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MODEL REDUCTION OF SWITCHED SYSTEMS BASED ON SWITCHING GENERALIZED GRAMIANS

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Abstract. *In this paper, a general method for model order reduction of discrete-time switched linear systems is presented. The proposed technique uses switching generalized gramians. It is shown that several classical reduction methods can be developed into the generalized gramian framework for the model reduction of linear systems and for the reduction of switched systems. Discrete-time balanced reduction within a specified frequency interval is taken as an example within this framework. To avoid numerical instability and to increase the numerical efficiency, a generalized gramian-based Petrov-Galerkin projection is constructed instead of the similarity transform approach for reduction. It is proven that the proposed reduction framework preserves the stability of the original switched system. The performance of the method is illustrated by numerical examples.*
Keywords: Model reduction, Switched systems, Gramian and stability

1. **Introduction.** The complexity of models is increasing in response to the ever-increasing need for the accurate mathematical modeling of physical as well as artificial processes for simulation and control. To maintain tractability, efficient computational prototyping tools are required to replace such complex models by simpler models that capture their dominant characteristics. Due to this fact, model reduction methods have become increasingly popular over the last two decades [1-3,40]. Such methods are designed to extract a reduced order state space model that adequately describes the behavior of the system in question.

Most of the studies on model order reduction to date have been devoted to linear systems. The few methods proposed for nonlinear systems are not strong compared with linear reduction methods.

On the other hand, most of the methods that have been proposed to date for the control and analysis of hybrid systems suffer from high computational burden when dealing with large-scale dynamical systems. This has motivated the study of model reduction for hybrid systems [4-17]. The model reduction problem for Markovian switched systems was studied in [16]. In Markov jump systems, the transition probabilities of the jumping process are important, and to date, almost all of the issues with Markov jump systems have been investigated assuming the knowledge of transition probabilities. However, the likelihood of obtaining complete knowledge on the transition probabilities is questionable, and the cost of doing so is likely high [17]. The method presented in [4] deals with the abstraction of both the continuous and discrete parts of hybrid dynamical systems and uses balanced residualization for the reduction of the continuous part. Application of the method to switched systems may not preserve stability, and non-elegant behavior may arise for general hybrid systems because of approximation error and possible guard/reset map overlap. It was shown in [5] that the dimension of the state space can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. This result does not indicate that the method is a strong tool for the reduction of affine systems because it is an exact reduction. While exact reduction is very elegant, the class of systems for which this procedure applies is quite small. This method only considers observability for investigating the importance of states to discard. The improvements and more details on these are available in [11]. The problem of model reduction for discrete-time switched systems is addressed in several papers [6,13-15]. In [6], two different approaches are proposed to solve this problem. The first approach casts the model reduction problem as a convex optimization problem, which solves the model reduction problem by using a linearization procedure. The second one, based on cone complementarity linearization, casts the model reduction problem as a sequential minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard (if not infeasible) to solve for a large-scale system. Not only is this method restricted to discrete-time switched systems, but it also does not provide any hints about the number of states that are suitable to retain before reduction. Similar methods have been developed for more general classes of discrete-time switched systems in [12,15]. In [13,14], this problem is investigated for discrete-time switched systems under average dwell time switching. Stability conditions based on dwell time and the average dwell time are among the stability problems of switched systems with respect to the restricted switching signal [31].

In [7,9], we proposed the generalized gramian framework for the model reduction of switched systems based on the common generalized gramians of the subsystems. This framework has been developed for controller reduction in [8,9]. The framework is shown to provide satisfactory approximations, and it preserves the stability of the original system under arbitrary switching signal but is over conservative. The method reported in [10] is based on the convex generalized gramian concept. Although this method is less conservative than its counterpart in [7,9], and by choosing suitable tuning parameters in the algorithm can be more accurate, the stability preservation is not guaranteed for all switching sequences in this method.

In this paper, we propose a framework for the model reduction of a switched system based on switching generalized gramians. This general framework can be categorized as gramian-based model reduction method. The balanced model reduction introduced in [18] is one of the most common gramian-based model reduction schemes.

To apply a balanced reduction, the system is first represented in a basis in which the states that are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states that have this property. The balanced model reduction method is modified and developed from different point of view [1,2]. One of the methods that are presented based on balanced model reduction is the method making use of the generalized gramian [19]. In this method, Lyapunov inequalities (rather than Lyapunov equations) are solved to compute generalized gramians. The physical interpretations of generalized gramians are similar to those of ordinary gramians. Generalized gramians are used to devise a technique for structure preserving model reduction methods in [20].

In this paper, we first show that the generalized method in [19] can be extended to various gramian-based reduction methods. We also modified the method in [19] to avoid numerical instability and to achieve higher numerical efficiency by building Petrov-Galerkin projection based on generalized gramians. We propose a method based on the balanced model reduction within a specific frequency bound in this framework. We generalized the framework to model reduction of switched system to determine the switching Petrov-Galerkin projection based on switching generalized gramians. This framework was developed in such a way that it preserves the stability of the original switched system for the arbitrary switching signal. Furthermore, it is a general method in the sense that different classical gramian-based methods can be developed for the reduction of switched systems within this framework. The feasibility of and the preservation of stability afforded by this algorithm was studied. It is shown that the proposed framework is less conservative than its preceding counterparts.

The paper is organized as follows. In the next section, we review the balanced reduction method and the balanced reduction technique based on the generalized gramian. Section 3 presents how different gramian-based methods can be approximated as generalized gramian based techniques. The balanced reduction within a specific frequency interval based on the generalized gramian is also presented in this section. This section ends with some remarks on the numerical implementation of the algorithm, and using projection for generalized gramian-based reduction methods is suggested instead of balancing and truncation. Section 4 is devoted to developing the switching generalized gramian-based reduction method for the model reduction of switched systems, followed by discussions on stability, feasibility, the algorithm parameters and error bounds. Section 5 presents our numerical results, and Section 6 concludes the paper.

The notation used in this paper is as follows: M^* denotes the transpose of matrix if $M \in \mathrm{R}^{n\times m}$ and the complex conjugate transpose if $M \in \mathrm{C}^{n\times m}$. The norm $\|\,.\,\|_\infty$ denotes the H_∞ norm of a rational transfer function. The standard notation $>$, $\geq$ ($<$, $\leq$) is used to denote the positive (negative) definite and semi-definite ordering of matrices.

2. **Balanced Truncation and Generalized Gramian.**

Balanced truncation is a well-known method used for the model reduction of dynamical systems (see for example [1,2]). The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G := (A, B, C, D) \tag{1}$$

where G is the transfer matrix with associated state-space representation,

$$\begin{cases} \eta x(t) = Ax(t) + Bu(t), \quad x(t) \in \mathrm{R}^n \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

where η is either the derivative operator $\eta f(t) = \frac{df(t)}{dt}$, $t \in \mathrm{R}$ or the shift $\eta f(t) = f(t+1)$, $t \in \mathrm{Z}$.

