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STUDY ON INPUT-OUTPUT RELATION WITH CHARACTERISTIC OF INCREASING MARGINAL REVENUE USING DEA MODEL

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Abstract. *This paper studies the input-output relation with characteristic of increasing marginal revenue. If traditional DEA model is still used as it is, there will be a fairly large deviation between the front surface of production possibility set determined by the DEA model and the real frontier production function, which makes the results of calculation unreasonable. We discuss the widespread existence of increasing marginal revenue, spell out a kind of method used for identification of increasing marginal revenue, transform the sample set through appropriate mathematic transformation to enable the non-convex real production possibility set to become convex, and on top of this, conduct the evaluation of the relative effectiveness of a unit using traditional DEA models, and solve the real front surface through a mathematic inverse transformation.*

Keywords: Data envelopment analysis, DEA model, Input-output analysis, Increasing marginal revenue, Production possibility set, Frontier production function

1. **Introduction.** Production frontier is an important concept used for the analysis and measurement of production effectiveness. The perceived description of production frontier means the construction of exterior boundary for all the possible combinations from the observed data of the known sample set, and so, the coordinates formed by all the observed data of input-output are located below or just fall over this boundary. When the area below the production frontier is a convex set, this area and its exterior boundary can be described and solved very well using DEA models. It can be seen from the definition above that production frontier is the description of the optimum production behaviors of an economic system, and represents the maximum production capacity. So, this production frontier is the real production function, and it is also called frontier production function. The technical efficiency of production activities can be evaluated to obtain the relative efficiency parameter by comparing the difference between actual production activities and production frontier. The study on the theory of production frontier and its application has been attracting much attention from the economic circles and has become an important research field in recent years.

In the case of only one input element, production function $f(x)$ with increasing marginal revenue satisfies: $f'(x) > 0$, $f''(x) > 0$. When the number of input elements in a production function is more than 1, the corresponding derivative formulas can be written as partial derivative or direction derivative formulas as appropriate. When the issue of increasing or decreasing marginal revenue is reviewed from the viewpoint of production possibility set in a DEA model, the assumed convexity of production possibility set actually means that the marginal revenue is assumed to be invariable or to decrease. The production possibility set, which corresponds to the nature of increasing marginal revenue, does not satisfy the convexity requirement, or in other words, a traditional DEA model is not fit for describing the input-output relation with characteristic of increasing marginal revenue. When there is increasing marginal revenue to a fairly large extent for an issue under investigation, the application of the existing DEA models will be subject to a very large limitation. The result will be made quite unreasonable if the existence of increasing marginal revenue is disregarded and if the DEA model is used at random.

When the front surface of production possibility set has the characteristic of increasing marginal revenue, its corresponding real production possibility set will not satisfy the convexity requirement. The results obtained using the classical DEA model will have a deviation that will be much larger than expected, because the value taken from the front surface determined by the existing DEA model might be much higher than the corresponding value taken from the best unit of the sample in the area with increasing marginal revenue. The result of calculation inevitably causes the relative efficiency parameter of the unit under review to be on the lower side, and thus fails to guarantee the fairness of evaluation result.

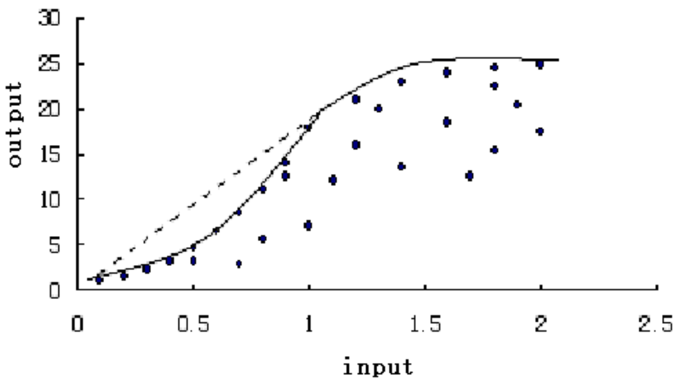

Figure 1. Schematic front surface of sample data with local increasing marginal revenue

As shown in Figure 1 above, when the input variable is taken from area $[0, 1]$ where increasing marginal revenue exists, the real production possibility set determined by the sample set is located in an area below the solid line, but it is very obvious that this area does not satisfy the convexity requirement. When the input variable is taken from area $[0, 1]$, the corresponding dotted line is the front surface determined by the traditional DEA model, and it is obviously irrational to use such a front surface as the criterion for the evaluation of a production unit, because the unit with input in area $[0, 1]$ cannot reach the output level of the so-called front surface represented by the dotted line, and it can reach the front surface represented by the solid line only. In other words, we should obtain the real front surface determined by the sample set in correspondence with the solid line. One of the basic assumptions for DEA model is the assumed convexity of production possibility set [1], which consists of all the convex combinations of samples, and inevitably includes the areas below the dotted line. Therefore, it is evident that the DEA model cannot be used to solve the input-output problem with characteristic of increasing marginal revenue in a reasonable way. How to use sample set and DEA model to study the case in which the real front surface determined by sample set has the characteristic of increasing marginal revenue has not been found yet, which is the main course of this paper.

If we have known the production function, this production function is also the effective front of production, and the real production possibility set consists of the areas below the production function. What we know for an actual application is just a sample set, and the production function is what we want to obtain. And when the number of input elements is more than 2, it is very difficult to review the corresponding effective front in a direct geometric way. If a single interconnected set T can be used to tightly envelop the sample set, and the effective front of set T can be easily mathematically solved, we can then take the effective front of set T as the production function determined by the sample set. When set T cannot tightly envelop the sample set, a deviation may be caused by doing so, the calculation result is inevitably not reasonable at this case, and this kind of irrationality cannot be easily noticed. Of course, if most of the real production possibility sets determined by the sample sets of actual applications are convex sets, the classical DEA models are ideal models. If the characteristic of increasing marginal revenue is widespread existence, the problem will become very conspicuous because a non-convex set cannot be expressed using a set of linear inequalities, and so, the problem cannot be solved reasonable using linear programming models. Therefore, we must study new models, and try to find the real solution to the problem, and of course, we cannot use the traditional DEA model at random. Current DEA model is a fairly complete theoretical system used for mathematic analysis, and much experience has been gained from actual applications. It is a natural and logic choice to convert the input-output problem with characteristic of increasing marginal revenue into a problem solvable using the traditional DEA model through some sort of mathematic transformation.

