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A NEIGHBORING EXTREMAL SOLUTION FOR OPTIMAL SWITCHED IMPULSIVE CONTROL PROBLEMS WITH LARGE PERTURBATIONS

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Abstract. *This paper presents an approach to compute the neighboring extremal solution for an optimal switched impulsive control problem with a pre-specified sequence of modes and a large perturbation in the initial state. The decision variables – the subsystem switching times and the control parameters – are subject to inequality constraints. Since the active status of these inequality constraints may change under the large perturbation, we add fractions of the initial perturbation separately such that the active status of the inequality constraints is invariant during each step, and compute the neighboring extremal solution iteratively by solving a sequence of quadratic programming problems. First, we compute a correction direction for the control in the perturbed system through an extended backward sweep technique. Then, we compute the maximal step size in this direction and derive the solution iteratively by using a revised active set strategy. An example problem involving a shrimp harvesting operation demonstrates that our solution approach is faster than the sequential quadratic programming approach.*
Keywords: Optimal control, Neighboring extremal, Impulsive control, Switched system, Sequential quadratic programming, Active set strategy

1. **Introduction.** Real-world optimal control problems are often nonlinear and far too complex to be solved analytically. Thus, numerical methods are indispensable for solving such problems. However, existing numerical solution methods (see, for example, [1, 2]) only compute open-loop optimal controls, which are sensitive to disturbances and modelling uncertainties. The neighboring extremal (NE) method was proposed in the 1960s [3] to construct a NE control in a feedback form for an unconstrained optimal control problem involving nonlinear dynamics. Assuming that a nominal optimal solution has been calculated offline, this method constructs an approximate optimal control online when the initial state and terminal condition are slightly perturbed. In this way, it gains robustness and computational efficiency. This NE method was extended in [4-8] to optimal control problems involving nonlinear continuous dynamics subject to continuous inequality constraints. This method was further extended in [9, 10] to singular control problems, and in [11] to constrained discrete-time optimal control problems. However, most of these NE methods [4-11] assume that the perturbations are small enough such that the solution's structure (i.e., the active status of inequality constraints) is unchanged after the perturbations, which limits the practical applications of these methods. In [12, 13],

this assumption was dropped, and the NE method was integrated with the model predictive control (MPC) to construct open-loop optimal control during each sampling interval. Thus, the online computational time was reduced a lot.

In this paper, we consider an optimal control problem for a class of switched impulsive systems subject to a large perturbation of the initial state. Switched impulsive systems are operated by switching between different subsystems or modes, and may exhibit instantaneous state jumps during the mode switching. Switched impulsive systems arise in areas such as circuits [14, 15]. Impulsive systems [16-20] are a special class of them, which have one mode. The optimal control problem for switched impulsive systems is to choose the sequence of the modes, the times to switch between the modes, and the parameters controlling the state jumps to minimize a given cost function subject to constraints. This paper assumes that the sequence of the modes is pre-specified. Furthermore, the structure of the solution – that is, the active status of the inequality constraints – may change with the perturbation. Hence, differentiating the active inequality constraints cannot be used directly, which hinders us from using the techniques in [4, 6] to derive the NE solution. One method for avoiding this difficulty is to solve a quadratic programming (QP) problem with linear inequality constraints directly [21-23], instead of a QP problem with linear equality constraints. However, if the perturbation is large, the approximation error in this formulation is also large. Hence, solving the original optimal control problem is needed when the accumulated approximation error is larger than a threshold. Another idea is using homotopy [12, 13]. Specifically, this idea is to perturb the system by fractions of the initial perturbation step-by-step such that, with respect to the current nominal solution and the fraction of the perturbation, the active status of the inequality constraints is invariant during each step [12, 13]. Then, the active set strategy [24] can be used, and the NE solution can be computed iteratively by solving a sequence of QP problems. In this paper, we follow the second idea. Note that the discrete system description in [12, 13] is not appropriate for the switched impulsive systems considered here. Further, interior point constraints must be considered for the switched impulsive control problem, the solution of which is much more involved than those in [12, 13]. To our knowledge, there are no NE solutions available for the switched impulsive control problem without requiring that the active status of the inequality constraints is unchanged after perturbations. Our solution procedure consists of two steps. In the first step, we compute a correction direction of the control for the perturbed system by solving an accessory minimum problem [3]. From the first-order necessary conditions for optimality (NCO), a multiple-point boundary-value problem (MPBVP) can be derived, which is more involved than the two-point boundary-value problem (TPBVP) considered in [3]. Then, if the related functions are differentiable and a symmetric matrix is invertible, this MPBVP can be solved using an extended backward sweep technique. In the second step, we compute the maximal step size in this correction direction to ensure that the active status of the inequality constraints is invariant, and derive the solution iteratively with a fraction of the initial perturbation added at each time by using the active set strategy. In order to accelerate the computation, we revise the standard active set strategy [24] in the iterative computation of the second step. Specifically, we allow an active inequality constraint to be dropped from the active set before the computation converges, and use simple inferential logics to avoid the active status of some inequality constraints from changing back and forth in successive steps. In summary, our algorithm starts from the known extremal solution associated with a nominal initial state and tries to approximate the optimal solution for another initial state by perturbing the initial state step-by-step. Since only a small fraction of the perturbation is introduced in each step, the computational procedure of our algorithm is more stable than that of the sequential quadratic programming (SQP) algorithm. Therefore, our algorithm converges faster and is not liable to be trapped in a suboptimal solution prematurely. Our numerical simulations also demonstrate that our algorithm performs better than the SQP algorithm, especially when the perturbation is large. Similar to [12, 13], our NE solution can find applications in constructing the MPC for the switched impulsive systems, once the time to compute the open-loop optimal control is longer than the switching interval.

The rest of this paper is organized as follows. In Section 2, after introducing the optimal control problem for the switched impulsive system, we present its NE problem with respect to the perturbation of the initial state. Then, we present our main result on the solution of this NE problem in Section 3. After that, we verify our NE solution by solving an example problem on the optimal shrimp harvesting in Section 4. Finally, in Section 5 we conclude the paper.

2. **Problem Formulation.** Consider the following switched impulsive system with $N+1$ subsystems:

$$\dot{x}(t) = f^i(x(t), t), \quad t \in (t_{i-1}, t_i), \quad i = 1, \ldots, N+1, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system's state at time t; $f^i : \mathbb{R}^n \times (t_{i-1}, t_i) \to \mathbb{R}^n$, $i = 1, \ldots, N+1$, are given functions; $t_i > 0$, $i = 1, \ldots, N$, are the subsystem switching times, and $t_0 \triangleq 0$ and $t_{N+1} \triangleq t_f > 0$. The subsystems are switched in a pre-specified sequence with index i from 1 to $N+1$, and these switchings are accompanied by instantaneous state jumps which are determined by

$$x(t_i^+) = \begin{cases} x^0, & i = 0, & \text{(2a)} \\ g^i\left(x(t_i^-), s_i, t_i\right), & i = 1, \ldots, N, & \text{(2b)} \end{cases}$$

where $x^0 \in \mathbb{R}^n$ is a given initial state, and $g^i : \mathbb{R}^{n+m} \times \mathbb{R}^+ \to \mathbb{R}^n$, $i = 1, \ldots, N$, are given functions that determine the state jumps at $t = t_i$ with $s_i \in \mathbb{R}^m$, $i = 1, \ldots, N$, being the parameters controlling the jumps. In (2), $x(t_i^+)$ and $x(t_i^-)$, $i = 1, \ldots, N$, denote, respectively, the limits of $x(t)$ from the right and left at $t = t_i$. Let

$$\psi^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right) \triangleq x(t_i^+) - g^i\left(x(t_i^-), s_i, t_i\right) = 0, \quad i = 1, \ldots, N. \tag{3}$$

In system (1)-(2), the terminal time t_f is fixed. The switching times t_i and parameters s_i, $i = 1, \ldots, N$, are decision variables, and they must satisfy the following constraints:

$$\begin{cases} a_i^j \le s_i^j \le b_i^j, & j = 1, \ldots, m, \ i = 1, \ldots, N, & \text{(4a)} \\ t_i - t_{i-1} \ge c, & i = 1, \ldots, N+1, & \text{(4b)} \end{cases}$$

where s_i^j is the jth element of the vector s_i, $a_i^j < b_i^j$ are given lower and upper bounds, and $c > 0$ is given minimum duration of a subsystem. For convenience, we write the linear constraints (4a) and (4b), respectively, as

$$\hat{\psi}^i(s_i) \triangleq \begin{bmatrix} a_i^1 - s_i^1 \\ s_i^1 - b_i^1 \\ \vdots \\ a_i^m - s_i^m \\ s_i^m - b_i^m \end{bmatrix} \le 0, \quad i = 1, \ldots, N, \tag{5a}$$

$$\eta\left(t_1, \ldots, t_N\right) \triangleq \begin{bmatrix} t_0 - t_1 + c \\ t_1 - t_2 + c \\ \vdots \\ t_{N-1} - t_N + c \\ t_N - t_f + c \end{bmatrix} \le 0, \tag{5b}$$

where the inequalities are componentwise.

Let

$$\sigma \triangleq \left[s_1^\top, \ldots, s_N^\top\right]^\top \quad \text{and} \quad \tau \triangleq [t_1, \ldots, t_N]^\top.$$

Let Δ be the set of all such σ satisfying (5a), and T be the set of all τ satisfying (5b).

For this switched impulsive system, a nominal optimal control problem can be formulated as follows.

Problem 2.1. *For the given system (1)-(2), find a control pair $(\sigma, \tau) \in \Delta \times T$ such that the cost function*

$$J(\sigma, \tau) \triangleq \Phi\left(x(t_f), t_f\right) + \sum_{i=1}^{N} \phi^i\left(x(t_i^+), x(t_i^-), t_i\right) \tag{6}$$

is minimized over $\Delta \times T$, where $\Phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, and $\phi^i : \mathbb{R}^{2n} \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \ldots, N$, are given functions.

Remark 2.1. *We can easily incorporate an integral term into (6) by introducing a dummy state variable. For example, consider the integral term, $\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} L^i\left(x(t), t\right) \mathrm{d}t$. It is clear that this term can be replaced by $y(t_f)$, where $y(t)$ satisfies the dynamics*

$$\dot{y}(t) = L^i\left(x(t), t\right), \quad t \in (t_{i-1}, t_i), \quad i = 1, \ldots, N+1,$$

and

$$y(t_i^+) = \begin{cases} 0, & i = 0, \\ y(t_i^-), & i = 1, \ldots, N. \end{cases}$$

For Problem 2.1, we need the following assumption.