Gramians for continuous time systems are given by the solutions to the Lyapunov equations:

$$\begin{aligned} AP + PA^* + BB^* = 0 \\ A^*Q + QA + C^*C = 0 \end{aligned} \tag{3}$$

and for discrete-time systems by

$$\begin{aligned} APA^* - P + BB^* = 0 \\ A^*QA - Q + C^*C = 0 \end{aligned} \tag{4}$$

For stable A, the equations produce unique positive definite solutions P and Q, called the controllability and observability gramians. In balanced reduction, the system is first transformed into a balanced structure in which gramians are equal and diagonal:

$$\begin{aligned} & P = Q = diag(\sigma_1 I_{k_1}, \ldots, \sigma_q I_{k_q}) \\ & \sum_{j=1}^{q} k_j = n \end{aligned} \tag{5}$$

where $\sigma_i > \sigma_{i+1}$; here, σ_i's are called Hankel singular values.

The reduced model can be easily obtained by truncating the states that are associated with the set of the lowest Hankel singular values. Applying the method to a stable, minimal G, if we keep all of the states associated to σ_m $(1 \leq m \leq r)$ and truncate the rest the reduced model, G_r will be minimal and stable and satisfies the following [1,2]:

$$\|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{q} \sigma_j \tag{6}$$

A closely related model reduction method is that presented in [19]. This method is based on generalized gramians. In this method, instead of Lyapunov Equations (3), the following Lyapunov inequalities are solved:

$$\begin{aligned} AP_g + P_g A^* + BB^* \leq 0 \\ A^*Q_g + Q_g A + C^*C \leq 0 \end{aligned} \tag{7}$$

For stable A, there are positive definite solutions P_g and Q_g called the generalized controllability and observability gramians. It should be noted that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method; the only difference is that in this algorithm the balancing and truncation are based on the generalized gramian instead of ordinary gramians. In this method, we have generalized Hankel singular values (γ_i), which are the diagonal elements of balanced generalized gramians instead of Hankel singular values σ_i, which are the diagonal elements of balanced standard gramians. The error bound (6) holds in terms of the generalized Hankel singular values (γ_i) instead of Hankel singular values (σ_i). It should be noted that $\gamma_i \geq \sigma_i$ [19]. Therefore the error in balanced reduction based on the generalized gramian is lower bounded by the error of ordinary balanced model reduction. To achieve more accurate results, we can find P_g and Q_g in (7), such that, $tr(Q_g)$ and $tr(P_g)$ are minimized.

3. **Generalized Gramian Framework for Gramian-Based Model Reduction Methods.** In this section, we first present a general framework to build generalized gramian-based reduction methods analogous to those created using gramians. Subsequently, we present generalized balanced reduction within a specific frequency bound within this framework, followed by a discussion on the numerical implementation of the algorithm based on projection.

3.1. Lyapunov equations, Lyapunov inequalities and reduction.

Lemma 3.1. *Suppose A is stable, Y is symmetric and*

$$\begin{aligned} A^*YA - Y &\leq 0 \\ A, Y &\in \mathrm{R}^{n\times n} \end{aligned} \tag{8}$$

holds. Then $Y \geq 0$, i.e., Y is positive semi-definite.

Proof: If $A^*YA - Y \leq 0$, there exists $M \geq 0$ such that:

$$A^*YA - Y + M = 0$$

On the other hand, for any stable A, the unique solution to the preceding is

$$Y = \sum_{k=0}^{\infty} (A^*)^k M A^k$$

In the above structure, $M \geq 0$; hence,

$$Y \geq 0$$

This lemma leads to the following proposition that makes the relation between Lyapunov equations and Lyapunov inequalities evident. Let $\Gamma : \mathrm{R}^{n\times n} \to \mathrm{R}^{n\times n}$ be an operator that is defined as:

$$\Gamma_{A,R}(X) := A^*XA - X + R \tag{9}$$

Proposition 3.1. *Suppose A is stable and X is the solution of the Lyapunov equation*

$$\Gamma_{A,Q}(X) = A^*XA - X + Q = 0 \tag{10}$$

where $Q \geq 0$. If a symmetric X_g satisfies

$$\Gamma_{A,Q}(X) = A^*X_gA - X_g + Q \leq 0 \tag{11}$$

then $X_g \geq X$.

Proof: Subtract (11)-(10) and apply Lemma 3.1 with $Y = X_g - X$.

Proposition 3.1 is a direct consequence of Lemma 3.1, which shows how ordinary gramians can be approximated by the generalized gramians. Balanced reduction based on generalized gramians, which we reviewed in the last section, is based on Proposition 3.1. While this method might be less accurate than its gramian based counterpart, the approximation error is still bounded.

By deriving the associated Lyapunov equations and relaxing them to inequalities, we can readily generalize other gramian-based reduction methods in this framework. In the following, we propose a generalized version of balanced reduction within a frequency bound.

3.2. **Generalized balanced reduction within frequency bound.** Over the past two decades, much attention has been devoted to balanced model reduction, which has been developed and improved from several points of view. The frequency-weighted balanced reduction method is one of the devised gramian-based techniques based on ordinary balanced truncation [1,2,21-23]. In this method, the model reduction is biased by frequency-dependent input/output weights. In many cases the input and output weights are not given; instead, the problem is to reduce the model over a given frequency range [1,2]. This problem can be addressed directly by balanced reduction within the frequency bound, which was first proposed in [24] and then modified in [2] to preserve the stability of the original system and to provide an error bound for approximation. In [25], a similar method is proposed for discrete-time systems and further improved in [26] to preserve stability and to provide a computable error bound. In this method, for a discrete-time dynamical system (1) the controllability gramian $P(\omega_1,\omega_2)$ and observability gramians $Q(\omega_1,\omega_2)$ within frequency range of operation $[\omega_1,\omega_2]$ are defined as follows:

$$\begin{aligned} P(\omega_1,\omega_2) &:= \frac{1}{2\pi}\int_{\omega_1}^{\omega_2}(I-Ae^{-j\omega})^{-1}BB^*(I-A^*e^{j\omega})^{-1}d\omega \\ Q(\omega_1,\omega_2) &:= \frac{1}{2\pi}\int_{\omega_1}^{\omega_2}(I-A^*e^{j\omega})^{-1}C^*C(I-Ae^{-j\omega})^{-1}d\omega \end{aligned} \tag{12}$$

where $0 \le \omega_1 < \omega_2 \le \pi$.

Due to the symmetry of the Fourier transform, the integration is carried out over $[\omega_1,\omega_2]$ and $[-\omega_2,-\omega_1]$, therefore, the gramians are always real.