The remaining portion of this paper is generally arranged as follows: Section 2 describes the widespread existence of increasing marginal revenue; Section 3 is the mathematic description of the characteristics of front surface of production possibility set with characteristic of increasing marginal revenue; Section 4 spells out a kind of algorithms used for identification of production possibility set with increasing marginal revenue; Section 5 presents the principles behind the mathematic transformation of production possibility set; Section 6 gives the method to determine the suitable transformation function; Section 7 tells how to establish the transformation function using sample set; Section 8 gives the solution to input-output problem with characteristic of increasing marginal revenue for an actual application problem; and the final part is the conclusions.

2. **Widespread Existence of Characteristic of Increasing Marginal Revenue.** It is already indicated through analysis of a huge amount of statistics in microeconomics that as the input scale varies, the output level of an enterprise exhibits a comparatively fixed characteristic of variation, i.e., when the input is relatively lower, the output with respect to input exhibits the characteristic of increasing marginal revenue; when the input level is greater than a certain value, the output with respect to input exhibits the characteristic of decreasing marginal revenue; when the input level is something around the joint between increasing marginal revenue and decreasing marginal revenue, the corresponding input scale is the rational input scale, the marginal product exhibits the characteristic of remaining generally invariable at this case. This characteristic of variation is of universal significance. When we conduct the evaluation of a number of enterprises in a trade for their input-output efficiency, it is very difficult to avoid the problem of increasing marginal revenue. It is quite often to find such enterprises, although they know how much is their optimum input scale, they have not reached their optimum input scale due to market demand or financing capability, or in other words, their input scale is at the level of increasing marginal revenue. Enterprises with relatively small assets scale are usually in the area with increasing marginal revenue, if the existence of increasing marginal revenue is disregarded in specific evaluation problems, especially when we use DEA models

as mathematic tools, the evaluation results obtained with small enterprises will deviate from our expectancy or have serious irrationality hidden.

DEA models have been widely used for evaluation of different accomplishment and efficiency. Some increasing marginal revenue problems have been encountered during the actual applications for evaluation of accomplishment and efficiency. We would like to use the scientific research evaluation problem as an example to illustrate the problems associated with the increasing marginal revenue. According to the thought line of input-output analysis, the scientific research evaluation parameters are usually divided into two categories: accomplishment and efficiency parameters (output parameters) and input parameters. The input parameters generally include the number of research personnel, research fund and investment in research equipment, etc. The accomplishment and efficiency parameters include economic performance parameters and social effect parameters: economic performance parameters mainly means revenue from transfer of scientific achievement, revenue from scientific service, and reduction in cost resulting from the use of scientific achievement, etc., and social effect mainly includes the publication of research papers, research reports, patents, monographs, and training of personnel, etc. The evaluation of scientific research accomplishment and efficiency cannot be that simple as the evaluation of enterprises. For enterprises, their outputs can be easily measured using the quantity of their outputs or value of their outputs, but the accomplishment and efficiency of scientific research institutions cannot be easily measured using values. The output of a scientific research institution is usually measured using multiple-parameter comprehensive evaluation method, to be more specific, first establish a weight for each output parameter using some sort of method, and then obtain the weighted sum of the output parameter to obtain the value of accomplishment and efficiency. Obviously, the issue of whether there is increasing marginal revenue in the evaluation of scientific research is closely related to the establishment of weight of output parameters. For the same sample, different weighting schemes can be established for different characteristics of marginal revenue.

Let us consider the scientific research accomplishment and efficiency problem with a number of input parameters and two output parameters. The input parameters include research fund, number of researchers and the value of laboratory equipment; two output parameters are the numbers of ordinary and high-level research papers respectively. For some experimental research fields, when the input of research fund fails to reach a certain level, it is very difficult to fulfill a high-level scientific research accomplishment due to the limitation of experimental conditions, and it is even impossible in some research fields. Only when the input of research fund reaches a certain level, it is possible to accomplish high-level scientific research, for example, the experimental study on high-energy physics, and high-precision cosmos observation and study. When the input of research fund fails to reach a certain level, the scientific accomplishment of a scientific research institution are mainly at a lower level (although it may be more in quantity), and so, the research papers published are mainly ordinary. When the input of research fund reaches a certain level, the scientific achievements of the scientific research institution include both ordinary and high-level research papers, and more high-level research papers are included in these research papers. Since weight is subjectively set for ordinary and high-level research papers, increasing marginal revenue inevitably occurs when weight is set large enough for high-level research papers. To set an unusually large weight for high-level research papers and high-level scientific achievements are also the special feature of scientific research evaluation in recent years, which reflects the policy guidance of the scientific research management department. Different from the special feature of the marginal revenue of enterprises, it is very likely the relation between the accomplishment and efficiency and the input scale of scientific research institutions exhibits the following special feature:

the marginal revenue remain invariable or decreasing when the input scale is relatively smaller, and there is increasing marginal revenue after the input scale becomes larger enough.

3. **Special Feature of Input-Output Relation with Characteristic of Increasing Marginal Revenue.** For production function $y = f(x)$ with single input x and single output y, first order derivative $y' = f'(x)$ is the marginal revenue, and in the significant range of input x, production function $f(x)$ satisfies the requirement of $f'(x) > 0$. It means decreasing marginal revenue when $f''(x) \le 0$, and it means increasing marginal revenue when $f''(x) > 0$.

Definition 3.1. *If $f''(x) > 0$ holds in the significant range of input x, it is claimed that the production function has a global characteristic of increasing marginal revenue; if $f''(x) > 0$ holds at some x, it is claimed that the production function has a local characteristic of increasing marginal revenue.*

Definition 3.2. *For production function $y = f(X)$ with multiple input X and single output y, we use set Ω to represent the significant range of input vector $X = (x_1, x_2, \dots, x_n)$, let $X = (x_1, x_2, \dots, x_n)$ vary in direction $\mathbf{l} = (l_1, l_2, \dots, l_n) \ge 0$. Of course, $\frac{df(X)}{d\mathbf{l}} \ge 0$ holds. If $\frac{d^2 f(X)}{d\mathbf{l}^2} \ge 0$ holds, it is claimed that the production function has the characteristic of increasing marginal revenue; if $\frac{d^2 f(X)}{d\mathbf{l}^2} \le 0$ holds for arbitrary $X \in \Omega$ and arbitrary direction $\mathbf{l} = (l_1, l_2, \dots, l_n) \ge 0$, it is claimed that the production function has the characteristic of decreasing marginal revenue.*

In the DEA theory, it is assumed that the sample set is (X_i, Y_i), $i = 1, 2, \dots, N$. X_i is n-dimensional input vector, and Y_i is m-dimensional output vector, the production possibility set formed by sample set under rational assumptions is defined as T, and the following expression holds:

$$T = \left\{ (X, Y) \left| \sum_{j=1}^{N} \lambda_j X_j \le X, \sum_{j=1}^{N} \lambda_j Y_j \ge Y, \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \ge 0 \right. \right\} \tag{1}$$

Set T is a convex set, because T is defined by the solution of a set of linear inequalities. In order to discriminate from T, we define the concept of real production possibility set for the general case.