Assumption 2.1. *f^i, $i = 1, \ldots, N+1$, Φ, and g^i, ϕ^i, $i = 1, \ldots, N$ are twice continuously differentiable with respect to each of their arguments. Furthermore, g^i, $i = 1, \ldots, N$, are bounded.*

Let $\nu_i \in \mathbb{R}^n$, $\hat{\nu}_i \in \mathbb{R}^{2m}$, $i = 1, \ldots, N$, and $\pi \in \mathbb{R}^{N+1}$ be, respectively, the Lagrange multipliers associated with the constraints (3), (5a) and (5b). Let $\lambda(t) \in \mathbb{R}^n$ be the costate. We define

$$\begin{aligned} \bar{\Phi}^i(x(t_i^+), x(t_i^-), s_i, \nu_i, \hat{\nu}_i, t_i) \triangleq{}& \phi^i\left(x(t_i^+), x(t_i^-), t_i\right) + \nu_i^\top \psi^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right) \\ &+ \hat{\nu}_i^\top \hat{\psi}^i(s_i), \qquad i = 1, \ldots, N, \\ H^i\left(x(t), \lambda(t), t\right) \triangleq{}& \lambda^\top(t) f^i\left(x(t), t\right), \quad i = 1, \ldots, N+1, \end{aligned}$$

and the augmented cost function

$$\begin{aligned} \bar{J}(\sigma, \tau) \triangleq{}& \Phi\left(x(t_f), t_f\right) + \sum_{i=1}^{N} \bar{\Phi}^i\left(x(t_i^+), x(t_i^-), s_i, \nu_i, \hat{\nu}_i, t_i\right) + \pi^\top \eta(\tau) \\ &+ \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left[H^i\left(x(t), \lambda(t), t\right) - \lambda^\top(t)\dot{x}(t)\right] \mathrm{d}t. \end{aligned} \tag{8}$$

Let $(\sigma^*, \tau^*) \in \Delta \times T$, where $\sigma^* \triangleq [s_1^{*\top}, \ldots, s_N^{*\top}]^\top$ and $\tau^* \triangleq [t_1^*, \ldots, t_N^*]^\top$, be an extremal solution to Problem 2.1 with respect to the initial state x^0. Furthermore, let $x^*(t)$ and $\lambda^*(t)$ be the corresponding state and costate, and let ν_i^*, $\hat{\nu}_i^*$, $i = 1, \ldots, N$, and π^* be, respectively, the Lagrange multipliers associated with the constraints (3), (5a) and (5b). We call this solution the nominal solution of Problem 2.1.

Let $\delta x(0) \in \mathbb{R}^n$ denote an arbitrary perturbation of the initial state. Specifically, the initial state after the perturbation becomes

$$x(0) = x^0 + \delta x(0). \tag{9}$$

Let $(\sigma, \tau) \in \Delta \times T$, $x(t)$, $\lambda(t)$, ν_i, $\hat{\nu}_i$, $i = 1, \ldots, N$ and π be a locally extremal solution to Problem 2.1 with the initial state perturbed according to (9). In what follows, we call this solution the perturbed solution. Assume that the nominal extremal control pair (σ^*, τ^*) is obtained by some computational software. The problem to be solved in this paper, i.e., the NE problem for the perturbed system, is defined as follows.

Problem 2.2. *Given the nominal extremal control pair $(\sigma^*, \tau^*) \in \Delta \times T$ of Problem 2.1 with respect to the initial state x^0, together with the corresponding state, costate and Lagrange multipliers, find the first-order approximation to the locally extremal solution $(\sigma, \tau) \in \Delta \times T$ of Problem 2.1 when the initial state is perturbed according to (9).*

For the rest of the paper, the objective is to present a method to solve this NE problem with respect to the initial perturbation, which may be considerably large.

3. **NE Solution for the Optimal Switched Impulsive Control Problem.** Before solving Problem 2.2, define

$$C(\sigma, \tau) \triangleq \begin{bmatrix} \hat{\psi}^{1\top}(s_1) & \cdots & \hat{\psi}^{N\top}(s_N) & \eta^\top(\tau) \end{bmatrix}^\top \leq 0$$

and its associated Lagrange multiplier vector,

$$\mu \triangleq \begin{bmatrix} \hat{\nu}_1^\top & \cdots & \hat{\nu}_N^\top & \pi^\top \end{bmatrix}^\top.$$

Then, $C(\sigma, \tau) \leq 0$ contains all the inequality constraints involved in Problem 2.1. Let $\mathcal{I} \triangleq \{1, 2, \ldots, (2m+1)N + 1\}$ be the index set of the inequality constraints $C(\sigma, \tau) \leq 0$, and let C^i and μ^i, $i \in \mathcal{I}$, be, respectively, the ith element of the vector C and the ith element of the vector μ. For these inequality constraints, we define the following active set:

$$\mathcal{A}(\sigma, \tau) \triangleq \{i \in \mathcal{I} \mid C^i(\sigma, \tau) = 0\}. \tag{10}$$

If the initial state perturbation is not small, the set $\mathcal{A}(\sigma, \tau)$ may be different from $\mathcal{A}(\sigma^*, \tau^*)$.

Since the active status of the inequality constraints may change after the initial perturbation, we cannot assume that all the differentials of the active inequality constraints are equal to zero, and the commonly used techniques in [4, 6] will fail. As a remedy, we first assume that the initial state perturbation is small such that the active set $\mathcal{A}$ is unchanged after the perturbation. This will enable us to compute a NE solution based on sensitivity analysis [3, 25], which serves as a correction direction for the control. Then, we will drop this assumption and design an algorithm to compute the NE solution when the active set changes by using this correction direction.

3.1. **Sensitivity analysis for small perturbations.** At the first step to solve Problem 2.2, we need the following assumption.

Assumption 3.1. *The perturbation $\delta x(0)$ is small enough such that the active set $\mathcal{A}(\sigma, \tau)$ is the same as $\mathcal{A}(\sigma^*, \tau^*)$.*

The following procedure is based on the idea of computing the NE solution by solving an accessory minimum problem [3], i.e., expanding the augmented cost function $\bar{J}(\sigma, \tau)$ in (8) to the second-order and the constraints to the first-order, and then solving the resulting QP problem. Before proceeding, we note that, for a continuously differentiable function $g(x, y) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$, the partial derivative $g_x \triangleq \partial g / \partial x$ is taken as a row vector, and the second-order partial derivative of g_{xy} is defined as $g_{xy} \triangleq (\partial^2 g)/(\partial x \partial y) = \partial(\partial g/\partial y)^\top/\partial x$. We now formulate the accessory minimum problem.

Expanding the augmented cost $\bar{J}(\sigma,\tau)$ along the nominal solution (σ^*,τ^*), $x^*(t)$, $\lambda^*(t)$, ν_i^*, $\hat{\nu}_i^*$, $i=1,\ldots,N$, and π^*, we have the following first-order term,

$$\begin{aligned}\delta\bar{J} = & \left[\lambda^{*\top}(t_f) - \Phi_x\right]\delta x(t_f) \\ & + \sum_{i=1}^{N}\left[\left(\lambda^{*\top}(t_i^{*+}) + \bar{\Phi}^i_{x_i^+}\right)\mathrm{d}x(t_i^{*+}) + \left(-\lambda^{*\top}(t_i^{*-}) + \bar{\Phi}^i_{x_i^-}\right)\mathrm{d}x(t_i^{*-}) + \bar{\Phi}^i_{s_i}\mathrm{d}s_i\right] \\ & + \sum_{i=1}^{N}\left[\bar{\Phi}^i_{t_i} + H^i\left(x^*(t_i^{*-}),\lambda^*(t_i^{*-}),t_i^*\right) - H^{i+1}\left(x^*(t_i^{*+}),\lambda^*(t_i^{*+}),t_i^*\right) + \pi^{*\top}\eta_{t_i}\right]\mathrm{d}t_i \\ & + \sum_{i=1}^{N+1}\int_{t_{i-1}^*}^{t_i^*}\left[\left(\dot{\lambda}^{*\top}(t) + H^i_x\right)\delta x(t)\right]\mathrm{d}t, \end{aligned} \tag{11}$$

where the subscripts x_i^+ and x_i^- denote, respectively, partial derivatives with respect to $x(t_i^+)$ and $x(t_i^-)$; $\delta x(t)$ denotes the first-order variation of $x(t)$ at time t; and all variables with prefix 'd' denote the corresponding differentials. In this section, the nominal control (σ^*,τ^*) is not required to be locally extremal. Then, $\delta\bar{J}=0$ if (σ^*,τ^*) is locally extremal, and $\delta\bar{J}\neq 0$ otherwise. In the latter case, let the corresponding costate $\lambda^*(t)$ and the Lagrange multipliers ν_i^* and $\hat{\nu}_i^*$, $i=1,\ldots,N$, satisfy the following equations:

$$\begin{cases} \dot{\lambda}^*(t) = -H_x^{i\top}\left(x^*(t),\lambda^*(t),t\right), \quad t\in(t_{i-1}^*,t_i^*), \quad i=1,\ldots,N+1, & \text{(12a)} \\ \lambda^*(t_f) = \Phi_x^\top(x^*(t_f),t_f), & \text{(12b)} \\ \lambda^*(t_i^{*+}) = -\bar{\Phi}^{i\top}_{x_i^+}\left(x^*(t_i^{*+}),x^*(t_i^{*-}),s_i^*,\nu_i^*,\hat{\nu}_i^*,t_i^*\right), \quad i=1,\ldots,N, & \text{(12c)} \\ \lambda^*(t_i^{*-}) = \bar{\Phi}^{i\top}_{x_i^-}\left(x^*(t_i^{*+}),x^*(t_i^{*-}),s_i^*,\nu_i^*,\hat{\nu}_i^*,t_i^*\right), \quad i=1,\ldots,N, & \text{(12d)} \end{cases}$$

where $x^*(t)$ is the state of the system (1)-(2) corresponding to the control (σ^*,τ^*). With (12), we have

$$\begin{aligned}\delta\bar{J} = & \sum_{i=1}^{N}\bar{\Phi}^i_{s_i}\mathrm{d}s_i + \sum_{i=1}^{N}\left[\bar{\Phi}^i_{t_i} + H^i\left(x^*(t_i^{*-}),\lambda^*(t_i^{*-}),t_i^*\right)\right. \\ & \left. - H^{i+1}\left(x^*(t_i^{*+}),\lambda^*(t_i^{*+}),t_i^*\right) + \pi^{*\top}\eta_{t_i}\right]\mathrm{d}t_i. \end{aligned} \tag{13}$$

Furthermore, by expanding the augmented cost function $\bar{J}$ to the second-order, we obtain the second-order variation

$$\begin{aligned}\delta^2\bar{J} = & \frac{1}{2}\delta x^\top(t_f)\Phi_{xx}\delta x(t_f) + \sum_{i=1}^{N}Q^i(\delta x(t_i^{*+}),\delta x(t_i^{*-}),\mathrm{d}s_i,\mathrm{d}t_i) \\ & + \sum_{i=1}^{N+1}\int_{t_{i-1}^*}^{t_i^*}\frac{1}{2}\delta x^\top H^i_{xx}\delta x\mathrm{d}t. \end{aligned} \tag{14}$$

In (14),

$$\begin{aligned}& Q^i(\delta x(t_i^{*+}),\delta x(t_i^{*-}),\mathrm{d}s_i,\mathrm{d}t_i) \\ & \triangleq \frac{1}{2}\begin{bmatrix}\mathrm{d}x(t_i^{*+}) \\ \mathrm{d}x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}t_i\end{bmatrix}^\top \begin{bmatrix}\bar{\Phi}^i_{x_i^+x_i^+} & \bar{\Phi}^i_{x_i^-x_i^+} & \bar{\Phi}^i_{s_ix_i^+} & \bar{\Phi}^i_{t_ix_i^+} \\ * & \bar{\Phi}^i_{x_i^-x_i^-} & \bar{\Phi}^i_{s_ix_i^-} & \bar{\Phi}^i_{t_ix_i^-} \\ * & * & \bar{\Phi}^i_{s_is_i} & \bar{\Phi}^i_{t_is_i} \\ * & * & * & \bar{\Phi}^i_{t_it_i}\end{bmatrix}\begin{bmatrix}\mathrm{d}x(t_i^{*+}) \\ \mathrm{d}x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}t_i\end{bmatrix}\end{aligned}$$

$$+ \delta x^\top(t_i^{*-})(H_x^i)^\top \mathrm{d}t_i - \delta x^\top(t_i^{*+})(H_x^{i+1})^\top \mathrm{d}t_i + \frac{1}{2}\dot{x}^\top(t_i^{*-})(H_x^i)^\top \mathrm{d}t_i^2$$
$$- \frac{1}{2}\dot{x}^\top(t_i^{*+})(H_x^{i+1})^\top \mathrm{d}t_i^2 + \frac{1}{2}\left(H_t^i - H_t^{i+1}\right)\mathrm{d}t_i^2, \tag{15}$$

where '$*$' denotes the symmetric term in a symmetric matrix. Here, note that

$$\begin{aligned}
\mathrm{d}x(t_i^{*\pm}) &= \delta x(t_i^{*\pm}) + \dot{x}(t_i^{*\pm})\mathrm{d}t_i \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{x_i^\pm}\right)^\top &= \bar{\Phi}^i_{x_i^+ x_i^\pm}\dot{x}^*(t_i^{*+}) + \bar{\Phi}^i_{x_i^- x_i^\pm}\dot{x}^*(t_i^{*-}) + \bar{\Phi}^i_{t_i x_i^\pm} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{s_i}\right)^\top &= \bar{\Phi}^i_{x_i^+ s_i}\dot{x}^*(t_i^{*+}) + \bar{\Phi}^i_{x_i^- s_i}\dot{x}^*(t_i^{*-}) + \bar{\Phi}^i_{t_i s_i} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{t_i}\right) &= \bar{\Phi}^i_{x_i^+ t_i}\dot{x}^*(t_i^{*+}) + \bar{\Phi}^i_{x_i^- t_i}\dot{x}^*(t_i^{*-}) + \bar{\Phi}^i_{t_i t_i}.
\end{aligned}$$

