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ENERGY-CONSTRAINT OPERATION STRATEGY FOR HIGH-SPEED RAILWAY

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Abstract. *With given trip time on each section, conventional train energy-efficient operation problem optimizes the reference speed profile such that the energy consumption for tracking the profile is minimized. Alternatively, this study aims to solve its anti-problem that minimizes the trip time under certain energy constraint, namely the train energy-constraint operation problem. In particular, this study focused on high-speed railway, for which the resistance mainly comes from the air friction. Firstly, we apply the Pontryagin maximum principle to prove that the optimal speed profile consists of four phases including acceleration, cruising, coasting and braking. Furthermore, we prove that the switching strategy among different phases is uniquely determined by the cruising speed, and then we solve the optimal cruising speed with an analytical approach. Finally, we prove some theorems on the energy-constraint operation strategy, which provide useful guidance for improving the train operation efficiency.*
Keywords: Energy-efficient operation, Energy-constraint operation, Pontryagin maximum principle, High-speed railway

1. **Introduction.** Energy and environmental concerns have made energy conservation and emission reduction important research issues in railway transportation. Most existing studies focused on the improvement of hardware conditions such as mass reduction, resistance reduction, space utilization, regenerative braking and energy storage. However, for a trip with fixed locomotive equipment and line conditions, the energy consumption is far from being fixed as the influences of trip time and speed profile. In other words, there is a need for novel approach for reducing the energy consumption. Traditionally, the trip time and speed profile are determined by the following process. Firstly, the shortest trip time is calculated and a buffer time (<5% with the shortest time) is added for treating the unpredictable delays on the way. Furthermore, a timetable optimization model [20, 22] is formulated to distribute trip time for an entire railway network under the shortest time constraint. Finally, for each section between two successive stations, an

energy-efficient operation model is formulated to optimize the speed profile such that the energy consumption is minimized under certain trip time constraint.

High-speed railway is a type of passenger railway transportation which runs significantly faster than the normal railway traffic. Specific definitions by the European Union include 200 km/h for upgraded track and 250 km/h for new track. At present, China has the world's longest high-speed railway network with about 8358 km of routes in service as of January 2011. Although the quick development of high-speed railway significantly improves the operation efficiency, the energy consumption also increases greatly, which reduces the operation effectiveness (Ghoseiri et al. [7]). It is proved that raising the top speed from 280 km/h to 350 km/h will increase the energy cost by about 60%. Therefore, a tradeoff study between trip time and energy consumption is very necessary. Train energy-efficient operation is an important aspect on such research, which has been well studied and widely applied [10, 11, 15, 18]. However, since high-speed railway operates significantly faster than the normal railway traffic, it is possible that once the trip time is fixed, the amount of energy consumption is too large to be accepted by the railway company from the view of operation cost. Especially, after the formulation of the carbon emission trading system in European Union, a railway company will be punished seriously once it exceeds the carbon emission allowance. Therefore, it is meaningful to study the anti-problem, i.e., minimizing the trip time under certain energy constraint such that the amount of energy consumption can be controlled accurately.

This study aims to formulate an optimal control model to describe the proposed energy-constraint problem. Since the energy appears as a constraint, we name it as energy-constraint operation model. Firstly, we apply the Pontryagin maximum principle to analyze the control phases for the optimal speed profile. Furthermore, we prove that the switching strategy among different phases is uniquely determined by the cruising speed, and then solve the optimal value of cruising speed with an analytical approach. Finally, we prove some properties on the energy-constraint operation strategy, and design a dichotomy algorithm to solve the energy-efficient operation problem.

The main contribution of this research is to propose a new type of tradeoff model between trip time and energy consumption, and solve the optimal speed profile with an analytical approach. We prove some theorems on the energy-constraint operation strategy, which can provide useful guidance for improving the train operation efficiency. In addition, the dichotomy algorithm for solving the energy-efficient speed profile has inherent accuracy and efficiency on computation time due to its analytical origin, which makes it possible to apply the algorithm to the automatic train operation system for calculating the reference speed profile onboard.

2. **Literature Reviews.** Train energy-efficient operation problem applies the optimal control theory to optimizing the reference speed profile such that the energy consumption for tracking the profile is minimized. Literature on this research may go back to the late 1970s, for example, Kokotović and Singh [16] formulated a nonlinear second-order optimal control model to minimize the electrical energy consumption by controlling the armature current. In 1980, Milroy [19] firstly proposed the minimization problem of mechanical energy consumption in his Ph.D. thesis. He showed that an energy-efficient speed profile for short trip has three phases including *acceleration*, *coasting* and *braking*. Furthermore, Lee et al. [17] discovered *cruising* as the fourth phase for longer trip. In 1990, Howlett [8] produced the first theoretical confirmation that an optimal speed profile should use an acceleration-cruising-coasting-braking phase sequence, and reformulated the problem to optimize the switching strategy among different phases. Then, he solved the simplified problem by showing that each possible strategy is determined by a single real parameter.