Definition 3.3. *Assuming the frontier production function with n-dimensional input $X = (x_1, x_2, \dots, x_n)$ and m-dimensional output $Y = (y_1, y_2, \dots, y_m)$ determined by sample set (X_i, Y_i), $i = 1, 2, \dots, N$, are describes as follows:*

$$y_i = f_i(x_1, x_2, \dots, x_n), \quad 1 \le i \le m, \quad X = (x_1, x_2, \dots, x_n) \in \Omega$$

Ω is a convex set. Set A is called the real production possibility set determined by the production functions above:

$$A = \{(X, Z) \,|X \in \Omega\, , Z = (z_1, z_2, \dots, z_m), 0 \le z_i \le f_i(X), i = 1, 2, \dots, m\} \tag{2}$$

If we have known the frontier production functions, the real production possibility set A are formed by the area below the frontier production function. What we know about an actual application is just a sample set, and the frontier production function is unknown. What is more, when the number of input elements is more than 2, it is very difficult to review the corresponding front surface through the geometric direct way. If it is possible to use a single interconnected set T to envelop the sample set tightly, and the front surface of this set T can be easily solved mathematically, we can take the front surface of T as the frontier production function determined by the sample set. When set

T cannot tightly envelop the sample set, a deviation will be caused by doing so. It can be seen from Figure 1 that when there is an obvious characteristic of increasing marginal revenue in the input-output relation, production possibility set T is much larger than the real production possibility set A, and the use of DEA model is obviously unreasonable at this case.

If we do not know the production functions, we have no way to determine the real production possibility set A. By definition of set A above, when the distribution of sample is dense enough, set A is the minimum single interconnected set including all the sample point (X_i, Y_i). Generally speaking, set T is larger than set A in common, only when $y_i = f_i(x_1, x_2, \ldots, x_n)$ are concave functions to all i, T and A are approximately the same. We have the following two theorems for the relation between T and A.

Theorem 3.1. *Assuming production function $y = f(x)$ is an univariate function, and $y = f(x)$ has second order continuous derivatives, $y = f(x)$ satisfies $f(a) > 0$, $f'(x) > 0$ in interval $[a, b]$, and $f''(x) > 0$ holds at some points, then plane area A formed by curves $y = f(x)$, $x = a$, $x = b$, $y = 0$ is not a convex set, where*

$$A = \{(x, y) \,|a \le x \le b, 0 \le y \le f(x)\}$$

Set A is the real production possibility set.

Proof: From the definition of convex set, all the point on the line connecting arbitrary two points within the convex set still belong to this set. Assuming $f''(x_1) > 0$ holds where $x_1 < b$, due to the continuity of $f''(x)$, certainly $\delta > 0$ holds, and so, $f''(x) > 0$ holds when $x \in [x_1, x_1 + \delta]$. We know, the following is the sufficient and necessary condition for function $y = f(x)$ being a second order differentiable convex function around x_1: to arbitrary $x \in [x_1, x_1 + \delta]$, $f(x_2) \ge f(x_1) + f'(x_1)(x_2 - x_1)$ holds.

We conduct Taylor expansion of function $y = f(x)$ at point x_1, and for $x_2 \in [x_1, x_1 + \delta]$, the following holds:

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + \frac{1}{2} f''(\xi)(x_2 - x_1)^2$$

$\xi \in [x_1, x_1 + \delta]$, according to the condition for theorem, $f''(\xi) > 0$, the following expression holds:

$$f(x_2) \ge f(x_1) + f'(x_1)(x_2 - x_1)$$

So, function $y = f(x)$ is a strict convex function around x_1. A strict convex function satisfies the following properties:

$$f\left(\frac{x_1 + (x_1 + \delta)}{2}\right) < \frac{1}{2}(f(x_1) + f(x_1 + \delta))$$

Check two points $(x_1, f(x_1))$ and $(x_1 + \delta, f(x_1 + \delta))$ in set A, and the middle point of line connecting these two points is

$$\left(\frac{x_1 + (x_1 + \delta)}{2}, \frac{1}{2}(f(x_1) + f(x_1 + \delta))\right)$$

From the definition of production possibility set and the inequality derived above, point $\left(\frac{x_1 + (x_1 + \delta)}{2}, \frac{1}{2}(f(x_1) + f(x_1 + \delta))\right)$ is not in set A. This indicates that set A is not in compliance with the definition of convex set. The proof of theorem finishes.

Theorem 3.2. *Assuming n element production function is $y = f(X) = f(x_1, x_2, \ldots, x_n)$, $X = (x_1, x_2, \ldots, x_n) \in \Omega$, and Ω is a n-dimensional convex set, and satisfies $f(x_1, x_2, \ldots, x_n) > 0$, to arbitrary direction $\mathbf{l} = (l_1, l_2, \ldots, l_n) \ge 0$, $\frac{df(X)}{d\mathbf{l}} > 0$ holds. If in place where*

$X_0 = (x_1^0, x_2^0, \dots, x_n^0) \in \Omega$, and for some direction $\mathbf{l}_0 = (l_1^0, l_2^0, \dots, l_n^0) \geq 0$, $\frac{d^2 f(X_0)}{d\mathbf{l}_0^2} > 0$ holds, set A is formed as follows:

$$A = \{(X; y)\,|X \in \Omega; 0 \leq y \leq f(X)\}$$

Set A is then not a convex set.