Then, by using Equations (12a), (12c) and (12d), Q^i can be rearranged as

$$\begin{aligned}
&Q^i(\delta x(t_i^{*+}), \delta x(t_i^{*-}), \mathrm{d}s_i, \mathrm{d}t_i) \\
&= \frac{1}{2}\begin{bmatrix} \delta x(t_i^{*+}) \\ \delta x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}t_i \end{bmatrix}^\top \begin{bmatrix} \bar{\Phi}^i_{x_i^+ x_i^+} & \bar{\Phi}^i_{x_i^- x_i^+} & \bar{\Phi}^i_{s_i x_i^+} & 0 \\ * & \bar{\Phi}^i_{x_i^- x_i^-} & \bar{\Phi}^i_{s_i x_i^-} & 0 \\ * & * & \bar{\Phi}^i_{s_i s_i} & \theta_i \\ * & * & * & \kappa_i \end{bmatrix} \begin{bmatrix} \delta x(t_i^{*+}) \\ \delta x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}t_i \end{bmatrix}
\end{aligned} \tag{16}$$

with

$$\begin{aligned}
\theta_i &\triangleq \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{s_i}\right)^\top, \\
\kappa_i &\triangleq \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{t_i}\right) + H^i_{t_i}\left(x^*(t_i^{*-}), \lambda^*(t_i^{*-}), t_i^*\right) - H^{i+1}_{t_i}\left(x^*(t_i^{*+}), \lambda^*(t_i^{*+}), t_i^*\right).
\end{aligned}$$

Let $\tilde{\psi}^i(s_i^*) = 0$, $i = 1, \ldots, N$, and $\tilde{\eta}(\tau^*) = 0$ denote, respectively, the active parts of the constraints (5a) and (5b) for the nominal control (σ^*, τ^*). Let $\tilde{\nu}_i^*$ and $\tilde{\pi}^*$ be, respectively, the Lagrange multipliers associated with $\tilde{\psi}^i(s_i^*) = 0$ and $\tilde{\eta}(\tau^*) = 0$. From Assumption 3.1, equations $\tilde{\psi}^i(s_i) = 0$, $i = 1, \ldots, N$, and $\tilde{\eta}(\tau) = 0$ hold at the control (σ, τ) for the perturbed problem, and none of the other inequality constraints are active. Considering that both the nominal solution and the perturbed solution satisfy the dynamics (1) and the constraints (3), and

$$\dot{\psi}^i = \psi^i_{x_i^+}\dot{x}(t_i^{*+}) + \psi^i_{x_i^-}\dot{x}(t_i^{*-}) + \psi^i_{t_i}\mathrm{d}t_i,$$

we can derive the following first-order approximations to the dynamics (1), and the constraints (3), (5a) and (5b) with respect to the nominal solution:

$$\delta\dot{x}(t) = f_x^i\delta x(t), \qquad t \in (t_{i-1}^*, t_i^*), \qquad i = 1, \ldots, N+1 \tag{17a}$$

$$0 = \psi^i_{x_i^+}\delta x(t_i^{*+}) + \psi^i_{x_i^-}\delta x(t_i^{*-}) + \psi^i_{s_i}\mathrm{d}s_i + \dot{\psi}^i\mathrm{d}t_i, \qquad i = 1, \ldots, N, \tag{17b}$$

$$0 = \tilde{\psi}^i_{s_i}\mathrm{d}s_i, \qquad i = 1, \ldots, N, \tag{17c}$$

$$0 = \sum_{i=1}^N \tilde{\eta}_{t_i}\mathrm{d}t_i. \tag{17d}$$

Let $\mathrm{d}\sigma \triangleq \left[\mathrm{d}s_1^\top, \ldots, \mathrm{d}s_N^\top\right]^\top$ and $\mathrm{d}\tau \triangleq [\mathrm{d}t_1, \ldots, \mathrm{d}t_N]^\top$. Define

$$\Delta\bar{J}(\mathrm{d}\sigma, \mathrm{d}\tau) \triangleq \delta\bar{J} + \delta^2\bar{J}. \tag{18}$$

The quadratic cost function (18) and linear constraints (17a)-(17d) form the following accessory minimum problem.

Problem 3.1. *Given a nominal control pair $(\sigma^*,\tau^*)\in\Delta\times T$ of Problem 2.1 with respect to the initial state x^0, together with the corresponding state, costate and Lagrange multipliers, where (σ^*,τ^*) may not be locally extremal and the costate satisfies (12), find the optimal control pair $(\mathrm{d}\sigma,\mathrm{d}\tau)$ such that the cost function (18) is minimized subject to the constraints (17a)-(17d) with $\delta x(0)$ being the initial state of (17a).*

Remark 3.1. *We incorporate some first-order terms (13) into the cost function (18) of the accessory minimum problem to force the corrected control $(\sigma,\tau)=(\sigma^*+\mathrm{d}\sigma,\tau^*+\mathrm{d}\tau)$ to be locally extremal. In this way, the accumulated approximation error in the iterative computation in Section 3.2 can be reduced.*

For this problem, we have the following NCO in addition to (17a)-(17d):

$$
\begin{aligned}
\delta\dot{\lambda}(t) &= -H^i_{xx}\delta x(t) - f_x^{i\top}\delta\lambda(t), \quad t\in(t^*_{i-1},t^*_i), \quad i=1,\dots,N+1, && \text{(19a)}\\
\delta\lambda(t_f) &= \Phi_{xx}\delta x(t_f), && \text{(19b)}\\
\delta\lambda(t_i^{*+}) &= -\bar{\Phi}^i_{x_i^+x_i^+}\delta x(t_i^{*+}) - \bar{\Phi}^i_{x_i^-x_i^+}\delta x(t_i^{*-}) - \bar{\Phi}^i_{s_ix_i^+}\mathrm{d}s_i - \psi^{i\top}_{x_i^+}\mathrm{d}\nu_i,\\
&\qquad\qquad i=1,\dots,N, && \text{(19c)}\\
\delta\lambda(t_i^{*-}) &= \bar{\Phi}^i_{x_i^+x_i^-}\delta x(t_i^{*+}) + \bar{\Phi}^i_{x_i^-x_i^-}\delta x(t_i^{*-}) + \bar{\Phi}^i_{s_ix_i^-}\mathrm{d}s_i + \psi^{i\top}_{x_i^-}\mathrm{d}\nu_i,\\
&\qquad\qquad i=1,\dots,N, && \text{(19d)}\\
0 &= \bar{\Phi}^i_{x_i^+s_i}\delta x(t_i^{*+}) + \bar{\Phi}^i_{x_i^-s_i}\delta x(t_i^{*-}) + \bar{\Phi}^i_{s_is_i}\mathrm{d}s_i + \psi^{i\top}_{s_i}\mathrm{d}\nu_i\\
&\quad + \tilde{\psi}^{i\top}_{s_i}\mathrm{d}\tilde{\nu}_i + \theta_i\mathrm{d}t_i + \bar{\Phi}^{i\top}_{s_i}, \qquad i=1,\dots,N, && \text{(19e)}\\
0 &= \theta_i^\top\mathrm{d}s_i + \dot{\psi}^{i\top}\mathrm{d}\nu_i + \kappa_i\mathrm{d}t_i + \tilde{\eta}_{t_i}^\top\mathrm{d}\tilde{\pi} + q_i, \quad i=1,\dots,N, && \text{(19f)}
\end{aligned}
$$

where $\delta\lambda(t)$ is the costate associated with $\delta x(t)$, and $\mathrm{d}\nu_i$, $\mathrm{d}\tilde{\nu}_i$, $i=1,\dots,N$, and $\mathrm{d}\tilde{\pi}$ are, respectively, the Lagrange multipliers associated with the constraints (17b)-(17d), and

$$
q_i \triangleq \bar{\Phi}^i_{t_i} + H^i\left(x^*(t_i^{*-}),\lambda^*(t_i^{*-}),t_i^*\right) - H^{i+1}\left(x^*(t_i^{*+}),\lambda^*(t_i^{*+}),t_i^*\right) + \pi^{*\top}\eta_{t_i}
$$

for $i=1,\dots,N$.

Rearranging (17a) and (19a), we have, for $t\in(t^*_{i-1},t^*_i)$, $i=1,\dots,N+1$,

$$
\begin{bmatrix}\delta\dot{x}(t)\\ \delta\dot{\lambda}(t)\end{bmatrix} = \begin{bmatrix} f_x^i & 0\\ -H^i_{xx} & -f_x^{i\top}\end{bmatrix}\begin{bmatrix}\delta x(t)\\ \delta\lambda(t)\end{bmatrix} = \begin{bmatrix} A^i & 0\\ C^i & -A^{i\top}\end{bmatrix}\begin{bmatrix}\delta x(t)\\ \delta\lambda(t)\end{bmatrix} \tag{20}
$$

where A^i and C^i are evaluated along the nominal solution.