To show the associated Lyapunov equations, we need to introduce more notations

$$F(\omega_1,\omega_2) := -\frac{\omega_2-\omega_1}{4\pi}I + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2}(I-Ae^{-j\omega})^{-1}d\omega \tag{13}$$

$$X(\omega_1,\omega_2) = F(\omega_1,\omega_2)BB^* + BB^*F(\omega_1,\omega_2)^* \tag{14}$$

$$Y(\omega_1,\omega_2) = C^*CF(\omega_1,\omega_2) + F(\omega_1,\omega_2)^*C^*C \tag{15}$$

The gramians satisfy the following Lyapunov equations [25,26]:

$$\begin{aligned} AP(\omega_1,\omega_2)A^* - P(\omega_1,\omega_2) + X(\omega_1,\omega_2) = 0 \\ A^*Q(\omega_1,\omega_2)A - Q(\omega_1,\omega_2) + Y(\omega_1,\omega_2) = 0 \end{aligned} \tag{16}$$

This method is modified in [26] to guarantee the stability and to provide a simple error bound. The modified version starts with the Schur decomposition of X and Y:

$$\begin{aligned} X(\omega_1,\omega_2) = U\Lambda U^* = U\,diag(\lambda_1,\ldots,\lambda_n)U^* \\ Y(\omega_1,\omega_2) = V\Delta V^* = V\,diag(\delta_1,\ldots,\delta_n)V^* \end{aligned} \tag{17}$$

where $UU^* = VV^* = I_n$, $|\lambda_i| \ge |\lambda_{i+1}| \ge 0$, $|\delta_i| \ge |\delta_{i+1}| \ge 0$.

It should be noted that since $X(\omega_1,\omega_2)$ and $Y(\omega_1,\omega_2)$ are real and symmetric, decompositions in the form (17) exist. Let

$$\begin{aligned} \hat{B} &:= U\,diag\left(|\lambda_1|^{1/2},\ldots,|\lambda_n|^{1/2}\right) \\ \hat{C} &:= diag\left(|\delta_1|^{1/2},\ldots,|\delta_n|^{1/2}\right)V^* \end{aligned} \tag{18}$$

The modified gramians satisfy the following Lyapunov equations instead of (16):

$$\begin{aligned} A\hat{P}(\omega_1,\omega_2)A^* - \hat{P}(\omega_1,\omega_2) + \hat{B}\hat{B}^* = 0 \\ A^*\hat{Q}(\omega_1,\omega_2)A - \hat{Q}(\omega_1,\omega_2) + \hat{C}^*\hat{C} = 0 \end{aligned} \tag{19}$$

For the generalization, we have the following inequalities:

$$\begin{aligned} A\hat{P}(\omega_1,\omega_2)A^* - \hat{P}(\omega_1,\omega_2) + \hat{B}\hat{B}^* \le 0 \\ A^*\hat{Q}(\omega_1,\omega_2)A - \hat{Q}(\omega_1,\omega_2) + \hat{C}^*\hat{C} \le 0 \end{aligned} \tag{20}$$

Then, the generalized modified balanced reduction within frequency bound can be obtained by simultaneously diagonalizing $\hat{P}_g(\omega_1,\omega_2)$ and $\hat{Q}_g(\omega_1,\omega_2)$ and then by truncating the states associated with the set of the least generalized Hankel singular values.

3.3. **Numerical issues.** Balanced transformation can be numerically ill-conditioned when dealing with systems with some nearly uncontrollable modes or some nearly unobservable modes. The difficulties associated with the computation of the required balanced transformation in [27] draw some attention to alternative numerical methods [28]. Balancing can be badly conditioned even when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian-based framework without balancing at all. The Schur method and square-root algorithms provide projection matrices to apply balanced reduction without balanced transformation [1,28]. This method can be easily applied to other gramian-based methods. In our generalized method, we can use the same algorithm by plugging generalized gramians into the algorithm instead of ordinary gramians.

4. Model Reduction of Switched Systems.

4.1. Model reduction of switched systems based on switching generalized gramians.

One of the most important subclasses of hybrid systems is linear switched systems [29]. A linear switched system is a dynamical system specified by the following equations:

$$\sum : \begin{cases} \eta x(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{cases} \tag{21}$$

where $x(t) \in \mathrm{R}^n$ is the state, $y(t) \in \mathrm{R}^p$ is the output, $u(t) \in \mathrm{R}^m$ is the input and $\sigma : \mathrm{R}^{\geq 0} \to \mathrm{K} \subset \mathrm{N}$ is the switching signal that is a piecewise constant map. K is the set of discrete modes which is assumed to be finite. For each $i \in \mathrm{K}$, A_i, B_i, C_i and D_i are matrices of appropriate dimensions. The indicator function is defined as:

$$\zeta_i(t) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the } i\text{th mode matrices } (A_i,\, B_i,\, C_i,\, D_i) \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

The switched system (21) can also be written as the following using indicator function:

$$\sum : \begin{cases} \eta x(t) = \sum\limits_{i=1}^{|K|} \zeta_i (A_i x(t) + B_i u(t)) \\ y(t) = \sum\limits_{i=1}^{|K|} \zeta_i (C_i x(t) + D_i u(t)) \end{cases} \tag{23}$$

In this section, we present a framework for the model reduction of the switched system described by (21). The aim is to find a projection that maps the state-space of a switched system to a lower dimensional subspace. Definition 4.1 describes the general definition of a Petrov-Galerkin projection.

Definition 4.1. *The Petrov-Galerkin projection for a dynamical system:*

$$\begin{cases} \eta x(t) = f(x(t), u(t)), \quad x \in \mathrm{R}^n \\ y(t) = g(x(t), u(t)) \end{cases} \tag{24}$$

is defined as a projection $\Pi = VW^*$*, where* $W^*V = I_k$*,* $V, W \in \mathrm{R}^{n\times k}$*,* $k < n$ *[1].*

The reduced order model produced using this projection is

$$\begin{cases} \eta \hat{x}(t) = W^* f(V\hat{x}(t), u(t)), \quad \hat{x} \in \mathrm{R}^k \\ y(t) = g\left(V\hat{x}(t), u(t)\right) \end{cases} \tag{25}$$

In our framework, we construct the aforementioned projection based on the switching generalized gramian that is defined as follows:

Definition 4.2. *Switching controllability (observability) generalized gramian for the dynamical system (21) is defined as:*

$$\Psi_g(t) = \sum_{i=1}^{|K|} \zeta_i(t) P_{g,i} \tag{26}$$

$P_{g,i}$ is the controllability (observability) generalized gramian associated with the ith mode of (21).

To develop a generalized gramian framework for the model reduction of switched linear systems, the generalized gramian reduction framework can be applied locally to reduce each subsystem independently. Unlike ordinary gramians, the generalized gramians are not unique; therefore, we can choose the generalized gramians for subsystems such that the reduction framework preserves important properties of the original system such as stability.