Proof: It is noticed that when $y = f(X)$ varies at point $X_0 = (x_1^0, x_2^0, \dots, x_n^0) \in \Omega$ along direction $\mathbf{l}_0 = (l_1^0, l_2^0, \dots, l_n^0) \geq 0$, $y = f(X)$ becomes an univariate function. The condition of theorem indicates the second order derivative of this univariate function around X_0 is greater than 0. According to Theorem 3.1, $y = f(X)$ around $t = 0$ on the straight line $X_0 + t\mathbf{l}_0$ is a convex function. The area below the corresponding curve is not a convex set, i.e., set $B = \{(X, y)\,|X = X_0 + t\mathbf{l}_0\,, t > 0, 0 \leq y \leq f(X)\}$ is not a convex set, but set A can be taken as the merging set of set B when $\mathbf{l}_0 = (l_1^0, l_2^0, \dots, l_n^0)$ has taken all the directions. So, set A certainly is not a convex set. The proof of theorem finishes.

The form of DEA model is a special linear programming, and the feasible set formed by linear restriction (production possibility set) certainly is a convex set. We know from Theorem 3.1 and Theorem 3.2 in this section that if the frontier production function has a certain type of characteristic of increasing marginal revenue, the real production possibility set A is not a convex set. The convex set containing real production possibility set is certainly larger than A. Thereby causing many input-output combinations, which are not possible to reach in reality, to be included in the production possibility set. A typical error is to judge the input-output combinations over the real front surface as invalid, thereby causing judgment errors in actual evaluation process. The intention of DEA model is to use a set which can tightly envelop the sample set to represent the production possibility set, and build up a linear programming model on the basis of the production possibility set. It can be seen from the analysis above that when the input-output relation has some sort of characteristic of increasing marginal revenue, the real production possibility set A is not a convex set. It may be too rough to use set T to envelop this real production possibility set A, thereby including many areas which should not be included.

When the problem we deal with have multiple input and multiple output, there is no way to review the characteristic of increasing marginal revenue of input-output relation in a geometric way, and it is not clear either whether the characteristic of increasing marginal revenue exists. The random use of DEA model may cause the excessive expansion of production possibility set in comparison with the real production possibility set, and this expanded portion may have already exceeded the possible input-output combination. Let the production unit be compared with the input-output combination which is totally impossible to achieve, and evaluate the input and output efficiency of the production unit in this way, this practice undoubtedly is very absurd. The key to the question is that we can not arbitrarily assume the widespread existence of the characteristic of decreasing marginal revenue, especially when we fairly arbitrarily select some input-output parameters for evaluation, we have no way to make sure the input-output relation has an global characteristic of decreasing marginal revenue, it is very dangerous to use DEA model at this case.

4. **A Kind of Methods Used for Identification of Production Possibility Set with Characteristic of Increasing Marginal Revenue.** Assuming sample set is (X_i, Y_i), $i = 1, 2, \dots, N$. X_i is a n-dimensional input vector, Y_i is a m-dimensional output

vector, and the following is an input-oriented DEA model:

$$\min \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{N} \lambda_j X_j \leq \theta X_{j_0} \\ \sum_{j=1}^{N} \lambda_j Y_j \geq Y_{j_0} \\ \sum_{j=1}^{N} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, 2, \ldots, N \end{cases} \tag{3}$$

Solution θ of model (3) is the relative validity parameter of sample (X_{j_0}, Y_{j_0}). If $\theta = 1$, sample (X_{j_0}, Y_{j_0}) is on the front surface, and if $\theta < 1$, it means that the sample (X_{j_0}, Y_{j_0}) is not on the front surface.

Output-oriented DEA model:

$$\max \rho$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{N} \lambda_j X_j \leq X_{j_0} \\ \sum_{j=1}^{N} \lambda_j Y_j \geq \rho Y_{j_0} \\ \sum_{j=1}^{N} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, 2, ..., N \end{cases} \tag{4}$$

Solution ρ of model (4) is the relative validity parameter of sample (X_{j_0}, Y_{j_0}). If $\rho = 1$, sample (X_{j_0}, Y_{j_0}) is on the front surface; and if $\rho > 1$, sample (X_{j_0}, Y_{j_0}) is not on the front surface. These two models above are simple in form and easy to use, and have a clear economic significance, so these DEA models of envelope type have found wide applications in reality.

In the case of single input and single output, watching the scattering point diagram can easily see the shape of real production frontier. If a segment of the front curve in the middle exhibits a progressive increase in slope, a judgment can be made with respect to the characteristic of increasing marginal revenue of the input-output relation, for example, the case of input in zone $[0, 1]$ as shown in Figure 1. In the case of two input elements, it is fairly difficult to judge whether there is increasing marginal revenue in the production frontier surface just by watching the scattering point diagram. When the number of input variables is more than 2, it is very difficult to judge whether the production possibility set has the characteristic of increasing marginal revenue.

The method proposed in this paper is the use of traditional DEA models to solve the front surface and to judge if the production possibility set has the characteristic of increasing marginal revenue by checking the distribution of sample points on the front surface. When the corresponding input-output relation has the characteristic of decreasing marginal revenue, its production possibility set satisfies the convexity requirement, and conforms to the conditions of traditional DEA models. Assuming the capacity of a sample set is large enough, according to the geometric direct significance of the front surface of production possibility set, there should be a number of sample points on the front surface or very close to the front surface at different input levels. When the corresponding input-output relation has the characteristic of increasing marginal revenue, there will be no sample points on the front surface or close to the front surface in some input scales. The analysis above provides us with a method, by which the existence of the characteristic of increasing marginal revenue can be judged by the calculating results of the traditional DEA models.

Assuming the input-output relation under study has two input variables and one output variable, and they are the total amount of fixed assets, labor force input (in number of persons) and output of enterprise. Tabulated in Table 1 below are the fictions primary sample data of 30 enterprises of an industry in 2010, these data are of course rational data in compliance with their economic significance.

Table 1. Primary sample data of 30 enterprises of an industry in 2010

No. of enterprise	Total amount of fixed assets (in 10k RMB)	Labor force input (in number of persons)	Annual output of enterprise (in 10k RMB)	No. of enterprise	Total amount of fixed assets (in 10k RMB)	Labor force input (in number of persons)	Annual output of enterprise (in 10k RMB)
1	100	19	40	16	790	49	220
2	205	24	60	17	1200	66	640
3	295	28	88	18	1605	83	740
4	380	32	128	19	2000	100	700
5	400	36	188	20	1820	92	620
6	595	41	260	21	1410	75	540
7	700	45	340	22	1000	58	280
8	810	49	440	23	1950	96	820
9	920	53	560	24	500	36	128
10	1050	58	720	25	1110	62	480
11	1210	66	840	26	1750	87	500
12	1420	75	920	27	1300	70	800
13	1590	83	960	28	1760	89	900
14	1790	92	980	29	700	45	112
15	2000	100	992	30	900	53	500

Traditional DEA models (3) and (4) are used to find the solution, and check through analysis to see if the production possibility set has the characteristic of increasing marginal revenue. In regard to the calculation result obtained using model (3), if the relative validity parameter of sample is greater than or equal to 0.9, this sample will then be considered to be close to the front surface, otherwise, this sample will not be considered to be close to the front surface. In regard to the calculation result obtained using model (4), if the relative validity parameter of sample is less than or equal to 1.1, this sample will be considered to be close to the front surface, otherwise, this sample will be considered not close to the front surface. In regard to a particular sample, if one of the calculation results obtained using models (3) and (4) is close to the font surface, we think the sample is close to the front surface.

