov functions. Some recent works are [4, 5, 6, 7, 8, 9], which provide sufficient conditions in terms of Linear Matrices Inequalities (LMIs), by using a multiple Lyapunov function in discrete-time. For continuous-time T-S systems, we can just cite [10, 11, 12, 13, 14]. This lack of results is caused by the difficulty of using a multiple Lyapunov function in the continuous-time case, as time-derivatives of the membership functions (MFs) appear in the evaluation of the derivative of the Lyapunov function. This makes it very difficult to obtain easily checkable conditions for its decrease. Some published works are limited to stability analysis study [15, 16]. For example, in [15] the authors proposed an approach based on reducing the global stability goals to an estimation of the region of attraction (using local asymptotic conditions), by solving a set of LMIs. While a new class of fuzzy Lyapunov functions, that depend not only on the fuzzy weighting functions of the TS fuzzy systems but also on their first order time derivatives, is proposed by [16]. Other works have been proposed to find stabilizing controllers for continuous-time T-S systems by using a multiple Lyapunov function [12, 14, 17]. To avoid the apparition of the time-derivatives of the MFs, [12] proposed a new type of Lyapunov function, expressed as the path integral of certain state function. An algorithm was given to find the corresponding feedback gain matrices. Unfortunately, this approach can give conservative results, due to the prescribed structure of the proposed Lyapunov function: the off-diagonal elements are common for all Lyapunov matrices. In [14], information on the MFs has been used to reduce conservatism in controller design. Recently, [18] has proposed to use a homogeneous polynomial matrix function to the nonquadratic Lyapunov functions, to derive more relaxed conditions; Unfortunately, this methodology is limited to discrete-time case and due to the derivative on time of MFs, so, as already pointed out in [18], it is difficult to extend their methodology to the continuous-time case. Finally, we can cite [11], where the authors proposed the upper bounds of the time-derivative of the MFs. Thus, to overcome the problem of selecting those upper bounds, the authors suggested a control law that depends on the time-derivative of MFS scheme. As a consequence, the methodology of [11] is limited to the case where the time derivatives of MFs depend only on the states.

To solve these difficulties, in this work, inspired by [11], an approach that considers the upper bounds of the time-derivative of the Lyapunov function as decision variables is proposed for both stability analysis and controller design, of continuous-time T-S models. The advantages of the proposed methodology are as follows. First, unlike [11, 19], which assume that each time-derivative of MF is bounded, the proposed approach considers the bounds of the Lyapunov matrices interpolated via the time-derivative of MFs, which leads to less conservative results, as it provides additional degrees of freedom. Second, it is not necessary to define a priori the bounds, as they are considered variables within an optimization problem. Third, the proposed methodology is not limited for stability analysis: as it is shown later, it can be extended to controller design.

This paper is organized as follows. Section 2 gives LMI-based stability conditions for autonomous T-S systems, using a cone complementary formulation. These conditions are applied for controller design in Section 3, where an example is also provided. Finally, Section 4 gives some conclusions.

2. Stability Conditions for Autonomous T-S Fuzzy Models.

This section develops the proposed stability analysis methodology for continuous-time T-S fuzzy systems, by using a multiple Lyapunov function.

A continuous-time T-S model [3] is described as a set of r rules, where each rule i uses l fuzzy sets $M_1^i, \ldots, M_l^i$ and fuzzy variables $z_1(t), \ldots, z_l(t)$, as follows:

$$\text{Rule } i: \text{ IF } z_1(t) \text{ is } M_1^i \text{ AND } \ldots \text{ AND } z_l(t) \text{ is } M_l^i \text{ THEN } \dot{x}(t) = A_i x(t) + B_i u(t).$$

The global T-S model is then structured as follows [3]:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)), \tag{1}$$

where A_i and B_i are constant matrices, $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control and $h_i(z(t))$'s are the *MFs* of the i^{th} rule (also denoted $h_i(z)$), obtained from the M_l^i.

The convex sum property $\sum_{i=1}^{r} h_i(z) = 1$ holds and $0 \leq h_i(z) \leq 1$. Moreover, it is assumed that the $h_i(z)$ are C^1 functions (continuous and derivable).

Consider the autonomous T-S system, corresponding to (1) when $u(t) = 0$:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z) A_i x(t). \tag{2}$$

Stability conditions for the T-S system (2) are now given.

Theorem 2.1. *Let* $\Upsilon_{ij}^k = \begin{bmatrix} -\frac{1}{r}\left(A_j^T P_i + P_i A_j\right) & \phi_k I \\ \phi_k I & Q_k \end{bmatrix}$. *If there exist scalars* $\phi_1 \geq 0, \dots, \phi_r \geq 0$ *and symmetric matrices* $P_1 > 0, \dots, P_r > 0$, $Q_1 > 0, \dots, Q_r > 0$, *feasible solution of the following Optimization Problem*

$$(OP_1)\begin{cases} \underset{\phi_i, P_i, Q_i}{\text{maximize}} \ \sum_{i=1}^{r} \phi_i \text{ subject to} \\ \Upsilon_{ii}^k > 0, \ \ 1 \leq i, \ \ k \leq r, \\ \Upsilon_{ij}^k + \Upsilon_{ji}^k \geq 0, \ \ 1 \leq k \leq r, \ \ 1 \leq i < j \leq r, \\ P_i Q_i = I, \quad 1 \leq i \leq r, \end{cases} \tag{3}$$

such that $\sum_{i=1}^{r} \left|\dot{h}_i(z)\right| P_i \leq \sum_{i=1}^{r} \phi_i^2 P_i, \forall t$, *then the T-S system (2) is asymptotically stable.*