At this point, it is possible to integrate different gramian-based reduction methods into this framework for the reduction of switched systems by finding the generalized controllability/observability gramian for each subsystem and constructing switching controllability/observability generalized gramians. The next step can be the simultaneous diagonalization of the switching generalized gramian and balancing and reduction of all subsystems based on Hankel singular values of the switching generalized gramian in each mode. To avoid numerical bad conditioning and to increase the efficiency of the method, we use the Schur or square-root algorithm instead of balancing; thus, direct Petrov-Galerkin projection matrices can be computed. This procedure is less conservative and provides more accurate results.

In the method that we proposed in [7,9], the stability of the original switched systems under arbitrary switching signal is guaranteed to be preserved due to the existence of a common quadratic Lyapunov function. This was the main cause of the conservatism. In our new framework, the generalized gramians are computed such that the existence of a piecewise quadratic Lyapunov function for the switched system is guaranteed, and the stability of the reduced switched system is consequently guaranteed. In the following, we first propose our general framework for the model reduction of a switched system.

Let the observability gramian Q_i and the controllability gramian P_i corresponding to a general gramian-based method for each subsystem be derived as the solutions to the following Lyapunov equations:

$$\Gamma_{A_i,M_i}(Q_i) = 0 \tag{27}$$

$$\Gamma_{A_i^*,N_i}(P_i) = 0 \tag{28}$$

where M_i, N_i are positive semi-definite.

To develop the gramian based reduction method for switched systems which preserves the stability of the original system, the switching controllability generalized gramian $\Psi_{cg}(t)$ and switching observability generalized gramian $\Psi_{og}(t)$ are obtained:

$$\Psi_{cg}(t) = \sum_{i=1}^{|K|} \zeta_i(t) P_{g,i} \tag{29}$$

$$\Psi_{og}(t) = \sum_{i=1}^{|K|} \zeta_i(t) Q_{g,i} \tag{30}$$

where the generalized observability gramian $Q_{g,i}$ and the generalized controllability gramian $P_{g,i}$ are the solutions to:

$$\begin{aligned} &\Gamma_{A_i^*,N_i}(P_{g,i}) < 0 \\ &\Gamma_{A_i,M_i}(Q_{g,i}) < 0 \\ &A_i^* Q_{g,j} A_i - Q_{g,i} < 0 \end{aligned} \tag{31}$$

for all $i \in \mathrm{K}$.

The next step is to simply construct a Petrov-Galerkin projection for each subsystem based on the switching gramians in each mode.

In the following, to clarify the proposed general framework we extend the generalized balanced reduction within a given frequency interval that was presented in previous section for model reduction of a switched linear system.

First, we must find the generalized controllability gramian $\hat{P}_{g,i}(\omega_1, \omega_2)$ and the generalized observability gramian $\hat{Q}_{g,i}(\omega_1, \omega_2)$ for each subsystem within a frequency domain satisfying (29)-(31) for all $i, j \in K$. In other words, the following LMI need to be solved:

$$A_i \hat{P}_{g,i}(\omega_1, \omega_2) A_i^* - \hat{P}_{g,i}(\omega_1, \omega_2) + \hat{B}_i \hat{B}_i^* < 0 \tag{32}$$

$$A_i^* \hat{Q}_{g,i}(\omega_1, \omega_2) A_i - \hat{Q}_{g,i}(\omega_1, \omega_2) + \hat{C}_i^* \hat{C}_i < 0 \tag{33}$$

$$A_i^* \hat{Q}_{g,j}(\omega_1, \omega_2) A_i - \hat{Q}_{g,i}(\omega_1, \omega_2) < 0 \tag{34}$$

The switching generalized gramians are:

$$\Psi_{cg}(t) = \sum_{i=1}^{|K|} \zeta_i(t)(P_{g,i}(\omega_1, \omega_2)) \tag{35}$$

$$\Psi_{og}(t) = \sum_{i=1}^{|K|} \zeta_i(t)(Q_{g,i}(\omega_1, \omega_2)) \tag{36}$$

If we substitute $\Psi_{cg}(t)$ and $\Psi_{og}(t)$ into the square-root algorithm we can directly obtain projectors associated with all subsystems for reduction. It should be noted that the results are the same as those of the balancing algorithm. A merit of the square-root method is that the method relies on the Cholesky factors of the gramians rather than the gramians themselves, which has advantages in terms of numerical stability.

4.2. **Stability and feasibility.** One of the important issues in model reduction is the preservation of the stability which needs to be studied. In other words, the question is whether the reduction technique can preserve the stability of the original model in approximation. In our proposed framework the stability of the original switched system is guaranteed to be preserved. To prove the stability preservation, we first need to recall a theorem on the stability of discrete time switched system from [30,31].

Theorem 4.1. *If there exist $|K|$ symmetric matrices $S_1, S_2, \ldots, S_{|K|}$ for a discrete-time dynamical system (21), satisfying:*

$$\begin{bmatrix} S_i & A_i^* S_j \\ S_j A_i & S_j \end{bmatrix} > 0 \qquad \forall (i,j) \in K \times K \tag{37}$$

then the switched system is asymptotic stable, and the associated Lyapunov function is given by

$$V(t, x(t)) = x(t)^* \left(\sum_{i=1}^{|K|} \zeta_i(t) S_i \right) x(t). \tag{38}$$

This theorem proposes a sufficient condition for the stability of a switched system based on the existence of a piecewise quadratic Lyapunov function, which is less conservative than the condition for stability based on a common Lyapunov function (see [30,31] for more details and proofs).

In the following proposition, we show that our framework for the reduction of switched system preserves stability.

Proposition 4.1. *If the discrete-time switched system described in (21) is stable, the generalized gramian based reduced order model is asymptotic stable.*

Proof: In the proposed method, we have

$$W_i^* V_i = I_k, \quad V_i, W_i \in \mathrm{R}^{n\times k},\ k < n$$
$$\begin{cases} \hat{A}_i = W_i^* A_i V_i \\ \hat{B}_i = W_i^* B_i \\ \hat{C}_i = C_i V \\ \hat{D}_i = D_i \end{cases} \tag{39}$$

which is a projected system matrices associated with the reduced order switched model.

$$\hat{\sum} : \begin{cases} \eta \hat{x}(t) = \sum_{i=1}^{|K|} \zeta_i \left(\hat{A}_i \hat{x}(t) + \hat{B}_i u(t) \right) \\ y(t) = \sum_{i=1}^{|K|} \zeta_i \left(\hat{C}_i \hat{x}(t) + \hat{D}_i u(t) \right) \end{cases} \tag{40}$$