Traditional DEA models (3) and (4) are used to find the solution, and the calculation results obtained are tabulated in Table 2 below. It can be seen from Table 2 that when the total amount of fixed assets is between 1.0 million and 9.2 million RMB, no sample is on the front surface, which means the existence of the characteristic of increasing marginal revenue.

We have the following two identification methods.

1) Major variable division statistics method: a major input variable, for example, the total amount of fixed assets in the example above, is chosen as X_{major}. According to the capacity of sample and the special features of the problem, we divide the value taking division into a number of subdivisions. These subdivisions may be of equal length. We use traditional DEA models to find the solution, and count the number of sample point

TABLE 2. Relative validity results obtained using DEA models (3) and (4)

No. of enterprise	Calculation result obtained using model (3) and (4)		If close to front surface	No. of enterprise	Calculation result obtained using model (3) and (4)		If close to front surface
	Input-oriented relative validity parameter θ	Output oriented relative validity parameter ρ			Input-oriented relative validity parameter θ	Output oriented relative validity parameter ρ	
1	1.0	1.0	Yes	16	0.598	2.442	No
2	0.839	1.928	No	17	0.809	1.301	No
3	0.777	2.052	No	18	0.715	1.297	No
4	0.751	1.889	No	19	0.569	1.417	No
5	0.766	1.363	No	20	0.568	1.581	No
6	0.771	1.526	No	21	0.636	1.697	No
7	0.801	1.389	No	22	0.565	2.459	No
8	0.856	1.253	No	23	0.674	1.202	No
9	0.921	1.122	Yes	24	0.668	2.565	No
10	1.0	1.0	Yes	25	0.713	1.599	No
11	1.0	1.0	Yes	26	0.522	1.938	No
12	1.0	1.0	Yes	27	0.904	1.093	Yes
13	1.0	1.0	Yes	28	0.817	1.081	No
14	1.0	1.0	Yes	29	0.514	4.218	No
15	1.0	1.0	Yes	30	0.856	1.233	No

close or not close to front surface with each subdivision, and check to see in this way if the production possibility set has the characteristic of increasing marginal revenue.

2) Input scale classification statistics method: Use the input parameter to build up a comprehensive index Z_{major} to reflect the input scale. Following the example of major variable division statistics method, we can check through analysis to see if the production possibility set has the characteristic of increasing marginal revenue.

5. **Theory behind Mathematic Transformation of Production Possibility Set.** In order to study the front surface of sample data with the characteristic of increasing marginal revenue, we chose the way of mathematic transformation, and try to convert a production possibility set without convexity into a production possibility set satisfying the convexity requirement, thereby enabling the existing DEA models to be used to solve the relative validity parameter of each unit under evaluation, and so, the front surface of real production possibility set can be solved through mathematic inverse transformation.

To describe in mathematic language, for given sample set B, $B = \{(x_i, y_i)|i = 1, 2, \cdots, N\}$, assuming the front surface determined by the given sample set has the characteristic of increasing marginal revenue, we search for a mathematic transformation function to enable the corresponding production possibility set of transformed sample set $B_G = \{(x_i, G(x_i)y_i)|i = 1, 2, \cdots, N\}$ to satisfy the convexity requirement, meanwhile, the mathematic transformation $G(x)$ satisfies some limitation conditions as well. We use the traditional DEA models to process the transformed sample set B_G, solve the relative validity parameter of each unit, and use inverse transformation $1/G(x)$ to solve the real front surface of the original problem. It can be seen from the definition of transformation above that transform means the different outputs of the same input x multiplied by the same value $G(x)$. For an output-oriented DEA model, such a mathematic transformation will not change the result of relative validity evaluation. For an input-oriented

DEA model, the change resulting from such a mathematic transformation in the result of relative validity evaluation will be very small.

The following are the rational requirements for transformation $G(x)$: 1) The production possibility set of the transformed sample set $B_G = \{(x_i, G(x_i)y_i)|i = 1, 2, \cdots, N\}$ satisfies the convexity requirement, so that the traditional DEA models can be used to find the solution; 2) The transformation function $G(x)$ must be as smooth as possible and satisfies $G(x) \geq 1$, so that the production possibility set can uniformly expand outwards in the zone with increasing marginal revenue to make the magnitude of deformation as small as possible while the convexity requirement is satisfied; 3) The transformation value at the maximum and minimum of input element x is 1. With regard to mathematic transformation $G(x)$, we have the following theorem.

Theorem 5.1. *If univariate function $f(x)$ over $[a, b]$ satisfies:*

$$f(x) > 0, \quad df(x)/dx > 0, \quad d^2f(x)/dx^2 > 0$$

Then the transformation function $G(x)$ exist over $[a, b]$, and $G(x)$ satisfies:

$$d(G(x)f(x))/dx > 0, \quad d^2(G(x)f(x))/dx^2 \leq 0, \quad G(a) = G(b) = 1$$

Proof: It is known from the condition given for the theorem that function $f(x)$ is a convex function with differentiable and monotonic increasing, and so, the following holds for $0 < \lambda < 1$:

$$f(\lambda a + (1 - \lambda)b) < \lambda f(a) + (1 - \lambda)f(b)$$

Assuming $x = \lambda a + (1 - \lambda)b$, then

$$\lambda = \frac{x - b}{a - b}, \quad 1 - \lambda = \frac{a - x}{a - b}$$

From the expression above, we have:

$$f(x) < \frac{f(a)}{(a - b)}(x - b) + \frac{f(b)}{(a - b)}(a - x)$$

Take

$$G(x) = \frac{\frac{f(a)}{(a-b)}(x - b) + \frac{f(b)}{(a-b)}(a - x)}{f(x)}$$

Obviously, $G(x)f(x)$ is a linear function with monotonic increasing over $[a, b]$, we have:

$$d(G(x)f(x))/dx = (f(b) - f(a))/(b - a) > 0$$

$$d^2(G(x)f(x))/dx^2 = 0, \quad G(a) = G(b) = 1$$

The proof of theorem finishes.