He also proved a special relationship between the cruising speed and the speed at which braking phase should begin, and the relationship was also found independently by Asnis et al. [1]. Although these researches have very restricting assumptions on track gradients, speed limits, tractive and braking efforts, their theoretical results lay the foundation of modern train control theory.

In 2000, Khmelnitsky [15] presented a complete study on the energy-efficient operation problem, in which track gradients and speed limits are both considered as arbitrary variable functions of the way, and the tractive and braking efforts are assumed to depend on the speed. In addition, the author also studied the effect of regenerative braking, and proved that if the energy can be fully recovered during the braking phase, the coasting phase will be interrupted on the optimal speed profile. Based on the analytical properties of the optimal speed profile, the author designed a numerical algorithm to solve the optimal switching strategy. In 2003, Liu and Golovitcher [18] proved that the speed profile for cruising and braking parts is determined by track gradients, speed limits, boundary conditions and a real parameter relating to the cruising speed, and these four sources also determine the preliminary locations for coasting and acceleration intervals. Based on such analysis, they developed a numerical solution approach to solve the optimal switching strategy. In addition, they discussed the minimization problem on operation cost which is assumed to be proportional to the mechanical work (energy consumption, power equipment amortization) and trip time (crew salary, passenger or goods delivery delays). In 2009, Howlett [11] provided an analytical solution approach for solving the optimal speed profile in the case that there are more than one steep slopes on the way; he first divided the route into small parts such that each part contains at most one steep slope, and then solved the precise switching strategy for each part by using a local energy minimization principle.

All above studies assume that the external force is continuous, i.e., we can choose any value between zero and the maximum for tractive and braking efforts. In 1989, Benjamin et al. [2] observed that the control mechanism on a typical diesel-electric locomotive is a throttle that can take only a finite number of positions. Each position determines a constant rate of fuel supply to the diesel motor and then determines a constant level of power supply to the wheels. After that, the Scheduling and Control Group at the University of South Australia carried out a systematic study on the discrete optimal control problem [2, 4, 5, 6, 9, 10]. For example, Cheng and Howlett [4, 5] pointed out that cruising is no longer a feasible phase since it is impossible to follow an arbitrary smooth speed with a finite number of traction output. In addition, the authors proved that any speed profile of continuous control can be approximated as closely as we please by a strategy with discrete control, i.e., the cruising phase may be well approximated by a frequent acceleration and braking process. In 1999, Cheng et al. [6] considered the solution of discrete optimal control problem with variable speed limits. In 2000, Howlett [10] presented a complete solution for such problem with variable gradients and speed limits, in which the Kuhn-Tucker equation is used to find the key equations that determine the optimal switching strategy.

Another aspect on the study of train energy-efficient operation is the application of evolutionary algorithm for solving the optimal switching strategy. Compared with the numerical algorithms designed on the basis of analytical properties, evolutionary algorithm has its advantage on solving the complex system. However, its computation accuracy and efficiency seriously limit its application in real-time control system. In 1997, Chang and Sim [3] formulated a multi-objective optimal control model for optimizing the riding comfort, punctuality and energy consumption, and designed a variable length chromosome genetic algorithm for solving the optimal coast control strategy. In 2005, Ke and Chen

[13] designed a genetic algorithm for assisting the design of fixed-block signalling system of mass rapid transit by optimizing the block layout and speed profile of each signalling block. In 2009, Ke et al. [14] furthermore designed a max-min ant colony optimization algorithm, which is proved to be efficient on improving the computation efficiency.

Recent studies on energy-efficient operation treat the trip time as one of the optimization objectives in a multi-objective optimal control problem. For example, Rémy et al. [21] formulated a multi-objective optimal control model for minimizing the energy consumption, trip time and delay time. Jong and Chang [12] proposed a compromise model for minimizing the linear combination of energy consumption and trip time. Compared with Liu and Golovitcher's work [18], the combination coefficient is assumed to be known depending on route, train and operation conditions. For example, for busy line where train flow is closed to line capacity, the weight on trip time is higher. On the contrary, if the travel demand is not too high and operation cost is a major concern, the weight on energy consumption is higher.

3. **Train Energy-Constraint Operation Model.**

This section aims to formulate a new type of tradeoff model between trip time and energy consumption. Since the energy appears as a constraint, we name it as energy-constraint operation model.