Now, we will solve Problem 3.1. Equations (17a)-(17d) and (19a)-(19f) form a linear MPBVP, which is more complicated than the TPBVP considered in [3]. In this section, we will extend the backward sweep technique in [3] to solve this MPBVP. First, define along the nominal solution

$$
\begin{aligned}
\Theta_i &\triangleq \begin{bmatrix} \mathrm{d}s_N^\top & \mathrm{d}\tilde{\nu}_N^\top & \mathrm{d}t_N & \cdots & \mathrm{d}s_i^\top & \mathrm{d}\tilde{\nu}_i^\top & \mathrm{d}t_i \end{bmatrix}^\top,\\
\Psi_i &\triangleq \begin{bmatrix} -\bar{\Phi}^N_{s_N} & 0 & -q_N - \tilde{\eta}_{t_N}^\top\mathrm{d}\tilde{\pi} & \cdots & -\bar{\Phi}^i_{s_i} & 0 & -q_i - \tilde{\eta}_{t_i}^\top\mathrm{d}\tilde{\pi} \end{bmatrix}^\top,\\
\Lambda_i &\triangleq \begin{bmatrix} \bar{\Phi}^N_{s_N} & 0 & q_N & \cdots & \bar{\Phi}^i_{s_i} & 0 & q_i \end{bmatrix}^\top
\end{aligned}
$$

for $i = N, \ldots, 1$. Next, consider the following group of auxiliary systems governed by switched impulsive differential equations:

$$\begin{cases} \dot{S}_{1,1}(t) = -S_{1,1}(t)A^i - A^{i\top}S_{1,1}(t) + C^i, & t \in (t_{i-1}^*, t_i^*), \quad i = N+1, \ldots, 1, \\ S_{1,1}(t_i^{*-}) = \Phi_{xx}, & i = N+1, \\ S_{1,1}(t_i^{*-}) = \psi_{x_i^-}^{i\top} S_{1,1}(t_i^{*+})\psi_{x_i^-}^i + \bar{\Phi}_{x_i^- x_i^-}^i, & i = N, \ldots, 1, \end{cases} \tag{21a}$$

$$\begin{cases} \dot{S}_{1,3j-1}(t) = -A^{i\top}S_{1,3j-1}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j-1}(t_i^{*-}) = \psi_{x_i^-}^{i\top} S_{1,1}(t_i^{*+})\psi_{s_i}^i + \bar{\Phi}_{s_i x_i^-}^i, & i = N-j+1, \\ S_{1,3j-1}(t_i^{*-}) = -\psi_{x_i^-}^{i\top} S_{1,3j-1}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{21b}$$

$$\begin{cases} \dot{S}_{1,3j}(t) = -A^{i\top}S_{1,3j}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j}(t_i^{*-}) = 0, & i = N-j+1, \\ S_{1,3j}(t_i^{*-}) = -\psi_{x_i^-}^{i\top} S_{1,3j}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{21c}$$

$$\begin{cases} \dot{S}_{1,3j+1}(t) = -A^{i\top}S_{1,3j+1}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j+1}(t_i^{*-}) = \psi_{x_i^-}^{i\top} S_{1,1}(t_i^{*+})\dot{\psi}^i, & i = N-j+1, \\ S_{1,3j+1}(t_i^{*-}) = -\psi_{x_i^-}^{i\top} S_{1,3j+1}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{21d}$$

where $j = 1, \ldots, N$, and all the coefficient matrices are evaluated along the nominal solution. Let Υ be a symmetric matrix defined by

$$\Upsilon \triangleq \begin{bmatrix} \Upsilon_{1,1} & \Upsilon_{1,2} \\ \Upsilon_{1,2}^\top & 0 \end{bmatrix}, \tag{22}$$

$$\Upsilon_{1,1} \triangleq \begin{bmatrix} S_{2,2}(0) & \ldots & S_{2,3N+1}(0) \\ * & \ddots & \vdots \\ * & * & S_{3N+1,3N+1}(0) \end{bmatrix},$$

$$\Upsilon_{1,2}^\top \triangleq \begin{bmatrix} 0 & 0 & \tilde{\eta}_{t_N} & \ldots & 0 & 0 & \tilde{\eta}_{t_1} \end{bmatrix},$$

where $S_{\alpha,\beta}(0)$, $\alpha = 2, \ldots, 3N+1$, $\beta = \alpha, \ldots, 3N+1$, are defined along the nominal solution by

$$\begin{cases} S_{3j-1,3j-1}(0) = \left(\psi_{s_{N-j+1}}^{N-j+1}\right)^\top S_{1,1}(t_{N-j+1}^{*+})\psi_{s_{N-j+1}}^{N-j+1} + \bar{\Phi}_{s_{N-j+1}s_{N-j+1}}^{N-j+1}, \\ S_{3j-1,3j}(0) = \left(\tilde{\psi}_{s_{N-j+1}}^{N-j+1}\right)^\top, \\ S_{3j-1,3j+1}(0) = \theta_{N-j+1} + \left(\psi_{s_{N-j+1}}^{N-j+1}\right)^\top S_{1,1}(t_{N-j+1}^{*+})\dot{\psi}^{N-j+1}, \\ S_{3j,3j}(0) = 0, \qquad S_{3j,3j+1}(0) = 0, \\ S_{3j+1,3j+1}(0) = \kappa_{N-j+1} + \left(\dot{\psi}^{N-j+1}\right)^\top S_{1,1}(t_{N-j+1}^{*+})\dot{\psi}^{N-j+1}, \\ S_{3k+r,3l-1}(0) = -S_{1,3k+r}^\top(t_{N-l+1}^{*+})\psi_{s_{N-l+1}}^{N-l+1}, \qquad S_{3k+r,3l}(0) = 0, \\ S_{3k+r,3l+1}(0) = -S_{1,3k+r}^\top(t_{N-l+1}^{*+})\dot{\psi}^{N-l+1} \end{cases} \tag{23}$$

for $j = 1, \ldots, N$, $k = 1, \ldots, N-1$, $l = k+1, \ldots, N$ and $r = -1, 0, 1$.

For Problem 3.1, we now have the following theorem.

Theorem 3.1. *Suppose that Assumptions 2.1 and 3.1 are satisfied, and the symmetric matrix Υ defined in (22) is invertible. Then, the solution to Problem 3.1, $(\mathrm{d}\sigma, \mathrm{d}\tau)$, and the multipliers $\mathrm{d}\tilde{\nu}_i$, $i = 1, \ldots, N$, and $\mathrm{d}\tilde{\pi}$ associated with the active inequality constraints* *are obtained by solving* $(\Theta_1, \mathrm{d}\tilde{\pi})$ *from equation*

$$\Upsilon \begin{bmatrix} \Theta_1 \\ \mathrm{d}\tilde{\pi} \end{bmatrix} = - \begin{bmatrix} \Lambda_1 \\ 0 \end{bmatrix} - \begin{bmatrix} S_{1,2}^\top(0) \\ \vdots \\ S_{1,3N+1}^\top(0) \\ 0 \end{bmatrix} \delta x(0). \tag{24}$$

Proof: We extend the backward sweep technique in [3] to solve the linear MPBVP (17a)-(17d) and (19a)-(19f). This is done by using the boundary conditions at $t = t_f$ and at $t = t_i^*$, $i = N, \ldots, 1$.

1) Boundary condition at $t = t_f$

Let

$$\delta\lambda(t) = S_{1,1}(t)\delta x(t), \quad t \in (t_N^*, t_f). \tag{25}$$

Then, (20) can be written as

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta\dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A^{N+1} \\ C^{N+1} - (A^{N+1})^\top S_{1,1} \end{bmatrix} \delta x(t). \tag{26}$$

Now, differentiating (25) with respect to t yields

$$\delta\dot{\lambda}(t) = \dot{S}_{1,1}(t)\delta x(t) + S_{1,1}(t)\delta\dot{x}(t). \tag{27}$$

Then, by substituting $\delta\dot{x}$ and $\delta\dot{\lambda}$ from (26) into (27), we obtain

$$\left[\dot{S}_{1,1} + S_{1,1}A^{N+1} + (A^{N+1})^\top S_{1,1} - C^{N+1}\right] \delta x(t) = 0. \tag{28}$$

To ensure that this identity is valid for arbitrary $\delta x(t)$, we must have

$$\dot{S}_{1,1} = -S_{1,1}A^{N+1} - (A^{N+1})^\top S_{1,1} + C^{N+1}, \quad t \in (t_N^*, t_f), \tag{29}$$

and

$$S_{1,1}(t_f) = \Phi_{xx}. \tag{30}$$

2) Boundary conditions at $t = t_i^*$, $i = N, \ldots, 1$

From (3), it is known that

$$\psi^i_{x_i^+} = I_n, \quad \bar{\Phi}^i_{x_i^+ x_i^+} = 0, \quad \bar{\Phi}^i_{x_i^+ x_i^-} = \bar{\Phi}^{i\top}_{x_i^- x_i^+} = 0, \quad \bar{\Phi}^i_{x_i^+ s_i} = \bar{\Phi}^{i\top}_{s_i x_i^+} = 0$$

hold for $i = N, \ldots, 1$. Then, we can rearrange Equations (19c)-(19e), (17b), (17c) and (19f) as

$$\begin{bmatrix} 0 \\ \delta\lambda(t_i^{*-}) \\ 0 \\ 0 \\ 0 \\ -\tilde{\eta}_{t_i}^\top \mathrm{d}\tilde{\pi} \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & \bar{\Phi}^i_{x_i^- x_i^-} & \bar{\Phi}^i_{s_i x_i -} & \psi^{i\top}_{x_i^-} & 0 & 0 \\ 0 & 0 & \bar{\Phi}^i_{x_i^- s_i} & \bar{\Phi}^i_{s_i s_i} & \psi^{i\top}_{s_i} & \tilde{\psi}^{i\top}_{s_i} & \theta_i \\ 0 & I_n & \psi^i_{x_i^-} & \psi^i_{s_i} & 0 & 0 & \dot{\psi}^i \\ 0 & 0 & 0 & \tilde{\psi}^i_{s_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_i^\top & \dot{\psi}^{i\top} & 0 & \kappa_i \end{bmatrix} \begin{bmatrix} \delta\lambda(t_i^{*+}) \\ \delta x(t_i^{*+}) \\ \delta x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}\nu_i \\ \mathrm{d}\tilde{\nu}_i \\ \mathrm{d}t_i \end{bmatrix}. \tag{31}$$

From the fourth block row of (31), we have

$$\delta x(t_i^{*+}) = -\psi^i_{x_i^-}\delta x(t_i^{*-}) - \psi^i_{s_i}\mathrm{d}s_i - \dot{\psi}^i \mathrm{d}t_i. \tag{32}$$

Note that, as time increases, the next boundary point is at $t = t_{i+1}^*$. Let

$$\begin{aligned}\delta\lambda(t_i^{*+}) &\triangleq S_{1,1}\left(t_i^{*+}\right)\delta x(t_i^{*+})\\&\quad+\sum_{j=1}^{N-i}\Big[S_{1,3j-1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1}+S_{1,3j}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}+S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\Big],\\&=-S_{1,1}\left(t_i^{*+}\right)\psi^i_{x_i^-}\delta x(t_i^{*-})-S_{1,1}\left(t_i^{*+}\right)\psi^i_{s_i}\mathrm{d}s_i-S_{1,1}\left(t_i^{*+}\right)\dot{\psi}^i\mathrm{d}t_i\\&\quad+\sum_{j=1}^{N-i}\Big[S_{1,3j-1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1}+S_{1,3j}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}+S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\Big].\end{aligned}\tag{33}$$

It is clear from the first block row of (31) that

$$\begin{aligned}\mathrm{d}\nu_i &= S_{1,1}\left(t_i^{*+}\right)\psi^i_{x_i^-}\delta x(t_i^{*-})+S_{1,1}\left(t_i^{*+}\right)\psi^i_{s_i}\mathrm{d}s_i+S_{1,1}\left(t_i^{*+}\right)\dot{\psi}^i\mathrm{d}t_i\\&\quad-\sum_{j=1}^{N-i}\Big[S_{1,3j-1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1}+S_{1,3j}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}+S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\Big].\end{aligned}\tag{34}$$

In (33) and (34), summations with upper limits less than lower limits are defined to be zero. Now, incorporate the following equation into (31),

$$\Psi_{i+1}=\begin{bmatrix} S_{2,1}(t_i^{*+}) & \cdots & S_{2,3(N-i)+1}(t_i^{*+})\\ \vdots & \ddots & \vdots\\ S_{3(N-i)+1,1}(t_i^{*+}) & \cdots & S_{3(N-i)+1,3(N-i)+1}(t_i^{*+})\end{bmatrix}\begin{bmatrix}\delta x(t_i^{*+})\\ \Theta_{i+1}\end{bmatrix}\tag{35}$$

for $i = N-1, \dots, 1$. After eliminating $\delta x(t_i^{*+})$, $\delta\lambda(t_i^{*+})$ and $\mathrm{d}\nu_i$ by, respectively, (32), (33) and (34), the expanded equation (31) becomes

$$\begin{bmatrix}\delta\lambda(t_i^{*-})\\ \Psi_i\end{bmatrix}=\Gamma^i\begin{bmatrix}\delta x(t_i^{*-})\\ \Theta_i\end{bmatrix},\tag{36}$$

where Γ^i is a block symmetric matrix with $(3(N-i)+4)\times(3(N-i)+4)$ blocks. The blocks of Γ^i are defined in Appendix A.