To review the existence of mathematic transformation $G(x)$ from the geometric direct viewpoint and on the basis of the sample set provided for the problem, we try to obtain the front surface using traditional DEA models, the output on this front surface is the upper limit of the output corresponding to the input. For the input-output problem with the characteristic of decreasing marginal revenue, the output on the front surface is achievable; but for the input-output problem with characteristic of increasing marginal revenue, the output on the front surface is not achievable in the area with the increasing marginal revenue.

We use $y(x)$ to represent the calculated output on the front surface obtained using output-oriented DEA model when the input vector is x, and we use $f(x)$ to represent the maximum output when the input vector is x. When $G(x) = y(x)/f(x)$, $G(x)$ is an ideal transformation function, because when (x, y) is on the real front surface, $y = f(x)$ hold, and the following holds at this case:

$$G(x)y = \frac{y(x)}{f(x)}f(x) = y(x)$$

When (x, y) is not on the real front surface, $y < f(x)$ holds, and the following holds at this case:

$$G(x)y < y(x)$$

Obviously, $G(x)$ transforms the real front surface into the front surface determined by the output-oriented DEA model. As the sample set is given in a limited scattering way, it is not an easy job to determine the maximum output $f(x)$, which corresponds to each input vector x during actual applications, and so, the determination of $G(x)$ is not easy, either.

6. **Methods to Determine the Suitable Transformation Function.** By formulating a series of transformation functions for selection, transform the sample set, use traditional DEA models to find the solution, use the way described in Section 4 to judge whether the transformation function taken are appropriate, and finally choose by sieving the most satisfactory transformation function. Next, we talk about how to obtain the transformation function separately when production possibility set has the characteristic of global or local increasing marginal revenue.

For a production possibility set with the characteristic of global increasing marginal revenue, it is assumed that the value of input variable is taken in interval $[a, b]$, both ends of the range are on the front surface. A class of parabolic functions is taken so that its value taken at the end of $[a, b]$ is 1, and it is parabola with downward opening, and it has different curves. The expression of transformation can be obtained through simple derivation:

$$f(x, \beta) = \beta x^2 - \beta(a + b)x + (1 + \beta ab) \tag{5}$$

where x is the main input variable X_{major} or the comprehensive index Z_{major}. It is not difficult to prove: $f(a, \beta) = 1$, and $f(b, \beta) = 1$, for parameter $\beta < 0$, $f(x, \beta)$ is a class of parabolic functions with downward opening. Choose a parameter $\beta < 0$, transform each sample point (x_i, y_i) in the sample set, and obtain the transformed sample set $(x_i, f(x_i, \beta)y_i)$, $i = 1, 2, \cdots, n$. We use traditional DEA models to solve the transformed sample set, and judge whether there is increasing marginal revenue in the way described in Section 4. If there is no increasing marginal revenue, this means an appropriate transformation has been found. It can be seen from the expression of transformation that the curve (transformation function) becomes smoother as the absolute value of β decreases. So, it is logic to choose a transformation curve with a smaller absolute value of β for appropriate transformation. Shown in Figure 2 are the transformation curve shapes of Formula (5) with regard to different β, where $a = 1$, $b = 10$.

For a production possibility set with the characteristic of local increasing marginal revenue, it is assumed that the value of input variable is taken from interval $[a, c]$, the ends of the zone are both on the front surface, and point b is an inflexion point on the front surface, i.e., there is a decreasing marginal revenue over $[b, c]$. The parabola functions at this case cannot satisfy our requirement very well. A class of 3rd order polynomial functions is taken to built the transformation function, the requirements is: its value taken at either end of $[a, b]$ is 1, it is a curve with downward opening over $[a, b]$, it has different curves and the derivative of the curve at point b is required to be 0. The following expression of transformation function can be obtained through simple derivation:

$$f(x, \beta) = \beta x^3 - \beta(a + 2b)x^2 + \beta(b^2 + 2ab)x + (1 - \beta ab^2) \tag{6}$$

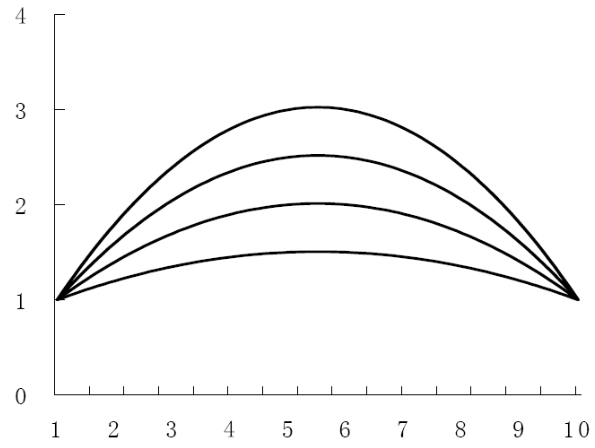

FIGURE 2. Transformation curve shapes of Formula (5) with regard to different β

It is required to take $f(x, \beta) = 1$ over $[b, c]$. It is not difficult to proof: $f(a, \beta) = 1$, $f(b, \beta) = 1$, $df(x, \beta)/dx\,|_{x=b} = 0$, and for $\beta > 0$, $f(x, \beta)$ is a class of 3rd order curves with downward opening, and $d^2 f(x, \beta)/dx^2 < 0$. Choose a parameter $\beta > 0$, transform each sample point (x_i, y_i) in the sample set, and obtain the transformed sample set $(x_i, f(x_i, \beta)y_i)$, $i = 1, 2, \cdots, N$. Shown in Figure 3 below is the transformation curve shapes of Formula (6) with regard to different β, where a=1 and b=10.