We know that $Q_{g,i}$ is the generalized observability gramian and that the original switched system satisfy (30) and (31); therefore $\forall (i,j) \in K \times K$,

$$A_i^* Q_{g,i} A_i - Q_{g,i} + M_i < 0, \quad Q_{g,i} > 0 \tag{41}$$
$$A_i^* Q_{g,j} A_i - Q_{g,i} < 0 \tag{42}$$

which is equivalent to

$$\begin{bmatrix} Q_{g,i} & A_i^* Q_{g,j} \\ Q_{g,j} A_i & Q_{g,j} \end{bmatrix} > 0 \qquad \forall (i,j) \in K \times K \tag{43}$$

based on the Schur complement inequality. The original system is asymptotic stable according to Theorem 4.1 and if we find $|K|$ symmetric matrices that satisfy (37) for the reduced order switched system, the reduced order switched model will be stable as well. From (41) and (42) we have

$$V_i^* \left(A_i^* Q_{g,i} A - Q_{g,i} + M_i \right) V_i < 0 \tag{44}$$
$$V_i^* \left(A_i^* Q_{g,j} A_i - Q_{g,i} \right) V_i < 0 \tag{45}$$

On the other hand, the outcome of the square-root algorithm for projection is [1]: $P_{g,i} W_i = V_i \sum_i$ and $Q_{g,i} V_i = W_i \sum_i$, where $\sum_i \in \mathrm{R}^{k\times k}$ is diagonal and positive definite, and we have

$$\begin{aligned} & V_i^* (A_i^* Q_{g,i} A - Q_{g,i} + M_i) V_i \\ &= V_i^* A_i^* Q_{g,i} A V_i - V_i^* Q_{g,i} V_i + V_i^* M_i V_i \\ &= V_i^* A_i^* W_i \textstyle\sum_i W_i^* A V_i - V_i^* W_i \sum_i + V_i^* M_i V_i \\ &= (W_i^* A V_i)^* \textstyle\sum_i (W_i^* A V) - \sum_i + V_i^* M_i V_i \end{aligned}$$

Hence

$$\hat{A}_i^* \sum\nolimits_i \hat{A}_i - \sum\nolimits_i + V_i^* M_i V_i < 0$$

and consequently,

$$\hat{A}_i^* \sum_i \hat{A}_i - \sum_i < 0 \tag{46}$$

Let $\Phi_{ij} := V_i^* Q_{g,j} V_i$; therefore, $Q_{g,j} = W_i \Phi_{ij} W_i^*$. We have from (40):

$$\begin{aligned} & V_i^*(A_i^* Q_{g,j} A_i - Q_{g,i}) V_i \\ &= V_i^* A_i^* Q_{g,j} A_i V_i - V_i^* Q_{g,i} V_i \\ &= V_i^* A_i^* W_i \Phi_{ij} W_i^* A_i V_i - V_i^* Q_{g,i} V_i \\ &= \hat{A}_i^* \Phi_{ij} \hat{A}_i - \sum_i \end{aligned}$$

Hence,

$$\hat{A}_i^* \Phi_{ij} \hat{A}_i - \sum_i < 0 \tag{47}$$

Note that $\Phi_{ij} = \Phi_{ij}^* > 0$ and $\Phi_{ii} = \sum_i$.

Let $S_j = \Phi_{ij}$ and $S_i = \sum_i$ according to Theorem 4.1. The reduced order switched model (40) is stable for an arbitrary switching sequence.

Our framework is said to be feasible if (41) and (42) are satisfied. These cannot be always satisfied. However, the framework is much less conservative compared to its counterparts in [7,9]. One way to improve the feasibility of the proposed model reduction method is to use the recently proposed extended notion of generalized gramians which are called extended gramians [32].

4.3. **Discussion on the method, features and restrictions.** The main feature of the proposed model reduction framework is the generality of the framework. In general, gramian-based model reduction algorithms for linear systems can be generalized for the model reduction of switched systems within this framework. This is demonstrated by the generalization of balanced reduction within a frequency interval for the model reduction of switched systems. One can extend other methods such as balanced reduction within a certain time interval in a similar way for the model reduction of switched system following the proposed framework. The structure of the reduction algorithm is such that it can easily be extended for switched controller reduction. This has already been done for the reduction framework based on common generalized gramians [8,9].

The advantage of the proposed method over those discussed in [4,10,11] is that it preserves stability for all switching signals, and the advantage of the presented method over those based on common generalized gramians lies in its conservatism. The proposed method is less conservative than the methods based on common generalized gramians. One unique feature of the proposed method with respect to the other LMI based model reduction methods for switched systems is that the Hankel singular values or more precisely in this case generalized Hankel singular values are computed and are available in the reduction procedure. This in particular is useful in design problems to tune the design parameters. A detailed example of such an application can be found in [42].

One of the drawbacks of the method is that it is not always feasible, and as it was suggested earlier one way to improve the feasibility of the proposed model reduction method is to use the recently proposed extended notion of generalized gramians which are called extended gramians [32]. Another drawback of the method is its computational complexity, which is the common problem in most LMI-based model reduction methods for switched systems. One way to improve the computational efficiency is to reduce the number of subsystems (discrete modes) before applying the method. To this end, discrete abstraction via pseudo-equivalence can be used [4].

One restriction of the proposed method is that the subsystems need to be stable for the method to be successfully applied. To overcome this restriction the generalized gramians for unstable systems need to be defined. This can be done in the manner similar to the method in [43], by solving two Riccati equations followed by two Lyapunov inequalities. It should be noted that in our method we must solve Lyapunov inequalities instead of Lyapunov equations because we are looking for generalized gramian instead of gramians.

5. **Numerical Example.** In this section, we first apply the proposed method for the reduction of two bimodal switched linear systems to illustrate the proposed method. The first example is of order 7, and the second is of order 25. The practical application of the method for fault-tolerant control and plug-and-play control is discussed. The proposed technique is applied for the model reduction of practical CD player example.

5.1. **Illustrative examples.**

Example 5.1. *7th-Order switched linear system: consider a single-input-single output switched linear of the form (21):*

$$A_1 = \begin{bmatrix} -0.334 & 0.3046 & -0.03543 & -0.07088 & 0.1474 & -0.2414 & -0.07635 \\ 0.1292 & -0.05956 & -0.03945 & 0.2164 & -0.3475 & -0.1074 & -0.2008 \\ -0.1205 & -0.02622 & -0.115 & -0.1031 & -0.05692 & -0.1377 & 0.02162 \\ -0.1308 & 0.01855 & -0.1999 & -0.6649 & -0.1376 & -0.0985 & -0.072 \\ -0.09125 & -0.3183 & -0.04991 & 0.1481 & -0.2894 & -0.1928 & 0.02208 \\ -0.3358 & 0.08599 & -0.05365 & 0.08062 & 0.07906 & -0.3054 & 0.01544 \\ -0.1247 & -0.1874 & 0.0197 & -0.01706 & 0.02899 & -0.01897 & 0.1089 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.02764 & 0.2331 & -0.3819 & 0.1918 & 0.1083 & -0.0531 & 0.412 \\ 0.2406 & -0.5743 & 0.06595 & 0.275 & -0.1156 & 0.3873 & 0.3771 \\ -0.3711 & 0.07406 & -0.3554 & 0.09365 & 0.2317 & 0.02326 & 0.3513 \\ 0.129 & 0.2794 & 0.1674 & 0.3015 & 0.1313 & 0.09701 & -0.05687 \\ 0.1283 & -0.1153 & 0.2107 & 0.1169 & 0.2967 & 0.3146 & -0.2963 \\ -0.01531 & 0.385 & -0.02076 & 0.09491 & 0.3066 & 0.2628 & -0.2449 \\ 0.4385 & 0.3797 & 0.3264 & -0.09338 & -0.2908 & -0.2239 & 0.3117 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.07866 \\ -0.6817 \\ -1.025 \\ -1.234 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.1497 \\ 0 \\ 0 \\ 1.535 \\ 0 \\ -1.347 \\ 0.4694 \end{bmatrix}$$