3.1. **Symbol systems.**

s	train position;
S	trip distance;
F	the maximum tractive effort per unit mass;
B	the maximum braking effort per unit mass;
E	energy level;
a	mechanical resistance coefficient;
c	aerodynamic resistance coefficient;
$v(s)$	speed profile, which is the state variable;
$R(v)$	basic resistance per unit mass, $R(v) = a + cv^2$;
$\alpha(s)$	relative tractive effort, which is the control variable, $\alpha(s) \in [0, 1]$;
$\beta(s)$	relative braking effort, which is the control variable, $\beta(s) \in [0, 1]$;
$E(v)$	the energy consumption for tracking speed profile v;
$T(v)$	the trip time for tracking speed profile v;
x_1	cruising point, the intersection point between acceleration and cruising profiles;
x_2	coasting point, the intersection point between cruising and coasting profiles;
x_3	braking point, the intersection point between coasting and braking profiles;
x_{ab}	the intersection point between acceleration and braking profiles;
u	cruising speed, the constant speed during the cruising phase;
w	braking speed, the speed at which the braking phase should begin.

3.2. **Assumptions.**

- Based on the original assumptions in Milroy's work, we assume that the track is flat, the tractive effort and the braking effort are constants, and there is no speed limit along the line.
- Without loss of generality, we assume that $v(s) > 0$ for all $s \in (0, S)$, that is, the train will not stop during its trip.
- We assume that there is no energy recovered from the braking process, that is, there is no application on the regenerative braking technique.

3.3. **Mathematical model.** Suppose that a train runs from one station to the next with trip distance S. Since there is no gradient on the line, the net acceleration of the train is $\alpha F - \beta B - R(v)$, and the motion equation is

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\alpha F - \beta B - R(v)}{v}. \tag{1}$$

Let v be a feasible speed profile with the boundary conditions $v(0) = v(S) = 0$. It is easy to prove that the trip time taken for tracking v is

$$T(v) = \int_0^S \frac{1}{v(s)}\mathrm{d}s, \tag{2}$$

and the energy consumption for tracking v is

$$E(v) = \int_0^S \alpha(s)F\mathrm{d}s. \tag{3}$$

The train energy-efficient operation problem optimizes the speed profile such that the energy consumption is minimized under certain trip time constraint. In this section, we study its anti-problem which minimizes the trip time such that the energy consumption is less than a given level, the mathematical model is formulated as follows:

$$\begin{cases} \min & \displaystyle\int_0^S \frac{1}{v(s)}\mathrm{d}s \\ \text{s.t.} & \displaystyle\int_0^S \alpha(s)F\mathrm{d}s \le E \\ & \displaystyle\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\alpha F - \beta B - R(v)}{v}, \quad \forall\, 0 \le s \le S \\ & 0 \le \alpha(s), \quad \beta(s) \le 1, \quad \forall 0 \le s \le S \\ & v(s) > 0, \quad \forall 0 < s < S, \quad v(0) = v(S) = 0. \end{cases} \tag{4}$$

For applying the Pontryagin maximum principle to analyze the optimal speed profile, we maximize the following Hamilton function

$$H(\alpha, \beta, v, p) = \frac{p(\alpha F - \beta B - R(v))}{v} - \frac{1}{v} - \lambda\alpha F \tag{5}$$

with conjugate condition

$$\frac{\mathrm{d}H(\alpha, \beta, v, p)}{\mathrm{d}p} = \frac{\mathrm{d}v}{\mathrm{d}s}, \tag{6}$$

where λ is a Lagrange multiplier, and p is the conjugate function. The argument breaks down into five cases.

Acceleration phase. If $p/v > \lambda$, the Hamilton function is maximized with $\alpha = 1$ and $\beta = 0$. In this case, the train motion equation is

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{F - R(v)}{v}. \tag{7}$$

According to the boundary condition $v(0) = 0$, the speed function is

$$v_a(s) = \sqrt{(F - a)(1 - \exp(-2cs))/c}, \tag{8}$$

which is strictly increasing with respect to s. Conversely, the displacement function is

$$s = \frac{1}{2c}\left(\ln(F - a) - \ln(F - R(v))\right). \tag{9}$$

Cruising phase. If $p/v = \lambda$, then the Hamilton function is maximized for all $\alpha = [0, 1]$ and $\beta = 0$. Since $\mathrm{d}H/\mathrm{d}p = 0$, it follows from the conjugate condition that

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\alpha F - R(v)}{v} = 0, \tag{10}$$

which implies that $\alpha = R(v)/F$. In this case, the train runs with a constant speed.

Coasting phase. If $0 < p/v < \lambda$, the Hamilton function is maximized with $\alpha = 0$ and $\beta = 0$, and the train motion equation is

$$\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{R(v)}{v}. \tag{11}$$

Partial braking phase. If $p/v = 0$, the Hamilton function is maximized for all $\beta \in [0, 1]$ and $\alpha = 0$. In this case, it follows from the conjugate condition and $\mathrm{d}H/\mathrm{d}p = 0$ that

$$\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\beta B + R(v)}{v} = 0,$$

which implies that $\beta = -R(v)/B < 0$. The contradiction proves that the optimal speed profile does not include this phase when the track is assumed to be flat.