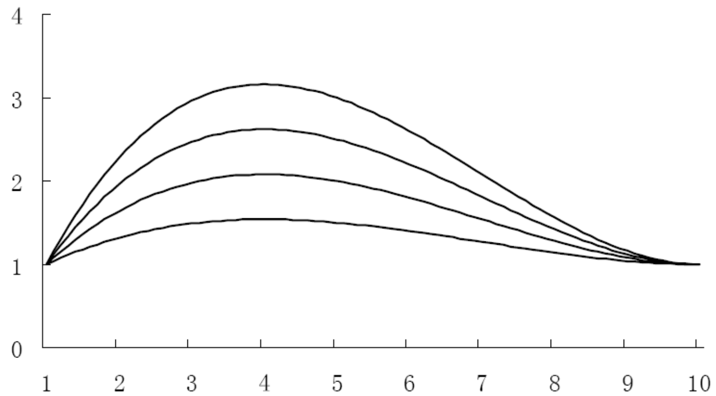

FIGURE 3. Transformation curve shapes of Formula (6) with regard to different β

7. **Obtaining Transformation Function for Sample Set with Characteristic of Increasing Marginal Revenue.** In this section, we will give a calculation example to illustrate how to obtain a transformation function. Tabulated in Table 3 is sample data with one input variable and one output variable. Figure 4 is the scattering diagram of input-output data tabulated in Table 3, which indicates the input-output relation has the characteristic of local increasing marginal revenue. The method suggested in this paper is used to find an appropriate 3rd order polynomial function. The value of β parameter obtained through several trial calculations is 7.5. For the result of transformation, see column 4-5 and column 9-10 of Table 3. Figure 5 is the scattering distribution diagram of transformed sample data.

Table 3. Obtaining transformation function and corresponding calculation results

Sample No.	Input variable	Output variable	Value of transformation function	Output variable after transformation	Sample No.	Input variable	Output variable	Value of transformation function	Output variable after transformation
1	0.1	1.0	1.61	1.61	16	0.8	5.5	1.24	6.82
2	0.2	1.5	1.96	2.94	17	1.2	16.0	1.00	16.00
3	0.3	2.2	2.10	4.63	18	1.6	18.5	1.00	18.50
4	0.4	3.2	2.08	6.66	19	2.0	17.5	1.00	17.50
5	0.5	4.7	1.94	9.11	20	1.8	15.5	1.00	15.50
6	0.6	6.5	1.72	11.18	21	1.4	13.5	1.00	13.50
7	0.7	8.5	1.47	12.52	22	1.0	7.0	1.00	7.00
8	0.8	11.0	1.24	13.64	23	1.9	20.5	1.00	20.50
9	0.9	14.0	1.07	14.95	24	0.5	3.2	1.94	6.20
10	1.0	18.0	1.00	18.00	25	1.1	12.0	1.00	12.00
11	1.2	21.0	1.00	21.00	26	1.7	12.5	1.00	12.50
12	1.4	23.0	1.00	23.00	27	1.3	20.0	1.00	20.00
13	1.6	24.0	1.00	24.00	28	1.8	22.5	1.00	22.50
14	1.8	24.5	1.00	24.50	29	0.7	2.8	1.47	4.12
15	2.0	24.8	1.00	24.80	30	0.9	12.5	1.07	13.34

Parameters related to transformation function: $a = 0$, $b = 1$, $\beta = 7.5$, and the following holds:

$$f(x, \beta) = \begin{cases} 7.5x^3 - 15x^2 + 7.5x + 1, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \end{cases}$$

It can be seen by comparing Figure 4 with Figure 5 that after being transformed using the transformation function chosen, the characteristic of local increasing marginal revenue of the sample set has been basically eliminated, and its front surface has become global decreasing marginal revenue.

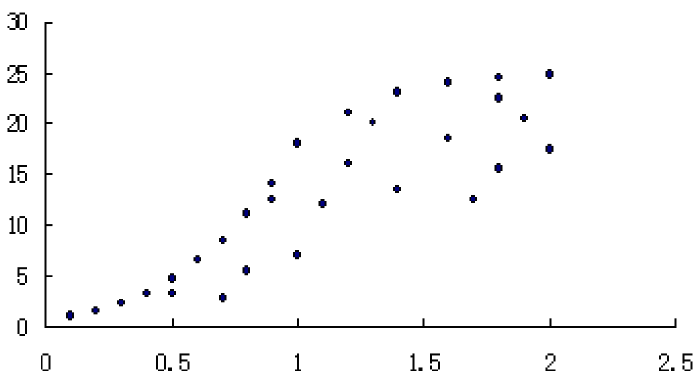

Figure 4. Front surface of sample data with local increasing marginal revenue

8. **Solution of Model Through Transformation of Sample Set.** Using the data listed in Table 1, we first use model (4) to find the solution to the problem, and give the results in the 5th and 6th columns of Table 4. The characteristic of increasing marginal revenue exists when the total amount of fixed assets is between one million RMB and nine million RMB. Through several trial calculations, we chose the following 3rd order

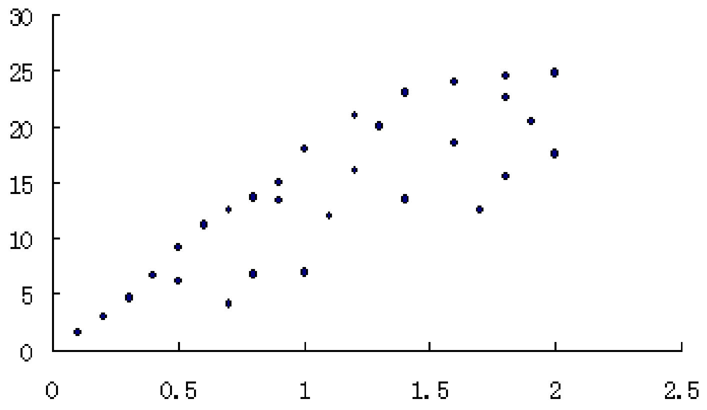

FIGURE 5. Front surface of sample data after transformation

multinomial as transformation function $G(x)$, major variable x is the total amount of fixed assets, $\beta = 9.467 \times 10^{-9}$ and the following holds:

$$G(x) = \begin{cases} 9.467 \times 10^{-9}x^3 - 1.988 \times 10^{-5}x^2 + 1.136 \times 10^{-2}x + 0.0533, & 0 \le x \le 950 \\ 1.0, & 950 \le x \le 2100 \end{cases}$$

The output of an enterprise is obtained by transforming the sample data in Table 1 using the function given above. The calculation result of enterprise output after transformation is listed in column 7 of Table 4. The longitudinal column of Table 4 is arranged in the order of magnitude from small to big of the total amount of fixed assets.