Proof: For stability analysis, we propose to use the following multiple Lyapunov function:

$$V(x(t)) = x^T(t) \sum_{i=1}^{r} h_i(z) P_i x(t). \tag{4}$$

Then, evaluating the rate of V along the trajectory gives:

$$\begin{aligned} \dot{V}(x(t)) =& x^T(t) \sum_{k=1}^{r} \dot{h}_k(z) P_k x(t) + \dot{x}^T(t) \sum_{i=1}^{r} h_i(z) P_i x(t) + x^T(t) \sum_{i=1}^{r} h_i(z) P_i \dot{x}(t) \\ =& x^T(t) \sum_{i=k}^{r} \dot{h}_k(z) P_k x(t) + x^T(t) \left[\sum_{i,j=1}^{r} h_i(z) h_j(z) (P_i A_j + A_j^T P_i)\right] x(t). \end{aligned} \tag{5}$$

One drawback of using directly the last equality in (5) is the apparition of the time-derivative of MFs $\dot{h}_k(z)$. To solve this problem, we assume that the first term is bounded as follows:

$$\begin{aligned} &\sum_{k=1}^{r} \left|\dot{h}_k(z)\right| P_k \leq \sum_{k=1}^{r} \phi_k^2 P_k, \forall t, \\ &P_1 = P_1^T > 0, P_2 = P_2^T > 0, \dots, P_r = P_r^T > 0 \text{ and } \phi_1 \geq 0, \phi_2 \geq 0 \dots, \phi_r \geq 0, \end{aligned} \tag{6}$$

where ϕ_k are decision variables to be computed, as well as the matrices P_i. As a consequence, $\dot{V}(x(t))$ can be bounded as follows:

$$\dot{V}(x(t)) \leq x^T(t) \left(\sum_{k=1}^{r} \phi_k^2 P_k + \sum_{i,j=1}^{r} h_i(z) h_j(z) \left(P_i A_j + A_j^T P_i\right)\right) x(t).$$

Hence, the stability of the system (2) is guaranteed by the following condition

$$\sum_{k=1}^{r} \phi_k^2 P_k + \sum_{i,j=1}^{r} h_i(z) h_j(z) \left(P_i A_j + A_j^T P_i\right) < 0. \tag{7}$$

Now, using the convex sum property $\sum_{i=1}^{r} h_i(z) = \sum_{j=1}^{r} h_j(z) = 1$, (7) can also be written as follows:

$$\sum_{k=1}^{r} \phi_k^2 P_k + \sum_{i=1}^{r} h_i(z)) \left(P_i A_i + A_i^T P_i\right) + \sum_{i<j\leq r} h_i(z) h_j(z) \left(P_i A_j + A_j^T P_i + P_j A_i + A_i^T P_j\right) < 0, \quad (8)$$

it is easy to check that the stability condition (8) is guaranteed by the following inequalities.

$$\begin{array}{l} \phi_k^2 P_k + \frac{1}{r}\left(A_i^T P_i + P_i A_i\right) < 0, \;\; 1 \leq i, \;\; k \leq r. \\ 2\phi_k^2 P_k + \frac{1}{r}\left(A_j^T P_i + P_i A_j + A_i^T P_j + P_j A_i\right) \leq 0, \;\; 1 \leq i < j \leq r, \;\; 1 \leq k \leq r. \end{array} \quad (9)$$

One of the objectives of this paper is to provide a methodology that makes possible the computation of the bounds ϕ_k, since it has been shown in the literature [11, 19] that the selection of those bounds is very difficult, especially when the MFs depend on nonmeasurable state variables. However, regarding conditions (9), the mixed products $\phi_k^2 P_k$ of unknown variables make hard the direct solution of (9). In this work, we suggest moving the nonlinearities in conditions (9) to equality conditions. To do this, let $P_k = Q_k^{-1}$ in (9) and take its Schur complement, to get (3).

Remark 2.1. *It should be noted that, compared with previous works in the literature that also consider upper bounds on the time derivative of the MFs* [11, 19]*, the proposed assumption (6) has some clear advantages:*

1. *In this work the ϕ_k in (6) are decision variables, that can be calculated as part of the proposed algorithm, so they do not need to be known a priori.*
2. *Assumption (6) considers only the upper bound of a sum of products, which leads to less conservative results, as there are more degrees of freedom.*
3. *If $\mathcal{S}$ is the set of $\phi_1, \ldots, \phi_r$ that fulfill (9), note that a larger size of $\mathcal{S}$ implies smaller conservatism of condition (9). Henceforth, conservatism can be significantly reduced by using the available degrees of freedom, enlarging $\mathcal{S}$, by maximizing its perimeter, i.e., by maximizing $\sum_{i=1}^{r} \phi_i$, as is proposed in Theorem 2.1.*
4. *The number of conditions to be satisfied in the optimization problem (OP_1) is equal to $r^2(r+1)/2 + r$.*
5. *Comparing with the discrete-time case, the number of stability condition is increased by r.*

To handle the nonconvex constraint $P_i Q_i = I$ in (3), we propose to use the cone complementarity formulation given by the following lemma [8].