Similar to (25), we assume that

$$\begin{bmatrix}\delta\lambda(t)\\ \Psi_i\end{bmatrix}=S^i(t)\begin{bmatrix}\delta x(t)\\ \Theta_i\end{bmatrix},\quad t\in(t_{i-1}^*,t_i^*),\tag{37}$$

where $S^i(t)$ is a $(3(N-i)+4)\times(3(N-i)+4)$ block symmetric matrix with its (α,β) term, $\alpha,\beta\in\{1,\dots,3(N-i)+4\}$, denoted by $S_{\alpha,\beta}$. Then, (20) can be written as

$$\begin{bmatrix}\delta\dot{x}(t)\\ \delta\dot{\lambda}(t)\end{bmatrix}=\begin{bmatrix}A^i & 0 & \cdots & 0\\ C^i-A^{i\top}S_{1,1} & -A^{i\top}S_{1,2} & \cdots & -A^{i\top}S_{1,3(N-i)+4}\end{bmatrix}\begin{bmatrix}\delta x(t)\\ \Theta_i\end{bmatrix}.\tag{38}$$

Now, differentiate (37) with respect to t, treating Ψ_i and Θ_i as constants. It follows that

$$\begin{aligned}\begin{bmatrix}\delta\dot{\lambda}\\ 0\end{bmatrix}&=\begin{bmatrix}\dot{S}_{1,1} & \cdots & \dot{S}_{1,3(N-i)+4}\\ * & \ddots & \vdots\\ * & * & \dot{S}_{3(N-i)+4,3(N-i)+4}\end{bmatrix}\begin{bmatrix}\delta x\\ \Theta_i\end{bmatrix}\\&\quad+\begin{bmatrix}S_{1,1} & \cdots & S_{1,3(N-i)+4}\\ * & \ddots & \vdots\\ * & * & S_{3(N-i)+4,3(N-i)+4}\end{bmatrix}\begin{bmatrix}\delta\dot{x}\\ 0\end{bmatrix}.\end{aligned}\tag{39}$$

Then, by substituting $\delta\dot{x}$ and $\delta\dot{\lambda}$, expressed by (38), into (39), we obtain that

$$\begin{bmatrix} \dot{S}_{1,1}+S_{1,1}A^i+A^{i\top}S_{1,1}-C^i & \dot{S}_{1,2}+A^{i\top}S_{1,2} & \cdots & \dot{S}_{1,3(N-i)+4}+A^{i\top}S_{1,3(N-i)+4} \\ * & \dot{S}_{2,2} & \cdots & \dot{S}_{2,3(N-i)+4} \\ * & * & \ddots & \vdots \\ * & * & * & \dot{S}_{3(N-i)+4,3(N-i)+4} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \Theta_i \end{bmatrix} = 0. \tag{40}$$

To ensure that this identity is valid for arbitrary $\delta x(t)$ and Θ_i, the following differential equations must be valid for $t \in (t_{i-1}^*, t_i^*)$:

$$\dot{S}_{1,1} = -S_{1,1}A^i - A^{i\top}S_{1,1} + C^i, \quad \dot{S}_{1,j} = -A^{i\top}S_{1,j}, \quad \dot{S}_{j,k} = 0, \tag{41}$$

where $i = N, \ldots, 1$, $j = 2, \ldots, 3(N-i)+4$, and $k = j, \ldots, 3(N-i)+4$, with boundary conditions

$$S_{\alpha,\beta}(t_i^{*-}) = \Gamma_{\alpha,\beta}^i, \quad \alpha = 1, \ldots, 3(N-i)+4, \quad \beta = \alpha, \ldots, 3(N-i)+4. \tag{42}$$

Now, (29) and (41) with boundary conditions (30) and (42), $i = N, \ldots, 1$, form the group of switched impulsive systems (21a)-(21d), which can be integrated backwards with time from $t = t_f^*$ to $t = t_0$ and subsystem index i from $N+1$ to 1 to obtain $S_{1,\alpha}(t_0)$, $\alpha = 1, \ldots, 3N+1$. The other terms $S_{\alpha,\beta}(t_0)$, $\alpha = 2, \ldots, 3N+1$, $\beta = \alpha, \ldots, 3N+1$, can be obtained by equations in (23) at $t = t_i^*$, $i = N, \ldots, 1$. Then, it follows from (37) (with $i = 1$ and $t_0^* = t_0 = 0$) and (17d) that $(\Theta_1, \mathrm{d}\tilde{\pi})$ can be solved from (24) for a given $\delta x(0)$ if the symmetric matrix Υ is invertible. □

Remark 3.2. *The results in [7, 8] demonstrate that the invertibility of Υ is equivalent to certain controllability of the linear system (17a) with perturbed boundary conditions (17b)-(17d). In practice, this NE solution can be computed in a numerically stable manner without explicitly inverting matrix Υ by using the triangular or orthogonal decomposition of Υ.*

Corollary 3.1. *Suppose that Assumptions 2.1 and 3.1 are satisfied, and the symmetric matrix Υ defined in (22) is invertible. Then, the solution of Problem 2.2 is given by*

$$\sigma \approx \sigma^* + \mathrm{d}\sigma \quad \text{and} \quad \tau \approx \tau^* + \mathrm{d}\tau, \tag{43}$$

where $(\mathrm{d}\sigma, \mathrm{d}\tau)$ is obtained by solving Θ_1 from Equation (24) with $\Lambda_1 = 0$.

Corollary 3.1 presents a solution of the NE problem, Problem 2.2, with a small initial perturbation satisfying Assumption 3.1.

Remark 3.3. *When $\delta x(0) = 0$, the solution $(\mathrm{d}\sigma, \mathrm{d}\tau)$, $\mathrm{d}\tilde{\nu}_i$, $i = 1, \ldots, N$, and $\mathrm{d}\tilde{\pi}$ derived from Theorem 3.1 will force the following equations to be satisfied:*

$$\bar{\Phi}_{s_i}^i = 0 \quad \text{and} \quad q_i = 0, \quad i = 1, \ldots, N. \tag{44}$$

If the perturbation is large, Assumption 3.1 may not be satisfied. In this situation, $(\mathrm{d}\sigma, \mathrm{d}\tau)$ computed in Theorem 3.1 presents a direction for the control correction after the perturbation. In the following subsection, Assumption 3.1 will be dropped. We first compute a step size in this correction direction, and then present an algorithm to compute the solution of Problem 2.2 iteratively.

3.2. **Sensitivity analysis for large perturbations.** Before presenting the method to compute the step size, we need some more notations. Define

$$\mathrm{d}C(\mathrm{d}\sigma, \mathrm{d}\tau) \triangleq \begin{bmatrix} \mathrm{d}\hat{\psi}^{1\top}(\mathrm{d}s_1) & \cdots & \mathrm{d}\hat{\psi}^{N\top}(\mathrm{d}s_N) & \mathrm{d}\eta^\top(\mathrm{d}\tau) \end{bmatrix}^\top,$$

where $(\mathrm{d}\sigma, \mathrm{d}\tau)$ is computed by Theorem 3.1 with respect to the nominal control (σ^*, τ^*), and

$$\mathrm{d}\hat{\psi}^i(\mathrm{d}s_i) \triangleq \hat{\psi}^i_{s_i}\mathrm{d}s_i, \quad i = 1, \ldots, N, \quad \text{and} \quad \mathrm{d}\eta(\mathrm{d}\tau) \triangleq \sum_{i=1}^{N} \eta_{t_i}\mathrm{d}t_i.$$

Let $\mathrm{d}C^i$, $i \in \mathcal{I}$, be the ith element of the vector $\mathrm{d}C$. Define

$$\mathrm{d}\mu \triangleq \begin{bmatrix} \mathrm{d}\hat{\nu}_1^\top & \cdots & \mathrm{d}\hat{\nu}_N^\top & \mathrm{d}\pi^\top \end{bmatrix}^\top$$

as the vector of the differentials of the associated multipliers, where the ith element of $\mathrm{d}\mu$, $\mathrm{d}\mu^i$, can be obtained from Theorem 3.1 if $i \in \mathcal{A}$ or is equal to zero if $i \in \mathcal{I}\backslash\mathcal{A}$. Then, set $\bar{k}$ as the maximal number of iterations, and let a variable with a superscript '(k)', $k = 0, 1, \ldots, \bar{k}$, denote the value of the corresponding variable computed in the kth iterative. Now, we will present a formula to compute the maximal step size for the control correction.

If the ith constraint $C^{i(k)} \le 0$ at the kth iteration is treated as inactive, i.e., $i \in \mathcal{I}\backslash\mathcal{A}^{(k)}$, then define

$$\bar{\alpha}^{i(k)} \triangleq \begin{cases} -C^{i(k)}/\mathrm{d}C^{i(k)}, & \text{if } C^{i(k)} < 0 \text{ and } \mathrm{d}C^{i(k)} > 0, & \text{(45a)} \\ \infty, & \text{if } C^{i(k)} < 0 \text{ and } \mathrm{d}C^{i(k)} < 0, & \text{(45b)} \\ 0, & \text{if } C^{i(k)} = 0 \text{ and } \mathrm{d}C^{i(k)} > 0, & \text{(45c)} \\ \infty, & \text{if } C^{i(k)} = 0 \text{ and } \mathrm{d}C^{i(k)} < 0. & \text{(45d)} \end{cases}$$

On the other hand, if $C^{i(k)} \le 0$ is treated as active, i.e., $i \in \mathcal{A}^{(k)}$, then define

$$\bar{\alpha}^{i(k)} \triangleq \begin{cases} -\mu^{i(k)}/\mathrm{d}\mu^{i(k)}, & \text{if } \mu^{i(k)} > 0,\ \mathrm{d}\mu^{i(k)} < 0 \text{ and } \gamma^{(k-1)} = 0, & \text{(46a)} \\ \infty, & \text{if } \mu^{i(k)} > 0,\ \mathrm{d}\mu^{i(k)} < 0 \text{ and } \gamma^{(k-1)} = 1, & \text{(46b)} \\ \infty, & \text{if } \mu^{i(k)} > 0 \text{ and } \mathrm{d}\mu^{i(k)} > 0, & \text{(46c)} \\ \mu^{i(k)}, & \text{if } \mu^{i(k)} \le 0 \text{ and } \alpha^{(k-1)} > 0, & \text{(46d)} \\ \infty, & \text{if } \mu^{i(k)} \le 0 \text{ and } \alpha^{(k-1)} = 0, & \text{(46e)} \end{cases}$$

where $\mu^{i(k)}$ is the ith element of the multiplier vector $\mu^{(k)}$. The variable $\alpha^{(k)}$ in (46) is the maximal step size in the correction direction $(\mathrm{d}\sigma^{(k)}, \mathrm{d}\tau^{(k)})$, which is defined by

$$\alpha^{(k)} \triangleq \begin{cases} 0, & \text{if } \bar{\alpha}^{(k)} \le 0, & \text{(47a)} \\ \bar{\alpha}^{(k)}, & \text{if } 0 < \bar{\alpha}^{(k)} < 1, & \text{(47b)} \\ 1, & \text{if } 1 \le \bar{\alpha}^{(k)} & \text{(47c)} \end{cases}$$

with $\bar{\alpha}^{(k)} \triangleq \min\{\bar{\alpha}^{i(k)} |\, i \in \mathcal{I}\}$. The purpose of incorporating the variable $\gamma^{(k)} \in \{0, 1\}$ in (46) will be explained after presenting our algorithm.