Braking phase. If $p/v < 0$, the Hamilton function is maximized with $\alpha = 0$ and $\beta = 1$, and the train motion equation is

$$\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{B + R(v)}{v}. \tag{12}$$

According to the boundary speed condition $v(S) = 0$, the speed function is

$$v_b(s) = \sqrt{(B + a)(\exp(2c(S - s)) - 1)/c}, \tag{13}$$

which is a strictly decreasing function. Conversely, the displacement function is

$$s = S + \frac{1}{2c}(\ln(B + a) - \ln(B + R(v))). \tag{14}$$

Based on above analysis, there are four possible phases defining the speed profile including acceleration, cruising, coasting and braking. In this study, we use x_1, x_2, x_3 to denote the cruising point, coasting point and braking point, respectively, and name the vector $\mathbf{x} = (x_1, x_2, x_3)$ as a switching strategy. Then the anti-problem is reformulated to optimize the switching strategy such that the trip time is minimized under certain energy constraint.

3.4. **The shortest trip time.** In this section, we consider the speed profile with the shortest trip time. If we use x_{ab} to denote the intersection point between acceleration profile and braking profile, it follows from (8) and (13) that

$$x_{ab} = \left(\ln((B + a)\exp(2cS) + (F - a)) - \ln(B + F)\right)/2c. \tag{15}$$

It follows from Howlett [11] that the operation strategy should first accelerate the train to x_{ab} with the maximum tractive effort and then decelerate the train to the terminate station with the maximum braking effort, which implies that the speed profile should be

$$v_{\min}(s) = \begin{cases} v_a(s), & \text{if } s \le x_{ab} \\ v_b(s), & \text{if } s > x_{ab}. \end{cases} \tag{16}$$

The trip time for tracking such a profile is

$$T_{\min} = \int_0^{x_{ab}} \frac{1}{v_a(s)}\mathrm{d}s + \int_{x_{ab}}^{S} \frac{1}{v_b(s)}\mathrm{d}s, \tag{17}$$

and the energy consumption for tracking such a profile is

$$Fx_{ab} = \frac{F}{2c} \ln \frac{(B+a)\exp(2cS) + (F-a)}{B+F}. \tag{18}$$

For any $E \geq Fx_{ab}$, it is clear that the optimal speed profile of the energy-constraint operation model is $v_{\min}$.

Theorem 3.1. *The minimum trip time is strictly decreasing with respect to the energy level on* $(0, Fx_{ab}]$. *For any* $E \leq Fx_{ab}$, *the optimal switching strategy satisfies*

$$\int_0^{x_1} F\,ds + \int_{x_1}^{x_2} R(v)\,ds = E. \tag{19}$$

Proof: For any $E_1 < E_2 \leq Fx_{ab}$, we use T_1 and T_2 to denote the minimum trip times. Let $\mathbf{x}$ be the optimal operation strategy with energy level E_1. It follows from $E_1 < E_2$ that there is a feasible switching strategy $\mathbf{y}$ with energy level E_2 satisfying $y_1 > x_1, y_2 > x_2$ and $y_3 < x_3$. Since $\mathbf{y}$ distributes a longer distance for acceleration and cruising phases, we have $v_{\mathbf{y}}(s) > v_{\mathbf{x}}(s)$ for all s, which implies that

$$T_2 \leq \int_0^S \frac{1}{v_{\mathbf{y}}(s)} \mathrm{d}s < \int_0^S \frac{1}{v_{\mathbf{x}}(s)} \mathrm{d}s = T_1.$$

Hence, the minimum trip time is strictly decreasing. Furthermore, Equation (19) follows immediately from the monotonicity of trip time. The proof is complete.

4. **Optimal Switching Strategy.** In this section, we consider the optimal switching strategy in the case of $E \leq Fx_{ab}$. It will be shown that each switching strategy (x_1, x_2, x_3) is uniquely determined by the cruising speed u. First, it follows from Equation (9) that the cruising point is

$$x_1 = \frac{1}{2c} \ln \frac{F-a}{F-R(u)}, \tag{20}$$

which is clearly increasing with respect to u. During the acceleration phase, the tractive effort keeps to be the maximum value F, and the energy consumption for accelerating the train is Fx_1. In cruising phase, the tractive effort imposed on the train is proved to be $R(u)$ for keeping it runs with the constant speed u. Since the total energy is limited by a given level E, and it is consumed only during the acceleration and cruising phases, the energy consumption during the cruising phase is

$$E - Fx_1 = E - \frac{F}{2c} \ln \frac{F-a}{F-R(u)}.$$

It is readily to prove that the cruising distance is

$$\frac{E - Fx_1}{R(u)} = \frac{E}{R(u)} - \frac{F}{2cR(u)} \ln \frac{F-a}{F-R(u)}$$

and then the coasting point can be calculated as

$$x_2 = \frac{E}{R(u)} - \frac{1}{2c} \frac{F-R(u)}{R(u)} \ln \frac{F-a}{F-R(u)}. \tag{21}$$