Lemma 2.1. *The equality constraint $P_i Q_i = I$ holds for $i = 1, \ldots, r$, if and only if there exist symmetric matrices $P_1 > 0, \ldots, P_r > 0$, $Q_1 > 0, \ldots, Q_r > 0$ such that the optimum of the following optimization problem (OP_2) is achievable and equal to $n \times r$:*

$$(OP_2) \begin{cases} \underset{P_i, Q_i}{\text{minimize}} \; \sum_{i=1}^{r} Tr(P_i Q_i) \text{ subject to} \\ \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} > 0, \;\; 1 \leq i \leq r. \end{cases} \quad (10)$$

Remark 2.2. *The main idea of the transformation in Lemma 2.1 is to move the nonlinearity from the constraints $P_iQ_i = I$ to an objective function subject to LMIs, as given by (10).*

Taking into account the previous considerations, one has a multi-objective or multi-criterion optimization problem ((OP_1) and (OP_2)), where the objective is to minimize at the same time $-\sum_{i=1}^{r}\phi_i$ and $\sum_{i=1}^{r}Tr(P_iQ_i)$. The most used approach for solving this multiobjective optimization problem is to combine the objective functions into a single objective, called the Aggregate Objective Function (AOF). A well-known combination is the weighting method, in which one specifies scalar weights for each objective to be optimized, and then combines them into a single function to be minimized. This can be done by fixing $0 < \beta < 1$ and then solving the following optimization problem, which can substitute (OP_1) in Theorem 2.1.

$$(OP_3)\begin{cases} \underset{\phi_i,P_i,Q_i}{\text{minimize}}\ \beta\sum_{i=1}^{r}Tr(P_iQ_i) - (1-\beta)\sum_{i=1}^{r}\phi_i \text{ subject to} \\ \Upsilon_{ii}^k > 0, \quad 1 \le i, \quad k \le r, \\ \Upsilon_{ij}^k + \Upsilon_{ji}^k \ge 0, \quad 1 \le k \le r, \quad 1 \le i < j \le r, \quad 1 \le k \le r, \\ \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} > 0, \quad 1 \le i \le r. \end{cases} \tag{11}$$

Note that, the number of stability conditions in (OP_3) is still the same as in (OP_1). It can be seen that the objective function $\sum_{i=1}^{r}Tr(P_iQ_i)$ is nonconvex, so it is proposed to use the following linearization algorithm for solving (OP_3).

Algorithm 1. *Fix a tolerance ϵ (for example, $\epsilon = 10^{-6}$) and $0 < \beta < 1$, and execute the following:*

- *Step 1: Set $P_i^0 = I$ and $Q_i^0 = I$, for $i = 1,\dots,r$.*
- *Step 2: Solve the LMI optimization:*
  $\underset{P_i,Q_i,\phi_i}{\text{minimize}}\ \beta\sum_{i=1}^{r}\mathbf{Tr}(P_i^lQ_i + Q_i^lP_i) - (1-\beta)\sum_{i=1}^{r}\phi_i$ *subject to the constraints (11).*
- *Step 3: Let P_i^*, Q_i^*, $i = 1,\dots,r$ be the optimal solution: if $\left|\sum_{i=1}^{r}\mathbf{Tr}(P_i^*Q_i^*) - n\times r\right| \le \epsilon$, then stop, otherwise set $P_i^{l+1} \longleftarrow P_i^*$, $Q_i^{l+1} \longleftarrow Q_i^*$, and repeat from Step 2.*

Remark 2.3. *In Algorithm 1, the weight β can be tuned to enlarge the perimeter $\sum_{i=1}^{r}\phi_i$.*

3. **Control Synthesis.**

In this section, we extend the methodology of the previous section to propose a control scheme based on a non classical PDC control law.

Thus, we consider stabilization using the following control law, previously used by [4]:

$$u(t) = \left(\sum_{i=1}^{r}h_i(z)K_i\right)\left(\sum_{j=1}^{r}h_j(z)P_j\right)^{-1}x(t). \tag{12}$$

The following result solves the stabilization problem for continuous-time T-S systems.