Based on the step size (47), our algorithm to compute the solution of Problem 2.2 (without Assumption 3.1) is presented as follows:

Algorithm 3.1.

1. *Set* $k = 0$. *Let* $x(0)^{(k)} = x^0$, $\delta x(0)^{(k)} = \delta x(0)$ *and* $(\sigma^{(k)}, \tau^{(k)}) = (\sigma^*, \tau^*)$, *where* (σ^*, τ^*) *is the nominal extremal solution corresponding to* x^0. *Set* $\gamma^{(k-1)} = \gamma_{\mathrm{n}}^{(k-1)} = 0$ *and* $\alpha^{(k-1)} = 1$.

2. *Compute* $x^{(k)}(t)$ *corresponding to the nominal control* $(\sigma^{(k)}, \tau^{(k)})$ *by solving Equations (1) and (2). With the* $x^{(k)}(t)$ *computed,* $\lambda^{(k)}(t)$ *and multipliers* $\nu_i^{(k)}$, $i = 1, \ldots, N$, *and* $\mu^{(k)}$ *can be computed by Equation (12) with (44) and* $\mu^{i(k)} = 0$ *for* $i \in \mathcal{I}\backslash\mathcal{A}^{(k)}$. *Then, with respect to this nominal solution, solve* $(\mathrm{d}\sigma^{(k)}, \mathrm{d}\tau^{(k)})$ *and* $\mathrm{d}\mu^{i(k)}$, $i \in \mathcal{A}^{(k)}$, *by Theorem 3.1. Let* $\mathrm{d}\mu^{i(k)} = 0$ *for* $i \in \mathcal{I}\backslash\mathcal{A}^{(k)}$.
3. *If* $\mathrm{d}\sigma^{(k)} = 0$ *and* $\mathrm{d}\tau^{(k)} = 0$, *go to the next step. Otherwise, go to Step 7.*
4. *Compute* $\beta^{(k)} \triangleq \min\{\mu^{i(k)} | \ i \in \mathcal{A}^{(k)}\}$. *If* $\beta^{(k)} \geq 0$, *terminate with* $\sigma = \sigma^{(k)}$ *and* $\tau = \tau^{(k)}$. *Otherwise, go to the next step.*
5. *Set* $\alpha^{(k)} = 0$. *Let* $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}\backslash\{l^{(k)}\}$, *where* $l^{(k)} = \{i \in \mathcal{I} | \ \mu^{i(k)} = \beta^{(k)}\}$.
6. *Let* $\gamma_{\mathrm{n}}^{(k)} = \gamma_{\mathrm{n}}^{(k-1)} + 1$. *If* $\gamma_{\mathrm{n}}^{(k)} = 2$ *and* $l^{(k)} = l^{(k-1)}$, *let* $\gamma^{(k)} = 1$. *Otherwise, let* $\gamma^{(k)} = 0$. *Set* $k = k + 1$, *and go to Step 2.*
7. *Compute* $\alpha^{(k)}$ *by (47) and* $l^{(k)} = \{i \in \mathcal{I} | \ \bar{\alpha}^{i(k)} = \bar{\alpha}^{(k)}\}$. *If* $\alpha^{(k)} = 1$, *let* $\gamma^{(k)} = \gamma_{\mathrm{n}}^{(k)} = 0$ *and go to Step 11. Otherwise, go to the next step.*
8. *Change the active status of the* $l^{(k)}$*th inequality constraint. That is, let* $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} \cup \{l^{(k)}\}$ *if* $l^{(k)} \in \mathcal{I}\backslash\mathcal{A}^{(k)}$, *and let* $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}\backslash\{l^{(k)}\}$ *if* $l^{(k)} \in \mathcal{A}^{(k)}$.
9. *If* $\alpha^{(k)} > 0$, *let* $\gamma^{(k)} = \gamma_{\mathrm{n}}^{(k)} = 0$ *and go to Step 11. If* $\alpha^{(k)} = 0$, *let* $\gamma_{\mathrm{n}}^{(k)} = \gamma_{\mathrm{n}}^{(k-1)} + 1$.
10. *If* $\gamma_{\mathrm{n}}^{(k)} = 2$ *and* $l^{(k)} = l^{(k-1)}$, *let* $\gamma^{(k)} = 1$. *Otherwise, let* $\gamma^{(k)} = 0$.
11. *Let*

$$\begin{cases} \sigma^{(k+1)} = \sigma^{(k)} + \alpha^{(k)}\mathrm{d}\sigma^{(k)}, \\ \tau^{(k+1)} = \tau^{(k)} + \alpha^{(k)}\mathrm{d}\tau^{(k)}, \end{cases} \tag{48a}$$

$$\begin{cases} x(0)^{(k+1)} = x(0)^{(k)} + \alpha^{(k)}\delta x(0)^{(k)}, \\ \delta x(0)^{(k+1)} = \delta x(0)^{(k)} - \alpha^{(k)}\delta x(0)^{(k)}. \end{cases} \tag{48b}$$

Set $k = k + 1$, *and go to Step 2.*

Remark 3.4. *The basic idea of the modification in (48a) is that there is at most one inequality constraint that will change its active status if the control* $(\sigma^{(k)}, \tau^{(k)})$ *is modified by a step size* $\alpha^{(k)}$ *along the direction* $(\mathrm{d}\sigma^{(k)}, \mathrm{d}\tau^{(k)})$. *From Equation (24), the initial state* $x(0)^{(k)}$ *and the initial perturbation* $\delta x(0)^{(k)}$ *should be, correspondingly, modified by (48b) if* Λ_1 *is small enough that can be neglected.*

Remark 3.5. *In order to accelerate the computation, we allow an active inequality constraint to be dropped from the active set before the computation converges. This will incur the zigzagging phenomenon. Specifically, the active status of an inequality constraint may oscillate back and forth in successive steps, which will slow down the computational procedure. Thus, we incorporate the variables* $\gamma^{(k)}$ *and* $\gamma_{\mathrm{n}}^{(k)}$, *and simple inferential logics into Algorithm 3.1. If the active status of one inequality constraint changes twice in two successive steps, we will prevent this change from continuing by setting* $\gamma^{(k)} = 1$ *and allowing the Lagrange multiplier associated with this constraint to become negative.*

Remark 3.6. *This algorithm will be reduced to an SQP algorithm* [26] *based on the active set strategy if we let* $x(0)^{(k)} = x^0 + \delta x(0)$ *and* $\delta x(0)^{(k)} = 0$ *for* $k = 0$, *and select an arbitrary feasible control as the initial nominal control.*

Remark 3.7. *Since constraints (5a) and (5b) are both linear, each intermediate solution* $(\sigma^{(k+1)}, \tau^{(k+1)})$ *computed by (48a) with* $\alpha^{(k)}$ *determined by (47) is feasible. Thus, in situation where computational efficiency is a critical consideration* [12, 13], *Steps 3-7 in Algorithm 3.1 can be replaced by "Compute* $\alpha^{(k)}$ *by (47) and* $l^{(k)} = \{i \in \mathcal{I} | \ \bar{\alpha}^{i(k)} = \bar{\alpha}^{(k)}\}$.

If $\alpha^{(k)} = 1$*, terminate with*

$$\begin{cases} \sigma = \sigma^{(k)} + \alpha^{(k)}\mathrm{d}\sigma^{(k)}, \\ \tau = \tau^{(k)} + \alpha^{(k)}\mathrm{d}\tau^{(k)}, \end{cases}$$

$$\begin{cases} x(0) = x(0)^{(k)} + \alpha^{(k)}\delta x(0)^{(k)}, \\ \delta x(0) = \delta x(0)^{(k)} - \alpha^{(k)}\delta x(0)^{(k)}, \end{cases}$$

and $k = k + 1$*. Otherwise, go to the next step". In this way, less iterations are needed to compute the perturbed solution, which is guaranteed to be feasible although not locally extremal.*

4. **A Numerical Example.** To verify our NE solution method, consider the optimal shrimp harvesting problem discussed in [20]. In this problem, $x(t)$ is the number of shrimp at time t and $y(t)$ is the average weight of shrimp (in grams) at time t, where t is measured in weeks. The shrimp population growth is modeled by the dynamics,

$$\begin{cases} \dot{x}(t) = -0.03x(t), & x(0) = 40000, \\ \dot{y}(t) = 3.5 - 0.00001x(t)y(t), & y(0) = 1. \end{cases} \tag{49}$$

Suppose that shrimp are harvested at times $t = t_i$, $i = 1, \ldots, N$, and s_i is the fraction of the total shrimp stock harvested at $t = t_i$. Then, we have the following jump conditions at each time $t = t_i$:

$$\begin{cases} x(t_i^+) = (1 - s_i)x(t_i^-), \\ y(t_i^+) = y(t_i^-), \end{cases} \quad i = 1, \ldots, N. \tag{50}$$

The revenue obtained by harvesting a fraction s_i of the shrimp stock is given by

$$py(t_i^-)s_i x(t_i^-) - h,$$

where $p \triangleq \$0.008$ is the price per gram of shrimp and $h = \$50$ is the fixed cost of harvesting. At the specified final time $t = t_f = 13.2$, all of the remaining shrimp are harvested. The harvesting times $\tau \triangleq [t_1, \ldots, t_N]$ and fractions $\sigma \triangleq [s_1, \ldots, s_N]$ are subject to the following constraints:

$$\begin{cases} 0.01 \le s_i \le 1, & i = 1, \ldots, N, \\ t_i - t_{i-1} \ge 0.01, & i = 1, \ldots, N+1, \ t_0 = 0, \ t_{N+1} = t_f. \end{cases} \tag{51}$$

The problem is to choose τ and σ to maximize the total revenue

$$R(\sigma, \tau) \triangleq \sum_{i=1}^{N} \left[py(t_i^-)s_i x(t_i^-) - h\right] + py(t_f)x(t_f) - h \tag{52}$$

subject to constraints (51).

To verify the effectiveness of Algorithm 3.1, we increase the lower bound of s_i in (51) to 0.3 and set $N = 2$ in the following two simulation cases. The indices of the different constraints are assigned as Table 1.

TABLE 1. The indices of the constraints

index	constraint	index	constraint
1	$s_1 \ge 0.3$	5	$t_0 - t_1 + 0.01 \le 0$
2	$s_1 \le 1.0$	6	$t_1 - t_2 + 0.01 \le 0$
3	$s_2 \ge 0.3$	7	$t_2 - t_3 + 0.01 \le 0$
4	$s_2 \le 1.0$		

4.1. **Computation from an interior point to a boundary point.** In this case, we set the initial state for $k = 0$ as

$$[x^{(k)}(0), y^{(k)}(0)]^\top = [40000, 1]^\top,$$

and let the initial state perturbation for $k = 0$ be

$$[\delta x^{(k)}(0), \delta y^{(k)}(0)]^\top = -75\%[40000, 1]^\top.$$

We solve Problem 2.2 iteratively by Algorithm 3.1, and the results are presented in Table 2. In Table 2, each row presents the intermediate result at iteration k, i.e., the step size $\alpha^{(k)}$, the current initial states $x^{(k)}(0)$ and $y^{(k)}(0)$, the controls $s_i^{(k)}$ and $t_i^{(k)}$, $i = 1, 2$, and the revenue $R^{(k)}$. $R^{(k)}$ is computed by (52) with the initial state $[x^{(k)}(0), y^{(k)}(0)]^\top$ and the controls $s_i^{(k)}$ and $t_i^{(k)}$.