Now, let us consider the braking point. In the coasting phase, it follows from motion Equation (11) and the boundary condition $v(x_2) = u$ that the displacement function is

$$s = \frac{E}{R(u)} - \frac{1}{2c} \frac{F-R(u)}{R(u)} \ln \frac{F-a}{F-R(u)} + \frac{1}{2c} \ln \frac{R(u)}{R(v)}. \tag{22}$$

Since braking point x_3 is the unique intersection point between coasting profile and braking profile, it follows from (14) and (22) that

$$x_3 = \frac{E}{R(u)} - \frac{1}{2c}\frac{F - R(u)}{R(u)} \ln \frac{F - a}{F - R(u)} + \frac{1}{2c} \ln \frac{g(R(u)) - R(u)}{B}, \tag{23}$$

and the braking speed w satisfies the following equation

$$R(w)(g(R(u)) - R(u)) = BR(u) \tag{24}$$

where function g is defined as

$$g(x) = (a + B) \exp \left(2c(S - E/x) + (F/x - 1)(\ln(F - a) - \ln(F - x))\right).$$

Based on the above analysis, it is proved that the cruising point, coasting point and braking point are all determined by the cruising speed. In what follows, we consider the optimization on cruising speed. First, it is necessary to analyze its minimum value and maximum value. Since the cruising distance is strictly decreasing with respect to u, the speed profile with the maximum cruising speed should exclude the cruising phase. On the other hand, the speed profile with the minimum cruising speed should exclude the coasting phase. If we use $u_{\max}$ to denote the maximum cruising speed, then we have

$$E = \frac{F}{2c}(\ln(F - a) - \ln(F - R(u_{\max}))), \tag{25}$$

which implies that the maximum cruising speed is

$$u_{\max} = \sqrt{(F - a)(1 - \exp(-2cE/F))/c}. \tag{26}$$

Furthermore, if we use $u_{\min}$ to denote the minimum cruising speed, it follows from Equations (9) and (14) that the cruising point is $\left(\ln(F - a) - \ln(F - R(u_{\min}))\right)/2c$ and the braking point is $S + \left(\ln(B + a) - \ln(B + R(u_{\min}))\right)/2c$. Then for satisfying the energy constraint, we have

$$\frac{F}{2c} \ln \frac{F - a}{F - R(u_{\min})} + R(u_{\min}) \left(S + \frac{1}{2c} \ln \frac{B + a}{B + R(u_{\min})} - \frac{1}{2c} \ln \frac{F - a}{F - R(u_{\min})}\right) = E.$$

Since the left part is strictly increasing with respect to $u_{\min}$, its value may be solved by using a dichotomy algorithm.

Since a speed profile generally concludes four phases, the trip time also consists of four parts. According to formulation (2), we have

$$T(v) = \int_0^{x_1} \frac{1}{F - R(v)} \mathrm{d}v + \int_{x_1}^{x_2} \frac{1}{u} \mathrm{d}s + \int_{x_2}^{x_3} -\frac{1}{R(v)} \mathrm{d}v + \int_{x_3}^{S} -\frac{1}{B + R(v)} \mathrm{d}v.$$

Since the speed profiles during the acceleration phase, coasting phase and braking phase are all strictly monotone with respect to the displacement, it follows from the theorem of integral transformation that $T(v)$ is

$$\int_0^u \frac{1}{F - R(x)} \mathrm{d}x + \frac{E}{uR(u)} - \frac{1}{2c}\frac{F}{uR(u)} \ln \frac{F - a}{F - R(u)} + \int_w^u \frac{1}{R(x)} \mathrm{d}x + \int_0^w \frac{1}{B + R(x)} \mathrm{d}x. \tag{27}$$

Take differential with u on both sides, we get

$$\frac{\mathrm{d}T}{\mathrm{d}u} = \left(\frac{1}{B + R(w)} - \frac{1}{R(w)}\right) \frac{\mathrm{d}w}{\mathrm{d}u} - \left(E - \frac{F}{2c} \ln \frac{F - a}{F - R(u)}\right) \frac{(R(u) + uR'(u))}{u^2 R^2(u)}. \tag{28}$$

According to Equation (24), we have

$$\left(\frac{1}{B + R(w)} - \frac{1}{R(w)}\right) \frac{\mathrm{d}w}{\mathrm{d}u} = R'(u) \frac{2cE - F \ln(F - a)/(F - R(u))}{R'(w) R^2(u)}. \tag{29}$$

Taking it into Equation (28), it follows from $R'(u) = 2cu$ that

$$\frac{\mathrm{d}T}{\mathrm{d}u} = \frac{1}{R^2(u)}\left(2cE - F\ln\frac{F-a}{F-R(u)}\right)\left(\frac{u-w}{w} - \frac{R(u)}{2(R(u)-a)}\right).$$