Theorem 3.1. *Let* $\Upsilon_{ij}^k \equiv \begin{bmatrix} -\frac{1}{r}\left(A_iP_j + B_iK_j + (A_iP_j + B_iK_j)^T\right) & \phi_k I \\ \phi_k I & Q_k \end{bmatrix}$. *If there exist scalars* $\phi_1 \geq 0, \ldots, \phi_r \geq 0$, *symmetric matrices* $P_1 > 0, \ldots, P_r > 0$, $Q_1 > 0, \ldots, Q_r > 0$ *and controller gains* $K_1, \ldots, K_r$, *feasible solution of the following optimization problem:*

$$(OP_4) \begin{cases} \underset{\phi_i, P_i, Q_i, K_i}{\text{maximize}} \displaystyle\sum_{i=1}^{r} \phi_i \text{ subject to} \\ \Upsilon_{ii}^k > 0, \;\; 1 \leq i, \;\; k \leq r, \\ \Upsilon_{ij}^k + \Upsilon_{ji}^k \geq 0, \;\; 1 \leq i < j \leq r, \;\; 1 \leq k \leq r, \\ P_iQ_i = I, \;\; 1 \leq i \leq r, \end{cases} \tag{13}$$

such that $\displaystyle\sum_{i=1}^{r} \left|\dot{h}_i(z)\right| P_i \leq \sum_{i=1}^{r} \phi_i^2 P_i$, $\forall t$, *then the T-S system (1) with the control law (12) is asymptotically stable.*

Proof: First, substituting (12) into (1), the closed-loop system can be written as follows:

$$\dot{x}(t) = (A_z + B_zK_zP_z^{-1})x(t), \tag{14}$$

where, by notation,

$$A_z \equiv \sum_{i=1}^{r} h_i(z)A_i, \;\; B_z \equiv \sum_{i=1}^{r} h_i(z)B_i, \;\; P_z \equiv \sum_{i=1}^{r} h_i(z)P_i, \;\; K_z \equiv \sum_{i=1}^{r} h_i(z)K_i. \tag{15}$$

For this system (14), consider the following Lyapunov function

$$V(x(t)) = x^T(t)P_z^{-1}x(t), \tag{16}$$

and evaluate the rate of V along the trajectory of the closed-loop system (14):

$$\dot{V}(x(t)) = x^T(t)\dot{P_z^{-1}}x(t) + \dot{x}^T(t)P_z^{-1}x(t) + x^T(t)P_z^{-1}\dot{x}(t).$$

By using the identity $P_zP_z^{-1} = I$, we obtain $\dot{P_z^{-1}} = -P_z^{-1}\dot{P}_zP_z^{-1}$, where $\dot{P}_z = \displaystyle\sum_{i=1}^{r} \dot{h}_i(z)P_i$. Thus, the rate of V is given by

$$\dot{V}(x(t)) = x^T(t)\left\{-P_z^{-1}\dot{P}_zP_z^{-1} + (A_z + B_zK_zP_z^{-1})^TP_z^{-1} + P_z^{-1}(A_z + B_zK_zP_z^{-1})\right\}x(t).$$

According to Assumption (6) one has $-\dot{P}_z \leq \displaystyle\sum_{k=1}^{r} \phi_k^2 P_k$, and by consequence we obtain

$$\dot{V}(x(t)) \leq x^T(t)\left\{P_z^{-1}\left(\sum_{k=1}^{r} \phi_k^2 P_k\right)P_z^{-1} + (A_z + B_zK_zP_z^{-1})^TP_z^{-1} + P_z^{-1}(A_z + B_zK_zP_z^{-1})\right\}x(t).$$

Hence, the stability of the closed-loop system is guaranteed by the following condition

$$P_z^{-1}\left(\sum_{k=1}^{r} \phi_k^2 P_k\right)P_z^{-1} + (A_z + B_zK_zP_z^{-1})^TP_z^{-1} + P_z^{-1}(A_z + B_zK_zP_z^{-1}) < 0, \tag{17}$$

which is equivalent to the following one (obtained by post- and pre-multiplying (17) by P_z

$$\sum_{k=1}^{r} \phi_k^2 P_k + (A_z P_z + B_z K_z)^T + (A_z P_z + B_z K_z) < 0 \tag{18}$$

(18) holds if

$$\phi_k^2 P_k + \frac{1}{r}\left((A_z P_z + B_z K_z)^T + (A_z P_z + B_z K_z)\right) < 0 \tag{19}$$

holds. By consequence, using the schur complement on (19) we obtain

$$\begin{aligned} &\begin{bmatrix} -\frac{1}{r}\left(A_z P_z + B_z K_z + (A_z P_z + B_z K_z)^T\right) & \phi_k I \\ \phi_k I & Q_k \end{bmatrix} \\ &= \left[\sum_{i=1}^{r} h_i(z)^2 \Upsilon_{ii}^k + \sum_{i<j=r} h_i(z) h_j(z)(\Upsilon_{ij}^k + \Upsilon_{ji}^k)\right] > 0, \end{aligned} \tag{20}$$

which completes the proof.

As before, we define $\mathcal{S}$ to be the set of all $\phi_1, \dots, \phi_r$ such that the stability conditions (13) are satisfied. Thus, we must maximize the size of the set $\mathcal{S}$ as given in Theorem 3.1. Using Lemma 2.1, we need to look for a solution to the following optimization problem (OP_5), which substitutes (OP_4) in Theorem 3.1, by selecting the tuning parameter $0 < \beta < 1$.

$$(OP_5)\left\{\begin{array}{l} \underset{\phi_i, P_i, Q_i}{\text{minimize}} \left(\beta \sum_{i=1}^{r} Tr(P_i Q_i) - (1-\beta)\sum_{i=1}^{r} \phi_i\right) \text{ subject to} \\ \Upsilon_{ii}^k > 0, \;\; 1 \le i, k \le r, \\ \Upsilon_{ij}^k + \Upsilon_{ji}^k \ge 0, \;\; 1 \le i < j \le r, 1 \le k \le r, \\ \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} > 0, \;\; 1 \le i \le r, \end{array}\right. \tag{21}$$

where $\Upsilon_{ij}^k \equiv \begin{bmatrix} -\frac{1}{r}\left(A_i P_j + B_i K_j + (A_i P_j + B_i K_j)^T\right) & \phi_k I \\ \phi_k I & Q_k \end{bmatrix}$.