$$C_1 = [\ 0 \quad 0 \quad 0.0558 \quad 0 \quad -0.465 \quad 0.371 \quad 0.7283\ ]$$

$$C_2 = [\ -0.9036 \quad 0 \quad -0.6275 \quad 0.5354 \quad 0.5529 \quad -0.2037 \quad -2.054\ ]$$

$$D_1 = 0, \quad D_2 = 0.1326$$

To reduce the switched system we solve LMI's (32)-(34) to compute the switching gramians over the frequency domain $[\omega_1, \omega_2] = [0.0001, 1]$. The switching observability generalized gramian is computed by solving (33) and (34):

$$\Psi_{og}(t) = \sum_{i=1}^{2} \zeta_i(t)(Q_{g,i}(\omega_1, \omega_2))$$

where

$$\hat{Q}_{g,1}(\omega_1,\omega_2) = \begin{bmatrix} 432.3740 & -2.0884 & 36.5183 & 22.1324 & -26.9003 & 27.7670 & 5.6355 \\ -2.0884 & 474.9520 & -15.9972 & 0.2649 & 32.5495 & -23.7194 & 2.1711 \\ 36.5183 & -15.9972 & 426.6650 & 3.4264 & -9.5463 & 7.0706 & -19.7154 \\ 22.1324 & 0.2649 & 3.4264 & 462.4184 & 8.0981 & 20.2084 & 9.2145 \\ -26.9003 & 32.5495 & -9.5463 & 8.0981 & 425.7597 & 29.0492 & -0.5272 \\ 27.7670 & -23.7194 & 7.0706 & 20.2084 & 29.0492 & 424.6442 & -4.5798 \\ 5.6355 & 2.1711 & -19.7154 & 9.2145 & -0.5272 & -4.5798 & 437.7793 \end{bmatrix}$$

$$\hat{Q}_{g,2}(\omega_1,\omega_2) = \begin{bmatrix} 438.1188 & -1.8337 & 37.8168 & 18.3223 & -32.8578 & 25.2266 & 14.1855 \\ -1.8337 & 474.0864 & -15.9894 & 2.5844 & 29.8504 & -25.6838 & 7.0294 \\ 37.8168 & -15.9894 & 427.4084 & 5.2123 & -10.4245 & 6.8873 & -20.0638 \\ 18.3223 & 2.5844 & 5.2123 & 457.5497 & 15.1847 & 24.8077 & 9.0909 \\ -32.8578 & 29.8504 & -10.4245 & 15.1847 & 434.9824 & 34.4873 & -14.7420 \\ 25.2266 & -25.6838 & 6.8873 & 24.8077 & 34.4873 & 431.6313 & -16.5326 \\ 14.1855 & 7.0294 & -20.0638 & 9.0909 & -14.7420 & -16.5326 & 456.5890 \end{bmatrix}$$

The switching controllability generalized gramian $\Psi_{cg}(t)$ is computed similarly by solving (32). The resulting fourth-order switched linear model obtained by applying the presented method is

$$A_{1r} = \begin{bmatrix} -0.7885 & 0.03459 & -0.1212 & -0.1066 \\ 0.1418 & -0.5086 & -0.3072 & -0.1373 \\ -0.008041 & 0.2888 & -0.5294 & -0.04207 \\ -0.05953 & -0.07607 & -0.1193 & 0.2978 \end{bmatrix}$$

$$A_{2r} = \begin{bmatrix} -0.9483 & 0.01865 & -0.01575 & -0.02349 \\ 0.01436 & 0.9308 & 0.04869 & -0.005533 \\ 0.01945 & 0.04652 & -0.9376 & 0.04768 \\ -0.03012 & 0.03389 & -0.03229 & 0.7319 \end{bmatrix}$$

$$B_{1r} = \begin{bmatrix} 0.4763 \\ -1.272 \\ -0.6668 \\ -2.061 \end{bmatrix}, \quad B_{2r} = \begin{bmatrix} 0.2252 \\ 0.3751 \\ 0.26 \\ 0.5943 \end{bmatrix}$$

$$C_{2r} = \begin{bmatrix} 0.9824 & -2.738 & -0.5533 & -0.9583 \end{bmatrix}$$

$$C_{1r} = \begin{bmatrix} -0.155 & -0.072 & 0.01531 & 0.1706 \end{bmatrix}$$

$$D_{1r} = 0, \quad D_{2r} = 0.1326$$

Figure 1 shows the generalized Hankel singular values of the first subsystem, and Figure 2 shows the generalized Hankel singular values of the second subsystem. The step response of the original and reduced order switched systems associated to the switching signal of Figure 3 is presented in Figure 4.

Figure 1 and Figure 2 show that most of the input/output information is in the four states of the original systems. The proposed method provides accurate results after reduction of 3 states of the original system (42.8 % of the states).

Example 5.2. *Bimodal Switched linear System of order 25: Consider a bimodal switched linear system of order 25. The original system is SISO and is reduced to order 17 using the proposed reduction method over* $[\omega_1, \omega_2] = [0.001, 1000]$.