For any $u \in [u_{\min}, u_{\max}]$, it is clear that the energy consumption during the acceleration phase is less than or equal to the total energy level, which implies that

$$2cE - F\ln\frac{F-a}{F-R(u)} \geq 0. \tag{30}$$

Define a function

$$h(u) = \frac{u-w}{w} - \frac{R(u)}{2(R(u)-a)}.$$

Since w is decreasing with respect to u, it is easy to prove that h is an increasing function. When u takes its minimum value $u_{\min}$, we have $w = u_{\min}$ and $h(u_{\min}) < 0$. Then the value of the optimal cruising speed depends on $h(u_{\max})$. It follows from (24) and (25) that

$$h(u_{\max}) = \sqrt{\frac{(a+B)\exp(2c(S-E/F)) + (F-a)\exp(-2cE/F) - F}{B}} \\ - \frac{a}{2(F-a)(1-\exp(-2cE/F))} - \frac{3}{2}.$$

It is easy to prove that $h(u_{\max}) \geq 0$ if and only if E is less than or equal to a threshold E^*. If $E > E^*$, then the trip time is decreasing on $[u_{\min}, u_{\max}]$, and the optimal value is obtained at $u_{\max}$. Otherwise, the trip time is convex, and the optimal value is obtained within the interval $[u_{\min}, u_{\max}]$ which satisfies $h(u) = 0$. Since function h is strictly increasing, the optimal value can be solved by using a dichotomy algorithm.

Algorithm 4.1. *For any $E < Fx_{ab}$, the algorithm for solving the optimal switching strategy is described as follows.*
Step 1. Calculate the minimum cruising speed $u_{\min}$ and the maximum cruising speed $u_{\max}$.
Step 2. Initialize ε a small positive real number.
Step 3. Set $l = u_{\min}$ and $r = u_{\max}$.
Step 4. Define $u = (l+r)/2$. If $r - l < \varepsilon$, return u as the optimal cruising speed.
Step 5. If $h(u) < 0$, reset $l = u$. Otherwise, reset $r = u$. Goto Step 4.
Step 6. Calculate the optimal switching strategy by Equations (20), (21) and (23).
Step 7. Calculate the braking speed w, and the minimum trip time by Equation (27).

Remark 4.1. *Compared with the conventional energy-efficient operation, the energy-constraint operation is capable of controlling the amount of energy consumption accurately, which is necessary and meaningful since the energy and environment problems are paid more and more attention to in the world. Especially, as the formulation of carbon emission trading system in European Union, a railway company will be punished seriously once it exceeds the carbon emission allowance.*

5. **Properties on the Optimal Cruising Speed.** In this section, we analyze the qualitative relations between the energy-constraint operation strategy and the energy level.

Theorem 5.1. *The optimal cruising speed is increasing with respect to the energy level.*

Proof: For simplicity, we use u^* to denote the optimal cruising speed. For any energy levels $E_1 < E_2$, the proof of $u_1^* < u_2^*$ breaks down into two cases. If $E_2 \leq E^*$, according to Equation (24), it is easy to prove that $h(u, E_1) > h(u, E_2)$ for all u, which implies that $u_1^* > u_2^*$. Otherwise, if $E_2 > E^*$, the optimal cruising speed u_2^* is obtained at the maximum value, then we have

$$u_1^* \leq \sqrt{(F-a)(1-\exp(-2cE_1/F))/c} < \sqrt{(F-a)(1-\exp(-2cE_2/F))/c} = u_2^*.$$

The proof is complete.

Theorem 5.2. *Let E^* be the threshold of energy level. Then we have*
(a) the acceleration distance is strictly increasing with respect to the energy level E;
(b) the cruising distance is zero when $E \geq E^*$;
(c) the coasting distance is strictly decreasing when $E \geq E^*$;
(d) the braking distance is strictly increasing with respect to the energy level E.

Proof: (a) Since Theorem 5.1 has proved that the optimal cruising speed is strictly increasing with respect to E, it is readily to prove that the acceleration distance $x_1 = (\ln(F-a) - \ln(F - R(u)))/2c$ is also a strictly increasing function.

(b) If $E \geq E^*$, the optimal cruising speed is $u_{\max}$. Then it follows from (25) that the cruising distance is

$$x_2 - x_1 = E/R(u) - F(\ln(F-a) - \ln(F - R(u)))/2cR(u) = 0.$$

(c) If $E \geq E^*$, it follows from (21) and (23) that the coasting distance is

$$x_3 - x_2 = (\ln((a+B)\exp(2c(S - E/F)) + (F-a)\exp(-2cE/F) - F) - \ln B)/2c$$

which is clearly decreasing with respect to E.

(d) According to (23), the braking distance is $S - x_3 = (\ln(B + R(w)) - \ln(B+a))/2c$. It is clear that the braking distance is increasing if and only if the braking speed w is increasing. If $E < E^*$, it follows from $h(u) = 0$ that $w = 2u/(3 + a/(R(u) - a))$, since u has been proved to be increasing with respect to E, the braking speed w is also increasing with E. If $E \geq E^*$, it follows from (24) that

$$R(w) = B\frac{F\exp(2cE/F) - (F-a)}{(a+B)\exp(2cS) - F\exp(2cE/F) + F - a},$$

which implies that w is also increasing with respect to E. The proof is complete.