Remark 3.1. *It is worth noting that all constraints in (21) are linear in P_i, Q_i, K_i and ϕ_i. Further, gains K_i are directly computed.*

Remark 3.2. *The number of LMIs to be satisfy in (OP_5) is the same as for uncontrolled system, which can be consider as one of the advantage of the proposed approach.*

Finally, the following linearization algorithm is proposed to solve (OP_5).

Algorithm 2. *Fix a tolerance ϵ (for example, $\epsilon = 10^{-6}$) and $0 < \beta < 1$, and execute the following:*

- *Step 1: Set $Q_i^0 = I$ and $P_i^0 = I$, for $i = 1, \dots, r$.*
- *Step 2: Solve the LMI optimization:*

  $$\underset{P_i, Q_i, \phi_i}{\text{minimize}} \; \beta \sum_{i=1}^{r} \mathbf{Tr}(P_i^l Q_i + Q_i^l P_i) - (1-\beta)\sum_{i=1}^{r} \phi_i \textit{ subject to the constraints (21).}$$

- *Step 3: Let P_i^*, Q_i^*, $i = 1, \dots, r$ be the optimal solution, if $\left|\sum_{i=1}^{r} \mathbf{Tr}(P_i^* Q_i^*) - n \times r\right| \le \epsilon$ then stop, otherwise set $Q_i^{l+1} \longleftarrow Q_i^*$, $P_i^{l+1} \longleftarrow P_i^*$, and repeat from Step 2.*

Remark 3.3. *Again, the weight* β *can be tuned to enlarge the perimeter* $\sum_{i=1}^{r} \phi_i$.

3.1. **First example.** To show the applicability of the proposed approach, the same model already studied by [11, 12, 15] is used:

$$\begin{array}{l} \text{Rule 1}: \ \text{IF } x_1(t) \text{ is } h_1(x_1(t)) \text{ THEN } \dot{x}(t) = A_1 x(t), \\ \text{Rule 2}: \ \text{IF } x_1(t) \text{ is } h_2(x_1(t)) \text{ THEN } \dot{x}(t) = A_2 x(t), \end{array} \tag{22}$$

where

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \tag{23}$$

and the membership functions h_1 and h_2 are defined as follows:

$$h_1 = (1 - \sin(x_1))/2, \quad h_2 = 1 - h_1 \tag{24}$$

3.1.1. *Stability analysis.* It is assumed that the model is valid in the region $R = \{x : |x_i| \leq \pi/2\}$. As shown by [11], quadratic stability, that consists in finding a common Lyapunov matrix for the all subsystems, fails for the TS model (30). Solving the optimization problem given in Algorithm 1, the following solution is obtained with $\beta = 0.29$, after only 3 iterations:

$$\begin{array}{ll} P_1 = \begin{bmatrix} 0.7708 & 1.2083 \\ 1.2083 & 2.9780 \end{bmatrix}, & P_2 = \begin{bmatrix} 1.0506 & 1.6464 \\ 1.6464 & 3.7071 \end{bmatrix}, \\ Q_1 = \begin{bmatrix} 3.5647 & -1.4463 \\ -1.4463 & 0.9226 \end{bmatrix}, & Q_2 = \begin{bmatrix} 3.1305 & -1.3903 \\ -1.3903 & 0.8872 \end{bmatrix}. \end{array} \tag{25}$$

Figure 1 depicts the trajectory of (30) from various initial points in R, using the proposed controller. For the sake of comparison, Figure 2 also presents the regions of attraction R_0 and Ω, estimated using the methodologies given by [15, 20], respectively. It is clearly shown that, using the proposed methodology, the stability region of (30) is significantly larger and reaches R.