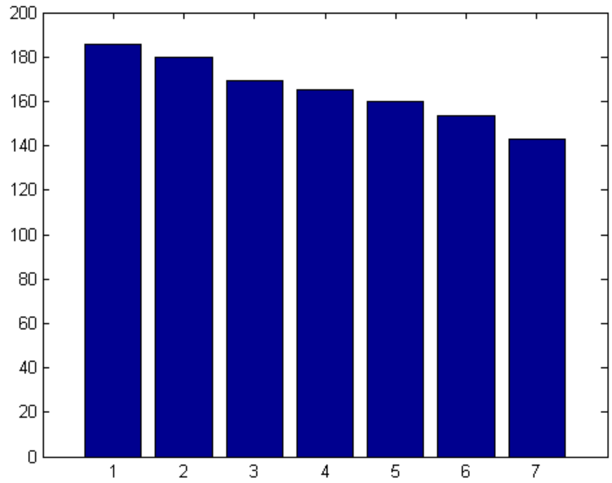

Figure 1. Generalized Hankel singular values (γ_i) of the first subsystem

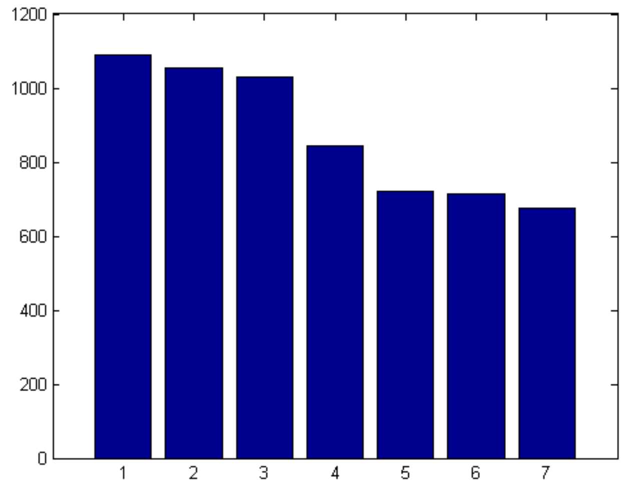

Figure 2. Generalized Hankel singular values (γ_i) of the second subsystem

The generalized Hankel singular values are shown in Figure 5 and Figure 6.

The step responses of the original and reduced order switched systems associated with the switching signal shown in Figure 7 is shown in Figure 8.

5.2. **Practical applications.** The focus of this section is on the applications of model reduction of switched systems. Methods for the model reduction of switched systems reduce the significant computational burden and complexity associated with the process of fault-tolerant control and plug-and-play control of large-scale systems. The objective of methods within the framework of plug-and-play process control and particularly fault-tolerant control is to establish control techniques which guarantee a certain performance through control reconfiguration upon the occurrence of faults or changes [33,34,41]. For this purpose, one natural way to model the system is to use a hybrid/switched systems framework. The procedure involves the assignment of one subsystem describing the system without fault and for each fault scenario a subsystem that models the faulty system. In this way, we obtain a switched system, and by designing a suitable controller we devise

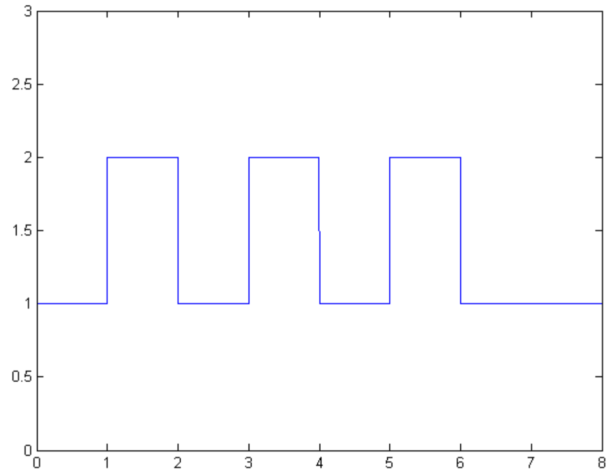

FIGURE 3. Randomly generated switching signal

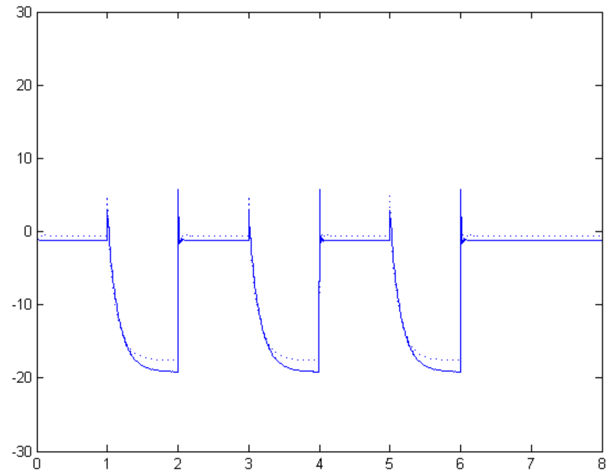

FIGURE 4. Step response of original switched linear system (solid line) and the reduced order model (dotted)

a fault-tolerant control. If the switched system is of high order, the design, computations and implementation of such controllers are complex. However, to maintain tractability model reduction of a switched system is required before designing a fault tolerant controller or plug-and-play controller. In the following, the model reduction of switched systems is applied in the reduction of a practical CD player model. This system is modeled as a bimodal switched system. One subsystem describes the system without fault, and the other describes the system with a sensor fault.

Example 5.3. *CD player example.*

The control of a CD player system is one of the well-known practical applications of model order reduction. The pattern of the CD player mechanism is shown in Figure 9. The control task is to achieve track following, which basically amounts to pointing the laser spot to the track of pits on a rotating CD. This mechanism involves a swing arm on which a lens is mounted by means of two horizontal leaf springs. The rotation of the

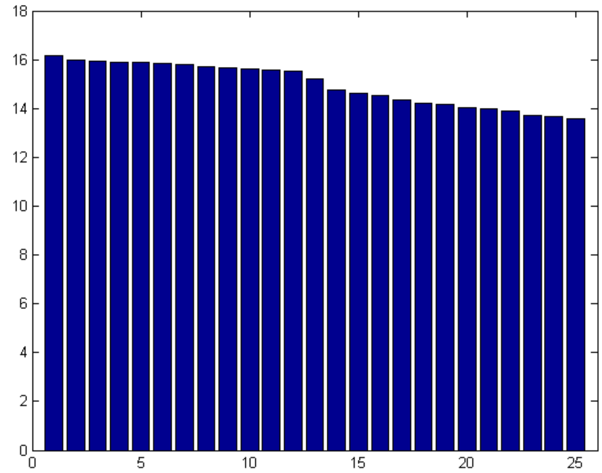

FIGURE 5. Generalized Hankel singular values (γ_i) of the first subsystem

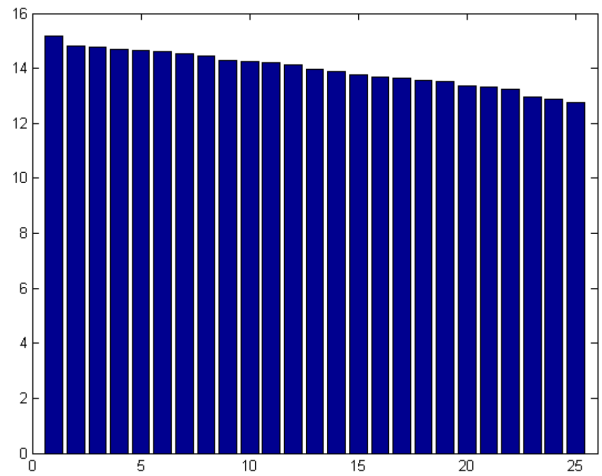

FIGURE 6. Generalized Hankel singular values (γ_i) of the second subsystem

arm in the horizontal plane enables the system to read the spiral-shaped disc-tracks, and the suspended lens is used to focus the laser spot on the disc. Due to the disc not being perfectly flat, and irregularities in the spiral shape of pits on the disc, a feedback system is needed. The higher the disc rate becomes, the stronger are the demands on the feedback controller. It is also required that the feedback system withstand some level of external shock. The challenge is to find a low-cost controller that can make the servo system faster and less sensitive to external shocks. In addition, it is required that all CD-players of a production set be equipped with the same type of controller.