Theorem 5.3. *The optimal cruising point is strictly increasing with respect to the energy level, and the optimal braking point is strictly decreasing with respect to the energy level.*

Proof: It follows immediately from conclusions (a) and (d) of Theorem 5.2.

Theorem 5.4. *If the energy-constraint operation model with energy level E has the minimum trip time T, then the energy-efficient operation model with trip time T has the minimum energy consumption E.*

Proof: Let v^* be the optimal speed profile of the energy-constraint operation model. First, it is clear that v^* is a feasible solution for the energy-efficient operation model. Furthermore, for any speed profile v' with a smaller energy consumption E', according to Theorem 3.1, we have

$$T(v') \geq \min_{E(v)\leq E'} T(v) > \min_{E(v)\leq E} T(v) = T$$

which implies that v' is unfeasible. Hence, E is the minimum energy consumption and v^* is the optimal solution of the energy-efficient operation model. The proof is complete.

According to this theorem, we may design a dichotomy algorithm to solve the energy-efficient operation problem based on the analytical solution approach for the energy-constraint operation problem.

Algorithm 5.1. *The algorithm for solving the energy-efficient switching strategy is described as follows.*
Step 1. Initialize ε a small positive real number.
Step 2. Set $l = 0$ and $r = Fx_{ab}$.
Step 3. Define an energy level $e = (l + r)/2$.
Step 4. Calculate the optimal switching strategy (x_1, x_2, x_3) and the minimum trip time t.
Step 5. If $r - l < \varepsilon$, return (x_1, x_2, x_3) as the energy-efficient switching strategy.
Step 6. If $t < T$, reset $r = e$. Otherwise, reset $l = e$.
Step 7. Define $e = (l + r)/2$, and goto Step 4.

6. **Numerical Example.** To examine the practical viability of this approach, we illustrate some numerical examples that are performed on a personal computer with processor speed 2.4 GHz and memory size 2 GB. The related parameters are set as follows: $F = 0.20$ N/kg; $B = 0.25$ N/kg; $a = 0.016$ N/kg and $c = 0.0000155$ N/kg(m/s^2).

Example 6.1. *In this example, we consider the trip from Beijing South station to Tianjin South station in Beijing-Shanghai high-speed railway line with trip distance $S = 131000$ m. We first consider the shortest trip time. According to Equation (15), it is easy to calculate that the intersection point between the acceleration profile and the braking profile is $x_{ab} = 114430$ m. For minimizing the trip time, the operation strategy should first accelerate the train to x_{ab} with the maximum power, and then decelerate the train with the maximum braking effort until the train stops at the terminate station. Taking such a strategy, the shortest trip time is calculated to be $T_{\min} = 1794.7$ s and the maximum energy level is 22886 J.*

Table 1. Energy-constraint operation strategy with different energy levels

Energy level		Optimal cruising speed			Optimal switching strategy			Trip time	
E (J)	%	$u_{\min}$ (m/s)	$u_{\max}$ (m/s)	u (m/s)	x_1 (m)	x_2 (m)	x_3 (m)	T (s)	%
22886	0	107.4	107.4	107.4	114430	114430	114430	1794.7	0
21742	5	100.6	107.0	107.0	108709	108709	118372	1799.6	0.27
20597	10	96.0	106.7	106.7	102982	102987	121340	1815.0	1.12
19453	15	91.8	106.2	103.9	77855	99007	123475	1842.2	2.58
18309	20	87.8	105.7	96.7	50128	101540	124030	1878.7	4.47
17165	25	83.8	105.1	92.7	41520	100898	124969	1921.3	6.59
16020	30	79.9	104.3	88.0	34106	101696	125558	1971.5	8.97
14876	35	76.0	103.4	82.8	27854	103855	125876	2030.5	11.6
13732	40	72.0	102.2	78.9	24019	103278	126636	2099.2	14.5

In what follows, we gradually reduce the energy level and calculate the optimal operation strategy including cruising speed, switching strategy and the trip time. The computational results are shown in Table 1. First, we record the optimal cruising speed in the second volume, and it is concluded that (a) the minimum cruising speed $u_{\min}$ and the maximum cruising speed $u_{\max}$ are both increasing with respect to the energy level, and they coincide at the maximum energy level; (b) the optimal cruising speed is strictly increasing with respect to the energy level. In addition, we illustrate these results by Figure 1 for showing the monotonicity clearly.