We begin Algorithm 3.1 with $k = 0$ and the nominal control corresponding to the initial state $[40000, 1]^\top$, i.e., $\sigma^* = [0.388, 0.454]^\top$ and $\tau^* = [4.270, 7.810]^\top$ with revenue $R^* = 3189$, which is derived by the method in [20]. It is clear that this solution has no active inequality constraints. Hence, it is an interior point in the feasible set. After 8 iterations, the initial state is perturbed to $[10000, 0.25]^\top$, and the corresponding control and revenue computed by Algorithm 3.1 coincide with the result derived by [20], i.e., $\sigma = [0.300, 0.300]^\top$ and $\tau = [10.123, 13.190]^\top$ with $R = 1442$. This solution satisfies that

$$0.3 - s_1 = 0.3 - s_2 = 0 \quad \text{and} \quad t_2 - t_3 + 0.01 = 0,$$

i.e., three inequality constraints are active. The active status of the inequality constraints at each iteration is illustrated in Figure 1, where, for example, the three horizontal lines with coordinates 1, 3 and 7 in the interval $[8, 9]$ denote that the 1st, 3rd and 7th constraints (defined in Table 1) are active at iteration $k = 8$.

So, starting from a nominal control corresponding to an interior point in the feasible set, Algorithm 3.1 computes the NE control for the perturbed problem quickly, which is on the boundary of the feasible set. Furthermore, the computational procedure terminates at iteration $k = 9$.

Table 2. Simulation results for Case 4.1 using Algorithm 3.1

k	$\alpha^{(k)}$	$x^{(k)}(0)$	$y^{(k)}(0)$	$s_1^{(k)}$	$s_2^{(k)}$	$t_1^{(k)}$	$t_2^{(k)}$	$R^{(k)}$
0	0.8013	40000	1.000	0.388	0.454	4.270	7.810	3189
1	0.3992	15962	0.399	0.300	0.350	5.906	9.117	1943
2	1.0000	13582	0.340	0.300	0.300	6.864	9.908	1761
3	0.5915	10000	0.250	0.300	0.300	9.035	12.454	1438
4	0.0000	10000	0.250	0.300	0.300	9.671	13.152	1442
5	0.0714	10000	0.250	0.300	0.300	9.671	13.152	1442
6	1.0000	10000	0.250	0.300	0.300	9.707	13.190	1442
7	1.0000	10000	0.250	0.300	0.300	10.115	13.190	1442
8	**1.0000**	**10000**	**0.250**	**0.300**	**0.300**	**10.123**	**13.190**	**1442**
9	—	10000	0.250	0.300	0.300	10.123	13.190	1442

In order to evaluate the computational speed of Algorithm 3.1, we run the simulation again using the SQP algorithm. In this simulation, we let

$$[x^{(k)}(0), y^{(k)}(0)]^\top = [10000, 0.25]^\top \quad \text{and} \quad [\delta x^{(k)}(0), \delta y^{(k)}(0)]^\top = [0, 0]^\top$$

at iteration $k = 0$, and set the initial nominal control as the same as that in the previous simulation. As stated in Remark 3.6, Algorithm 3.1 is now reduced to an SQP algorithm. The results are listed in Table 3 and Figure 2. After 20 iterations, the control obtained by

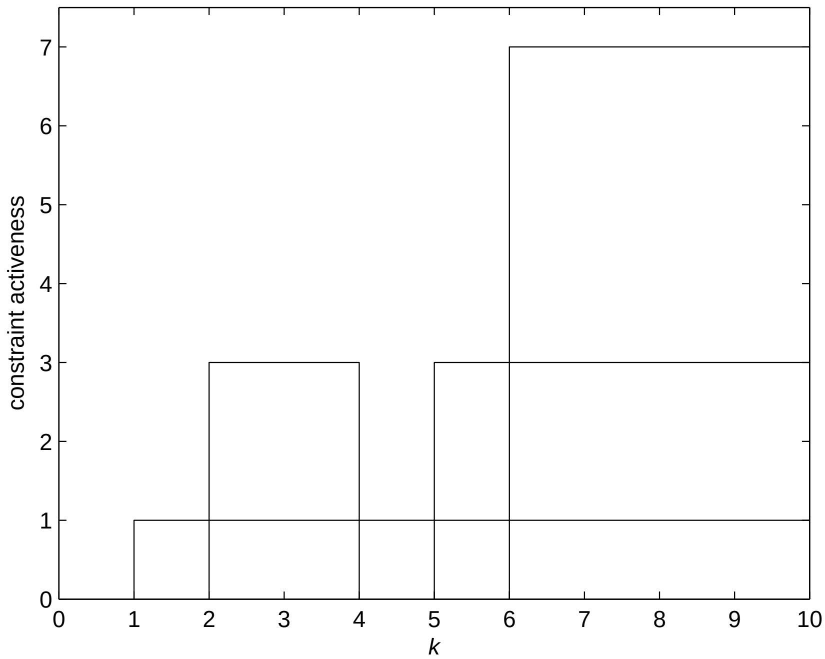

FIGURE 1. The active status of the constraints in Case 4.1 using Algorithm 3.1

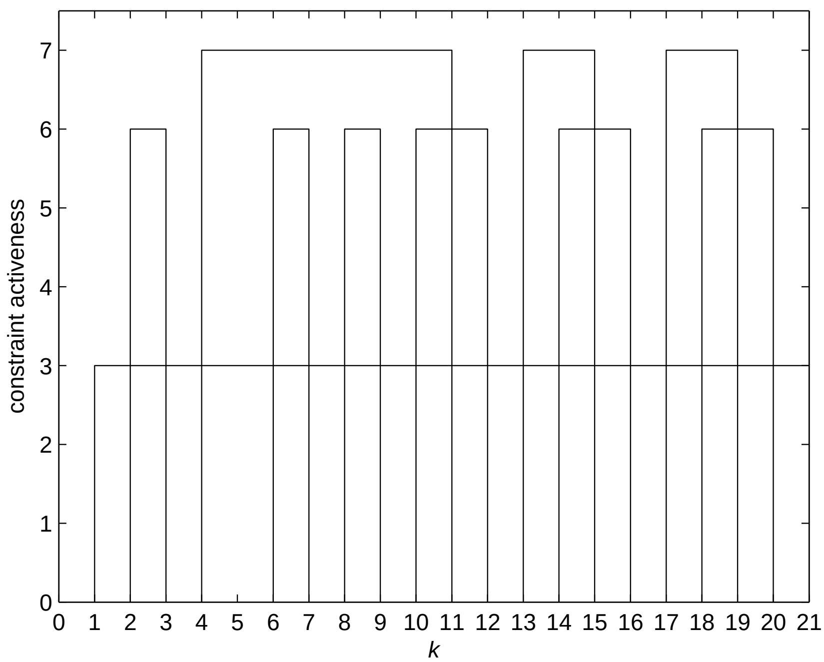

FIGURE 2. The active status of the constraints in Case 4.1 using the SQP algorithm

this SQP algorithm is trapped into a solution inferior to the NE solution for the perturbed system. Compared with Algorithm 3.1, the SQP algorithm converges much more slowly and seems unable to reach the correct NE solution.

Table 3. Simulation results for Case 4.1 using the SQP algorithm

k	$\alpha^{(k)}$	$x^{(k)}(0)$	$y^{(k)}(0)$	$s_1^{(k)}$	$s_2^{(k)}$	$t_1^{(k)}$	$t_2^{(k)}$	$R^{(k)}$
0	0.6935	10000	0.250	0.388	0.454	4.270	7.810	1239
1	0.2869	10000	0.250	0.467	0.300	8.065	9.662	1379
2	0.6323	10000	0.250	0.875	0.300	12.635	12.645	1409
3	0.0404	10000	0.250	0.694	0.300	12.868	12.878	1420
4	1.0000	10000	0.250	0.694	0.300	12.920	13.190	1423
5	0.1630	10000	0.250	0.666	0.300	13.176	13.190	1423
6	0.0034	10000	0.250	0.666	0.300	13.180	13.190	1423
7	0.0000	10000	0.250	0.665	0.300	13.180	13.190	1423
8	0.0000	10000	0.250	0.665	0.300	13.180	13.190	1423
9	0.0000	10000	0.250	0.665	0.300	13.180	13.190	1423
10	0.2978	10000	0.250	0.665	0.300	13.180	13.190	1423
11	0.0000	10000	0.250	0.538	0.300	13.180	13.190	1424
12	0.0000	10000	0.250	0.538	0.300	13.180	13.190	1424
13	0.0000	10000	0.250	0.538	0.300	13.180	13.190	1424
14	0.0547	10000	0.250	0.538	0.300	13.180	13.190	1424
15	0.0000	10000	0.250	0.522	0.300	13.180	13.190	1424
16	0.0000	10000	0.250	0.522	0.300	13.180	13.190	1424
17	0.0000	10000	0.250	0.522	0.300	13.180	13.190	1424
18	0.0010	10000	0.250	0.522	0.300	13.180	13.190	1424
19	0.0000	10000	0.250	0.521	0.300	13.180	13.190	1424
20	—	10000	0.250	0.521	0.300	13.180	13.190	1424

In this case, the initial nominal control corresponds to an interior point in the feasible set, while the NE control for the perturbed problem corresponds to a boundary point. The active set during the entire computational procedure is enlarged. To further examine Algorithm 3.1, we consider an inverse problem in the next simulation, where the computation starts from a boundary point and ends at an interior point.

4.2. **Computation from a boundary point to an interior point.** In this case, we set the initial state and the initial state perturbation at step $k = 0$ as

$$[x^{(k)}(0), y^{(k)}(0)]^\top = [10000, 0.25]^\top \quad \text{and} \quad [\delta x^{(k)}(0), \delta y^{(k)}(0)]^\top = [30000, 0.75]^\top,$$

and select the NE solution $\sigma^* = [0.300, 0.300]^\top$ and $\tau^* = [10.123, 13.190]^\top$ derived in the previous case as the initial nominal control. That is, we exchange the start point and end point in the previous case. The desired NE control in this case should be the initial nominal one in Case 4.1. The simulation results are shown in Table 4 and Figure 3.

From Table 4 and Figure 3, we find that the computation is not straightforward. The active status of the 4th constraint changes repeatedly from iteration $k = 5$ to iteration $k = 7$. However, the algorithm obtains a solution in the 19th step, which is very close to the locally extremal one corresponding to the initial state $[40000, 1]^\top$. So, in the case when the computation is from a boundary point to an interior point, Algorithm 3.1 also has a good performance.