From a practical point of view, a high-order model is needed to describe the vibrational behavior of an electro-mechanical system over a large frequency range to anticipate the interaction with a controller of a possibly high-bandwidth. In many examples, the behavior of electrodynamics is predicted by means of finite-element and sub-structuring methods [35].

First, the mechanism is divided into structural parts, which are modeled by finite-element discretization. The resulting model contains 60 vibration modes. In other words,

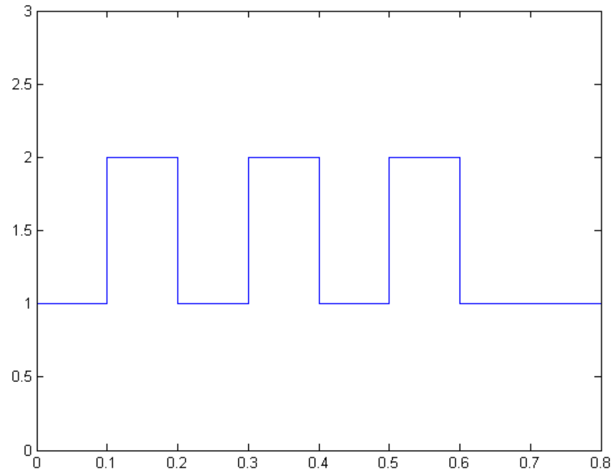

FIGURE 7. Switching signal

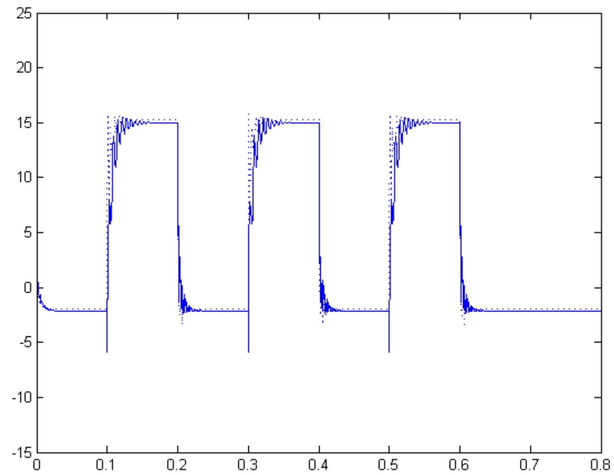

FIGURE 8. Step response of original switched linear system (solid line) and the reduced order model which is of order 17 (dotted)

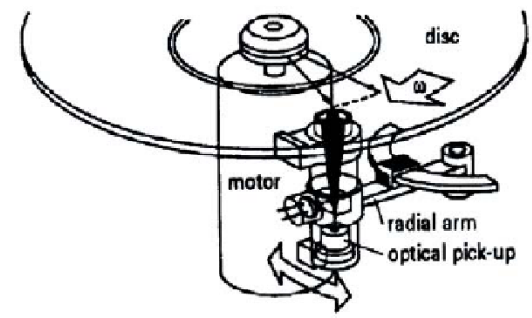

FIGURE 9. CD player [35]

the model is a continuous model of order 120 [36-39]. The model is converted to a discrete model using FOH (first-order hold) sampling with a sampling time of 0.1. Due to a sensor

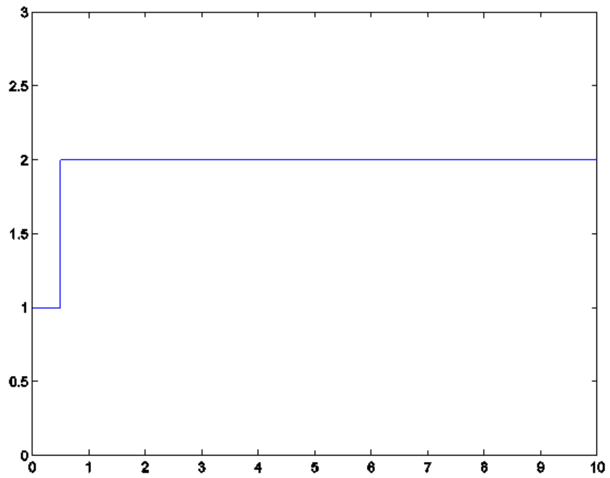

FIGURE 10. Switching signal

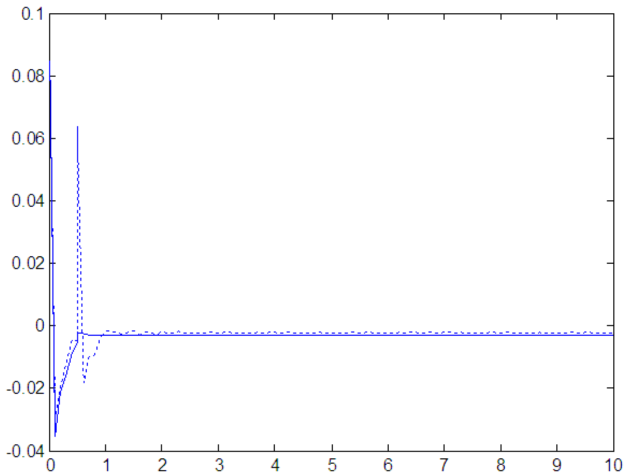

FIGURE 11. Step response of original switched linear system (solid line) and the reduced order model which is of order 30 (dotted)

fault, the sensor gain falls to 40% of the actual gain. The model describing the system with this fault scenario is extracted and converted into a discrete model using FOH with a sample time of 0.1. The overall system is a bimodal switched system of order 120, which is then reduced within $[\omega_1, \omega_2] = [0.001, 10]$.

The step responses of the original and reduced order switched systems associated with the switching signal shown in Figure 10 are shown in Figure 8. The reduced order switched system is of order 30. In other words, 75% of the states are reduced. Nonetheless, apart from around the switching instant that we have some bounded transient error the approximation is fairly accurate.

6. **Conclusions.**

A general framework for the model order reduction of switched linear dynamical systems has been presented. In this paper we have reformulated the frequency-domain balanced reduction method within the generalized gramian framework; generally, however, various gramian based reduction methods can be reformulated within the proposed framework easily and can be applied for the reduction of switched systems. It has been shown that the proposed framework preserves the stability of the original system. The method is much less conservative than previous methods based on common generalized gramians. The method can be further extended for the reduction of switching controllers and for closed-loop model reduction with embedded switching which will be addressed in future studies.

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