The transformed problem is solved using model (4), and the calculation results are given in the 8th and 9th columns of Table 4. These results indicate that after the sample set is transformed using the transformation function chosen, it can be guaranteed that there are always samples close to the front surface on different scales (spacing of 3 million RMB), and the corresponding production possibility set is a convex set. Therefore, the 8th column of Table 4 is the more rational index for the relative validity evaluation. Using model (4) for No. 5 enterprise, the corresponding relative validity value is 1.363, far away from front surface. However, after transformation, the calculation result is 1.0, which indicates No. 5 enterprise is actually on the front surface. As another example, the relative validity value of No. 3 enterprise obtained using model (4) is 2.052, far from the front surface, however, after transformation, the calculation result is 1.305, which indicates No. 3 enterprise is not far from the front surface. The data in the 8th column of Table 4 should be taken as the real output-oriented relative validity index, and it can be seen from them that in the area of increasing marginal revenue, the relative validity index given by traditional DEA model is always on the large side; in the area of decreasing marginal revenue, the relative validity index given by the traditional DEA model is generally in agreement with the calculation result obtained after transformation.

Check to see a unit i under evaluation, the relative validity index obtained using an output-oriented DEA model for the transformed sample is ρ_i. If $\rho_i = 1$, unit i is on the real front surface; if $\rho_i > 1$, unit i is not on the real front surface, and the input-output combinations on the real front surface corresponds to unit i is $(x_i, \rho_i y_i)$. For example, for No. 16 enterprise, it is directly obtained using model (4) that $\rho_{16} = 2.442$, this means the output on front surface for input element $(790, 49)$ is $220 \times 2.442 = 537.2$. For transformed sample set, it is obtained using model (4) that $\rho_{16} = 2.052$, this means for input element $(790, 49)$, the output on real front surface is $220 \times 2.052 = 451.4$.

Denote the output on real front surface at x_i as $\tilde{y}_i$, and for sample (x_i, y_i) becomes $(x_i, G(x_i)y_i)$ through mathematic transformation, and the corresponding output-oriented

Table 4. Calculation results obtained using transformation function and model (4)

Enterprise No.	Total amount of fixed assets (in 10k RMB)	Labor number of persons	Output of enterprise (10k RMB)	Calculation result obtained using model (4)		Calculation result after transformation of sample set		
				Relative validity index	Is it close to front surface	Transformation result of enterprise output	Relative validity index	Is it close to front surface
1	100	19	40	1.0	Yes	40	1.0	Yes
2	205	24	60	1.928	No	97.7	1.434	No
3	295	28	88	2.052	No	168.7	1.305	No
4	380	32	128	1.889	No	258.4	1.162	No
5	400	36	188	1.363	No	380.2	1.0	Yes
24	500	36	128	2.565	No	249.2	1.526	No
6	595	41	260	1.526	No	459.9	1.0	Yes
7	700	45	340	1.389	No	513.8	1.014	Yes
29	700	45	112	4.218	No	169.3	3.077	No
16	790	49	220	2.442	No	283.4	2.052	No
8	810	49	440	1.253	No	546.8	1.065	Yes
30	900	53	500	1.233	No	538	1.195	No
9	920	53	560	1.122	No	587.9	1.095	Yes
22	1000	58	280	2.459	No	280	2.459	No
10	1050	58	720	1.0	Yes	720	1.0	Yes
25	1110	62	480	1.599	No	480	1.599	No
17	1200	66	640	1.301	No	640	1.303	No
11	1210	66	840	1.0	Yes	840	1.0	Yes
27	1300	70	800	1.093	Yes	800	1.093	Yes
21	1410	75	540	1.697	No	540	1.697	No
12	1420	75	920	1.0	Yes	920	1.0	Yes
13	1590	83	960	1.0	Yes	960	1.0	Yes
18	1605	83	740	1.297	No	740	1.297	No
26	1750	87	500	1.938	No	500	1.938	No
28	1760	89	900	1.081	Yes	900	1.081	Yes
14	1790	92	980	1.0	Yes	980	1.0	Yes
20	1820	92	620	1.581	No	620	1.581	No
23	1950	96	820	1.202	No	820	1.202	No
15	2000	100	992	1.0	Yes	992	1.0	Yes
19	2000	100	700	1.417	No	700	1.417	No

relative validity index is ρ_i, i.e., $\rho_i G(x_i)y_i = G(x_i)\tilde{y}_i$ holds, and the following holds as well:

$$\tilde{y}_i = (\rho_i G(x_i)y_i)/G(x_i) = \rho_i y_i \tag{7}$$

Formula (7) means that we transform y_i first to obtain $G(x_i)y_i$, let i take all the sample to obtain the transformed sample set, and obtain ρ_i using model (4) for the transformed sample set, obtain the output $\rho_i G(x_i)y_i$ on real front surface, and then obtain the output $\rho_i y_i$ on real front surface through mathematic inverse transformation $1/G(x_i)$. The derivation above is applicable to each unit.

9. **Conclusions.**

In traditional DEA model theory, the characteristic of decreasing marginal revenue is a fundamental assumption, under which the concept of production possibility set can be easily introduced, so that the effective front of production possibility set can be studied using linear programming models in different forms, the distance between each unit under evaluation and the effective front can be studied for different significances so that the relative validity can be evaluated. If the input-output relation has the characteristic of increasing marginal revenue, the real production possibility set is no longer a convex set. It is inevitably an excessively rough choice to use the production possibility set determined by DEA models because it includes many unachievable input-output combinations, and these combinations are used as standards for evaluation of other units for their relative validity, this is obviously not rational, and even absurd. The key to the problem also lies in the widespread existence of increasing marginal revenue in the input-output relation.

This paper spells out a kind of method used for identification of increasing marginal revenue, which is based on the statistic analysis of the number of sample points in the area on different input scales and the number of sample points on the front surface. By introducing the mathematic transformation principle, an appropriate mathematic transformation is chosen to transform a sample set for the purpose of making the sample set to a convex set through the transformation. On top of this basis, use the traditional DEA model to find the solution, thereby making each unit under evaluation full in compliance with actual relative validity index, and the real front surface can be obtained through mathematic inverse transformation.

The study presented in this paper is just a preliminary exploitation in this particular field. Such issues as identification of increasing marginal revenue, and solution of real surface and real relative validity index, need further study by different approaches. As far as the issue of more input and more output is concerned, the concept of increasing marginal revenue itself also needs serious classification and definition. As the applications of DEA models are increasingly wide, and there is a huge number of research papers on this particular aspect. The authors of this paper believe that many of the actual problems have the hidden characteristic of increasing marginal revenue, and so, it is difficult to avoid the irrationality of corresponding calculation and evaluation results. This is an issue meriting attention; maybe some of the research work needs to be rechecked for the characteristic of increasing marginal revenue.

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