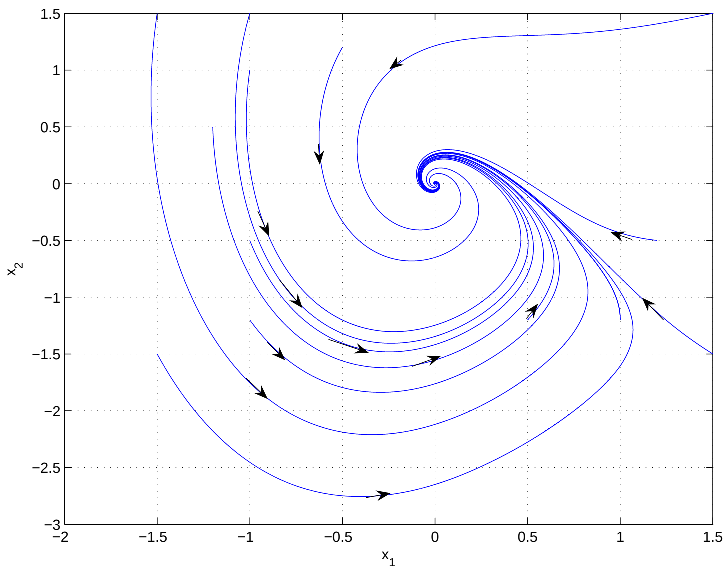

FIGURE 1. System trajectories using the proposed approach

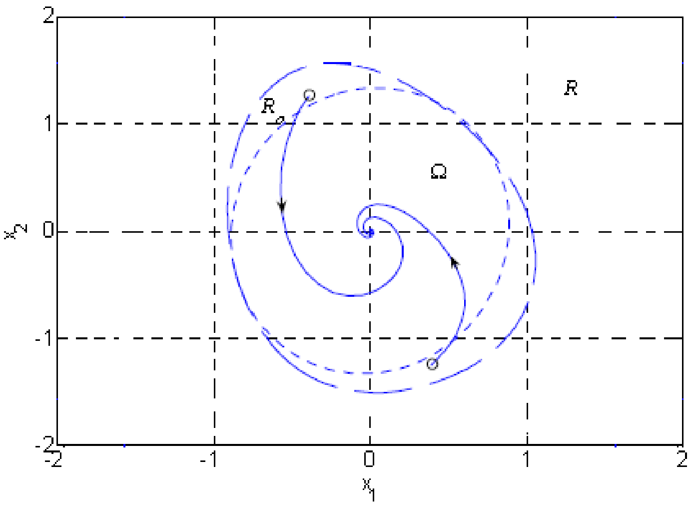

FIGURE 2. Estimated regions of attraction using [15] (R_0), [20] (Ω) and the proposed methodology (R)

3.1.2. *Controller design.* For the same system, assume that

$$B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \tag{26}$$

It should be noted that the methodology proposed by [11] cannot solve this control problem, since the time derivative of $h_i(x_1(t))$ cannot be calculated from only the states. In contrast, using Algorithm 2 stabilizing controller gains for the system (30) can be obtained as follows:

$$K_1 = [-0.6090 \ -2007.6], \quad K_2 = [-8.9432 \ -3227.9]. \tag{27}$$

Thus, a stabilizing control law is given by

$$u(t) = (h_1K_1 + h_2K_2)(h_1P_1 + h_2P_2)^{-1}x(t). \tag{28}$$

Figure 3 shows the evolution of the state variables from several initial conditions using the control law (28). It can be seen that asymptotically stability for the closed-loop system is guaranteed by the proposed approach.

3.2. **Second example.** Consider an inverted pendulum on a cart which is described as below. This model was also considered by [21]

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{g\sin(x_1(t)) - amlx_2^2(t)\sin(2x_1(t)/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))}, \end{aligned} \tag{29}$$

where $x_1(t)$ denotes the angle (in radians) of the pendulum from the vertical, $x_2(t)$ is the angular velocity, $g = 9.8m/s^2$ is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, u is the force applied to the cart (in newtons) and $a = 1/m + M$.

We can describe this nonlinear system as a T-S system composed of two subsystems, related through two rules: the first rule corresponds to the operation point about $x_1 = 0$, whereas the second one corresponds to x_1 around $\pm pi/2$. Thus, similar to [21] one can

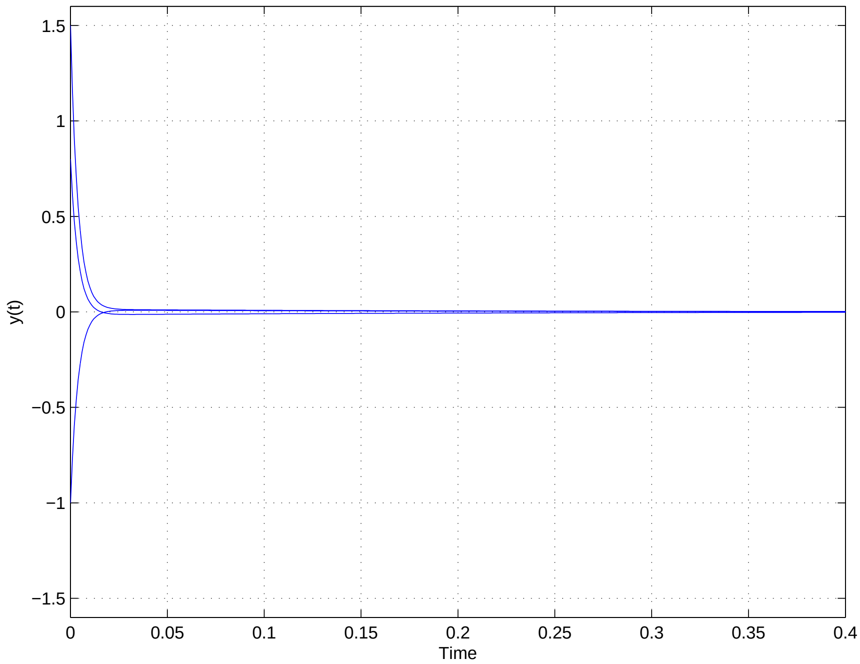

FIGURE 3. Control result

describe this system by the following two rules:

$$\begin{array}{l} \text{Rule 1: if } x_1(t) \text{ is about } 0, \text{ then } \dot{x}(t) = A_1x(t) + B_1u(t), \\ \text{Rule 2: if } x_1(t) \text{ is about } \pm pi/2(|x_1| < \pi/2), \text{ then } \dot{x}(t) = A_2x(t) + B_2u(t), \end{array} \tag{30}$$

where

$$\begin{array}{l} A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix}, \\ A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\theta^2)} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -\frac{a\theta}{4l/3-aml\theta^2} \end{bmatrix}, \end{array} \tag{31}$$

and $\theta = \cos(88\,\text{deg})$. The membership functions for Rules 1 and 2 are shown in Figure 4.

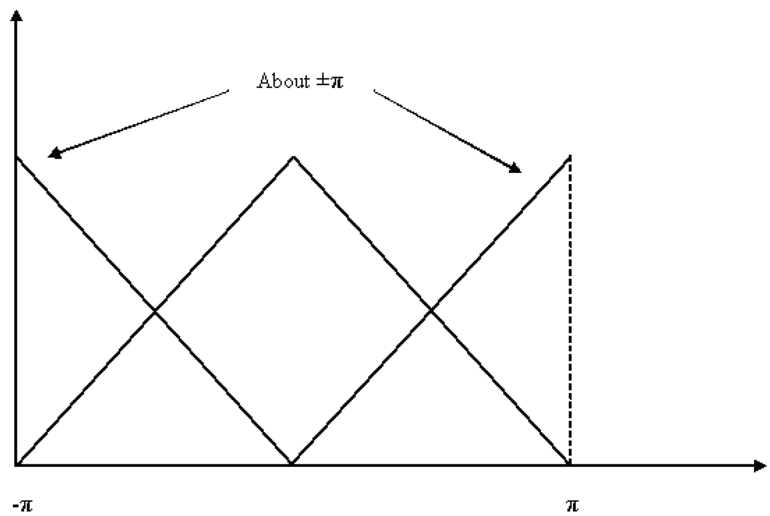

FIGURE 4. Membership functions the inverted pendulum

Using Algorithm 2 for (31), the following controller matrices are obtained after only 1 iteration, executed in 0.6 sec, using LMITools running on a Pentium IV computer:

$$\begin{array}{l} P_1 = \begin{bmatrix} 0.9852 & -0.1752 \\ -0.1752 & 1.0469 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.9854 & -0.1753 \\ -0.1753 & 1.0469 \end{bmatrix}, \\ K_1 = [100 \;\; 6.62 \times 10^5], \quad K_2 = [20 \;\; 2.35 \times 10^6]. \end{array} \tag{32}$$

Figure 5 illustrates some simulations of the closed-loop behavior of the system with the developed TS controller, from the initial conditions $x_1 = 65\,\text{deg}, 75\,\text{deg}, 85\,\text{deg}$, always with $x_2 = 0$, that correspond to the pendulum starting at rest from different angles. It can be seen that, effectively, the system is stabilized by the proposed controller.

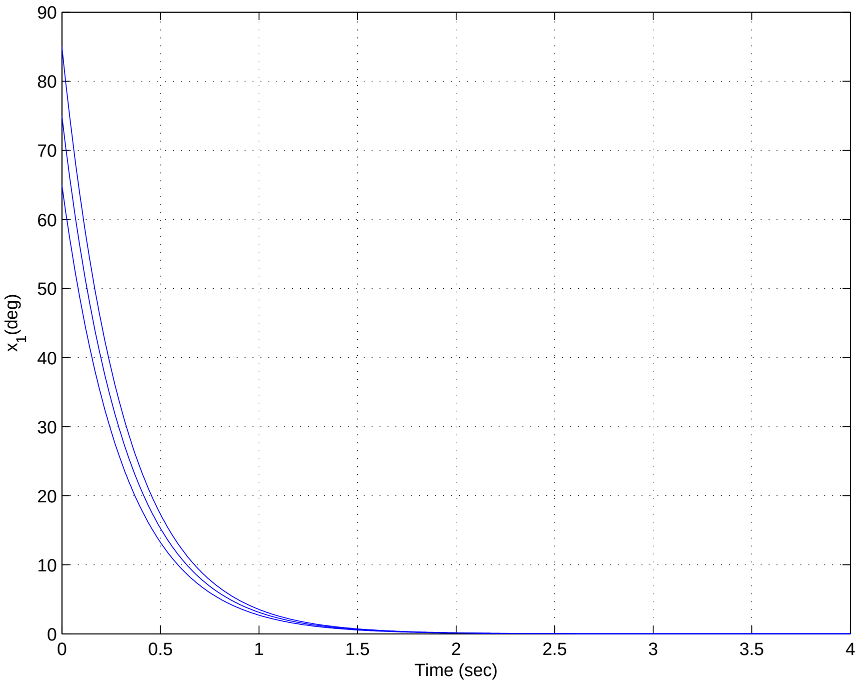

Figure 5. Evolution of the angle using the proposed controller design, starting from rest at different positions

4. **Conclusions.** This article has proposed a new methodology for stability analysis and controller design for continuous-time Takagi-Sugeno systems. The proposed approach gives LMI-based algorithms for checking stability and designing stabilizing controllers, following a Non-PDC structure. These algorithms are based on conditions less conservative than those in previous works. Indeed, the provided result is derived without fixing in advance the bounds on the time-derivatives of the Membership Functions. The efficiency of the proposed approach is shown through two examples from the literature.