Similar to Case 4.1, we run the simulation again using the SQP algorithm. In this simulation, we set

$$[x^{(k)}(0), y^{(k)}(0)]^\top = [40000, 1]^\top \quad \text{and} \quad [\delta x^{(k)}(0), \delta y^{(k)}(0)]^\top = [0, 0]^\top$$

for $k = 0$, and select $\sigma^* = [0.300, 0.300]^\top$ and $\tau^* = [10.123, 13.190]^\top$ as the initial nominal control. The computation converges more slowly than the previous simulation using

TABLE 4. Simulation results for Case 4.2 using Algorithm 3.1

k	$\alpha^{(k)}$	$x^{(k)}(0)$	$y^{(k)}(0)$	$s_1^{(k)}$	$s_2^{(k)}$	$t_1^{(k)}$	$t_2^{(k)}$	$R^{(k)}$
0	0.0251	10000	0.250	0.300	0.300	10.123	13.190	1442
1	0.5602	10752	0.269	0.300	0.300	9.692	13.190	1517
2	0.0000	27137	0.678	0.300	0.300	0.010	1.993	2000
3	0.2285	27137	0.678	0.300	0.300	0.010	1.993	2000
4	0.3029	30075	0.752	0.300	0.300	0.831	2.954	2360
5	0.0000	33082	0.827	0.300	1.000	1.545	10.343	2405
6	0.0000	33082	0.827	0.300	1.000	1.545	10.343	2405
7	0.9167	33082	0.827	0.300	1.000	1.545	10.343	2405
8	0.0000	39424	0.986	0.300	1.000	3.769	13.190	2821
9	1.0000	39424	0.986	0.300	1.000	3.769	13.190	2821
10	0.0000	40000	1.000	0.911	1.000	7.567	13.190	2595
11	0.0013	40000	1.000	0.911	1.000	7.567	13.190	2595
12	1.0000	40000	1.000	0.881	0.300	6.790	13.190	2687
13	0.0000	40000	1.000	0.491	0.300	0.482	13.190	2402
14	1.0000	40000	1.000	0.491	0.300	0.482	13.190	2402
15	0.0000	40000	1.000	0.628	0.300	3.113	9.441	3021
16	1.0000	40000	1.000	0.628	0.300	3.113	9.441	3021
17	1.0000	40000	1.000	0.455	0.500	3.882	7.323	3163
18	1.0000	40000	1.000	0.390	0.453	4.193	7.748	3189
19	**1.0000**	**40000**	**1.000**	**0.388**	**0.454**	**4.270**	**7.809**	**3189**
20	—	40000	1.000	0.388	0.454	4.272	7.809	3189

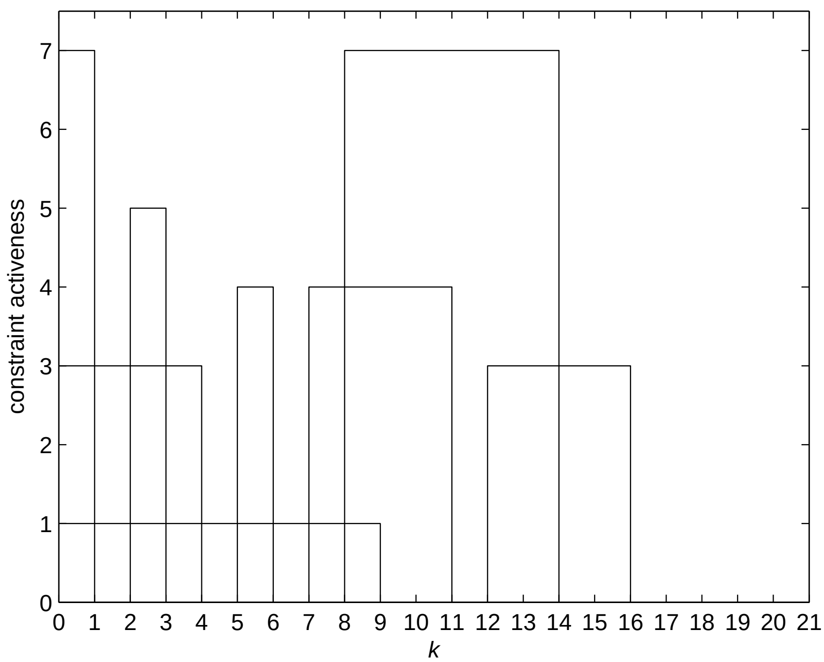

FIGURE 3. The active status of the constraints in Case 4.2 using Algorithm 3.1

TABLE 5. Simulation results for Case 4.2 using the SQP algorithm

k	$\alpha^{(k)}$	$x^{(k)}(0)$	$y^{(k)}(0)$	$s_1^{(k)}$	$s_2^{(k)}$	$t_1^{(k)}$	$t_2^{(k)}$	$R^{(k)}$
0	0.0000	40000	1.000	0.300	0.300	10.123	13.190	2698
1	0.1337	40000	1.000	0.300	0.300	10.123	13.190	2698
2	0.0000	40000	1.000	1.000	0.300	1.711	13.190	1338
3	1.0000	40000	1.000	1.000	0.300	1.711	13.190	1338
4	0.0000	40000	1.000	0.816	0.300	3.188	13.190	2687
5	1.0000	40000	1.000	0.816	0.300	3.188	13.190	2687
6	0.3646	40000	1.000	0.619	0.300	4.072	5.747	3033
7	0.9913	40000	1.000	0.562	0.300	4.203	6.866	3118
8	0.0000	40000	1.000	0.300	0.817	3.729	10.747	2940
9	1.0000	40000	1.000	0.300	0.817	3.729	10.747	2940
10	0.7266	40000	1.000	0.615	0.694	5.572	11.249	3061
11	0.1179	40000	1.000	0.606	0.796	5.550	13.190	3075
12	0.0000	40000	1.000	0.604	0.743	5.522	13.190	3076
13	0.0080	40000	1.000	0.604	0.743	5.522	13.190	3076
14	0.0000	40000	1.000	0.604	0.740	5.521	13.190	3076
15	0.0001	40000	1.000	0.604	0.740	5.521	13.190	3076
16	0.0000	40000	1.000	0.604	0.740	5.521	13.190	3076
17	1.0000	40000	1.000	0.604	0.740	5.521	13.190	3076
18	0.0000	40000	1.000	0.584	0.359	5.320	13.190	3078
19	0.0000	40000	1.000	0.584	0.359	5.320	13.190	3078
20	—	40000	1.000	0.584	0.359	5.320	13.190	3078

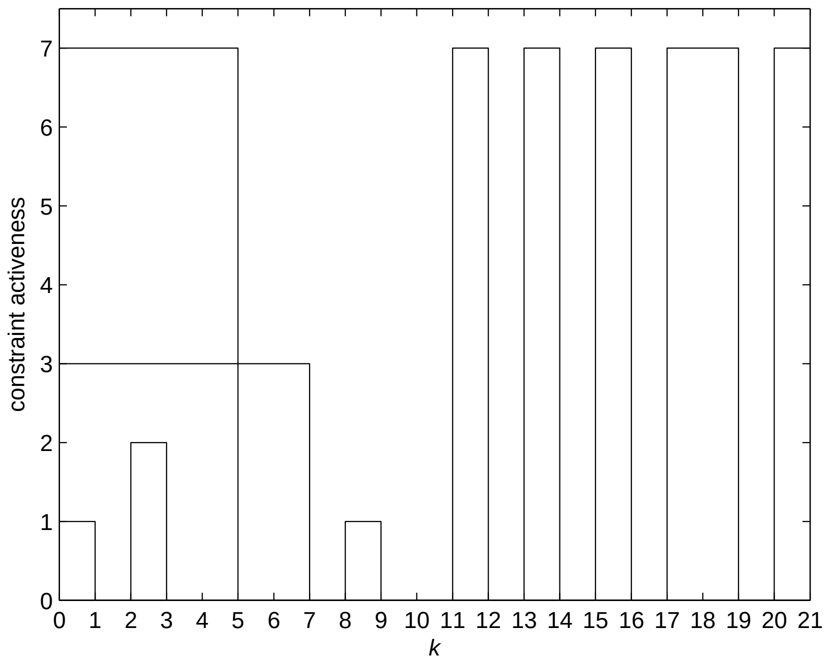

FIGURE 4. The active status of the constraints in Case 4.2 using the SQP algorithm

Algorithm 3.1. In fact, it yields the correct NE solution after 30 iterations. Table 5 and Figure 4 present the computational results of the first 20 iterations.

From the previous two simulation cases, we find that Algorithm 3.1 is better than the SQP algorithm in terms of convergence. In Algorithm 3.1, the initial state is perturbed step-by-step. So, the computational procedure is more stable and is less liable to be trapped into a suboptimal solution. In our other simulations, e.g., in the cases where the initial state is perturbed by $[\delta x(0), \delta y(0)]^\top = \pm 65\%[40000, 1]^\top$, Algorithm 3.1 also does better. In the case with a perturbation $[\delta x(0), \delta y(0)]^\top = -65\%[40000, 1]^\top$, where the active set is enlarged, both algorithms yield the NE solution after 5 iterations; while in the case with a perturbation $[\delta x(0), \delta y(0)]^\top = 65\%[40000, 1]^\top$, where the active set is reduced, Algorithm 3.1 and the SQP algorithm obtain the NE solution after 11 iterations and 16 iterations, respectively. So, the convergence of Algorithm 3.1 is still faster.

In order to accelerate the computation, we can also terminate the iterative when $\alpha^{(k)} = 1$. As stated in Remark 3.7, the solution will be feasible although not locally extremal.

5. **Conclusions.** In this paper, we have developed an iterative approach by using a homotopy to compute the NE solution for a class of optimal switched impulsive control problems with a large initial perturbation and inequality constraints on the switching times and parameters. The example in Section 4 demonstrates that, compared with the SQP, our approach can obtain the NE solutions more quickly even when the active set changes. Further work would be of considerable importance if the iterative procedure can be optimized and the computational efficiency can be increased accordingly.

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Appendix A. **Definition of the matrix Γ^i.** The block terms of the symmetric matrix Γ^i are defined as follows:

$$\Gamma^i_{1,1} \triangleq \psi^{i\top}_{x_i^-} S_{1,1}(t_i^{*+}) \psi^i_{x_i^-} + \bar{\Phi}^i_{x_i^- x_i^-},$$

$$\Gamma^i_{1,3j-1} \triangleq \begin{cases} \psi^{i\top}_{x_i^-} S_{1,1}(t_i^{*+}) \psi^i_{s_i} + \bar{\Phi}^i_{s_i x_i^-}, & \text{if } i = N-j+1, \\ -\psi^{i\top}_{x_i^-} S_{1,3j-1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{1,3j} \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ -\psi^{i\top}_{x_i^-} S_{1,3j}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{1,3j+1} \triangleq \begin{cases} \psi^{i\top}_{x_i^-} S_{1,1}(t_i^{*+}) \dot{\psi}^i, & \text{if } i = N-j+1, \\ -\psi^{i\top}_{x_i^-} S_{1,3j+1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{3j-1,3j-1} \triangleq \begin{cases} \psi^{i\top}_{s_i} S_{1,1}(t_i^{*+}) \psi^i_{s_i} + \bar{\Phi}^i_{s_i s_i}, & \text{if } i = N-j+1, \\ S_{3j-1,3j-1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{3j-1,3j} \triangleq \begin{cases} \tilde{\psi}^{i\top}_{s_i}, & \text{if } i = N-j+1, \\ S_{3j-1,3j}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{3j-1,3j+1} \triangleq \begin{cases} \theta_i + \psi^{i\top}_{s_i} S_{1,1}(t^{*+}_i)\dot{\psi}^i, & \text{if } i = N-j+1, \\ S_{3j-1,3j+1}(t^{*+}_i), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j,3j} \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ S_{3j,3j}(t^{*+}_i), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j,3j+1} \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ S_{3j,3j+1}(t^{*+}_i), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+1,3j+1} \triangleq \begin{cases} \kappa_i + \dot{\psi}^{i\top} S_{1,1}(t^{*+}_i)\dot{\psi}^i, & \text{if } i = N-j+1, \\ S_{3j+1,3j+1}(t^{*+}_i), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l-1} \triangleq \begin{cases} -S^\top_{1,3k+r}(t^{*+}_i)\psi^i_{s_i}, & \text{if } i = N-l+1, \\ S_{3k+r,3l-1}(t^{*+}_i), & \text{if } i = N-l,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l} \triangleq \begin{cases} 0, & \text{if } i = N-l+1, \\ S_{3k+r,3l}(t^{*+}_i), & \text{if } i = N-l,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l+1} \triangleq \begin{cases} -S^\top_{1,3k+r}(t^{*+}_i)\dot{\psi}^i, & \text{if } i = N-l+1, \\ S_{3k+r,3l+1}(t^{*+}_i), & \text{if } i = N-l,\dots,1, \end{cases}$$

where $j = 1,\dots,N-i+1$, $k = 1,\dots,N-i$, $l = k+1,\dots,N-i+1$, and $r = -1,0,1$. Note that block terms $\Gamma^i_{3k+r,3l-1}$, $\Gamma^i_{3k+r,3l}$ and $\Gamma^i_{3k+r,3l+1}$ exist only when $i \le N-1$.