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REFERENCES

[1] W.-J. Wang and C.-H. Huang, Model-based fuzzy control application to a self-balancing two-wheeled inverted pendulum, *IEEE International Symposium on Intelligent Control, Part of 2009 IEEE Multi-Conference on Systems and Control*, Saint Petersburg, Russia, 2009.

[2] M. Nachidi, F. Rodríguez, F. Tadeo and J. Guzmán, Takagi-Sugeno control of nocturnal temperature in greenhouses using air heating, *ISA Transactions*, vol.50, pp.315-320, 2011.

[3] T. Takagi and M. Sugeno, Fuzzy identification of systems and its applications to modelling and control, *IEEE Transactions on Fuzzy Systems, Man, and Cybernetics*, vol.15, no.1, pp.116-132, 1985.

[4] T. M. Guerra and L. Vermeiren, LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form, *Automatica*, vol.40, no.5, pp.823-829, 2004.

[5] B. Ding, H. Sun and P. Yang, Further studies on LMI-based relaxed stabilization conditions for nonlinear systems in Takagi-Sugeno's form, *Automatica*, vol.42, no.3, pp.503-508, 2006.

[6] V. S. Denysenko, A. A. Martynyuk and V. I. Slynḱo, Stability analysis of impulsive Takagi-Sugeno systems, *International Journal of Innovative Computing, Information and Control*, vol.5, no.10(A), pp.3141-3155, 2009.

[7] X. Ban, X. Z. Gao, X. Huang and H. Yin, Stability analysis of the simplest Takagi-Sugeno fuzzy control system using popov criterion, *International Journal of Innovative Computing, Information and Control*, vol.3, no.5, pp.1087-1096, 2007.

[8] M. Nachidi, A. Benzaouia, F. Tadeo and M. A. Rami, LMI-based approach for output-feedback stabilization for discrete time Takagi-Sugeno systems, *IEEE Transactions on Fuzzy Systems*, vol.16, no.5, pp.1188-1196, 2008.

[9] L. Wang and X. Liu, New relaxed stabilization conditions for fuzzy control systems, *International Journal of Innovative Computing, Information and Control*, vol.5, no.5, pp.1451-1460, 2009.

[10] A. Jadbabaie, A reduction in conservatism in stability and l_2 gain analysis of Takagi-Sugeno fuzzy systems via linear matrix inequalities, *Proc. of the 14th IFAC World Congress*, Beijing, China, pp.285-289, 1999.

[11] K. Tanaka, T. Hori and H. O. Wang, A multiple Lyapunov approach to stabilization of fuzzy control systems, *IEEE Transactions on Fuzzy Systems*, vol.11, no.4, pp.582-589, 2003.

[12] B.-J. Rhee and S. Won, A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design, *Fuzzy Sets and Systems*, vol.157, no.9, pp.1211-1228, 2006.

[13] T. M. Guerra, A. Kruszewski and J. Lauber, Discrete Tagaki-Sugeno models for control: Where are we? *Annual Reviews in Control*, vol.33, no.1, pp.37-47, 2009.

[14] M. Bernal, T. M. Guerra and A. Kruszewski, A memebership-fucnction-dependent approach for stability analysis and controller synthesis of Takagi-Sugeno models, *Fuzzy Sets and Systems*, vol.160, no.19, pp.2776-2795, 2009.

[15] T. M. Guerra and M. Bernal, A way to escape from the quadratic framework, *Fuzz-IEEE*, 2009.

[16] D. H. Lee, J. B. Park and Y. H. Joo, A new fuzzy Lyapunov function for relaxed stability condition of continuous-time Takagi-Sugeno fuzzy systems, *IEEE Transactions on Fuzzy Systems*, vol.19, no.4, pp.785-791, 2011.

[17] H. Zhang and X. Xie, Relaxed stability conditions for continuous-time ts fuzzy-control systems via augmented multi-indexed matrix approach, *IEEE Transactions on Fuzzy Systems*, vol.19, no.3, pp.478-492, 2011.

[18] J. Cai, S. Hu and H. Tao, Fuzzy static output feedback guaranteed cost reliable control for uncertain nonlinear systems with time-delay, *International Journal of Innovative Computing, Information and Control*, vol.4, no.12, pp.3409-3420, 2009.

[19] P. Apkarian and P. Gahinet, A convex characterization of gain-scheduling H_∞ controllers, *IEEE Transactions on Automatic Control*, vol.40, no.5, pp.853-864, 1995.

[20] H. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey, USA, 2002.

[21] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*, Wiley, New York, 2001.