The third volume records the optimal switching strategy. As the decreasing of the energy level, it is observed that the optimal cruising point is decreasing and the optimal braking point is increasing. However, there is no obvious regularity on the variation of optimal coasting point. Based on the optimal switching strategy, we furthermore calculate the distances for acceleration, cruising, coasting and braking phases, the results are illustrated

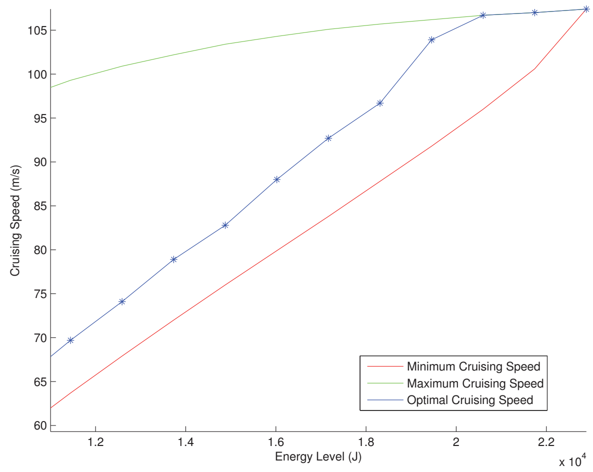

FIGURE 1. Relation between the optimal cruising speed and the energy level

by Figure 2. As the increasing of energy level, it is easy to conclude that the acceleration distance and braking distance are strictly increasing.

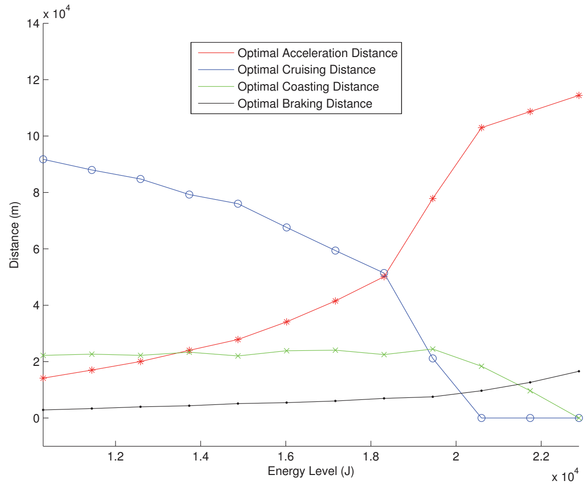

FIGURE 2. Distances for the acceleration, cruising, coasting and braking phases

The last volume records the trip time, which shows that as the decreasing of energy level, the minimum trip time is strictly increasing. However, its growth rate is far less than the reduction rate of energy consumption when the energy level nears its maximum value. For example, for saving energy 10%, the increase on trip time is only 1.12%.

Finally, we consider the solution of the energy-efficient operation model by performing Algorithm 5.1. We increase the trip time from its minimum value 1794.7 *s to* 2243.4 *s gradually, and then record the optimal switching strategy by Table 2. The last volume shows the computation time. Note that the computation time is less than 2.0 s even if the buffer time reaches 20% of the shortest trip time, which implies that the algorithm has potential applications to the real-time automatic train operation system.*

Example 6.2. *Nowadays, the energy and environment problems are paid more and more attention to in the world. In 2010, the Chinese energy policy is to reduce energy intensity by 20%. Take the Beijing-Shanghai high-speed railway line G11 for example, for achieving such a target with fixed locomotive equipments and line conditions, we should adjust the*

TABLE 2. Energy-efficient operation strategy with different trip times

Trip time		Optimal switching strategy			Energy consumption		Computation time
T (s)	%	x_1 (m)	x_2 (m)	x_3 (m)	E (J)	%	(s)
1794.7	0	114430	114430	114430	22886	0	1.0
1796.5	0.1	110900	110900	117000	22181	3.08	1.1
1798.3	0.2	109480	109490	117900	21898	4.32	1.1
1800.1	0.3	108400	108410	118550	21681	5.27	1.1
1801.9	0.4	107490	107500	119070	21500	6.06	1.2
1803.7	0.5	106700	106700	119500	21341	6.75	1.3
1812.8	1	103590	103590	121060	20718	9.47	1.4
1831.3	2	99251	99256	122900	19851	13.26	1.4
1850.2	3	68443	99304	123710	19181	16.19	1.5
1869.5	4	55988	100000	124120	18573	18.85	1.6
1889.2	5	48663	100590	124440	18005	21.33	1.7
1994.1	10	31669	102030	125780	15558	32.02	1.9

current timetable (see Table 3) to decrease the train speed on each section. For instance, the current energy consumed on the trip from Beijing to Jinan is 39212 J per unit mass, which must be reduced to 31370 J for achieving the target. By performing Algorithm 4.1, it is calculated that the trip time should be adjusted from 5520 s to 6111 s. As a result, the average speed reduces to 66.44 m/s and the top speed reduces to 71.08 m/s. The detailed results on the adjusted timetable is shown by Table 4.

TABLE 3. The current timetable of G11

	Trip time (s)	Average speed (m/s)	Top speed (m/s)	Energy (J)
Beijing-Jinan	5520	73.55	79.81	39212
Jinan-Nanjing	7920	77.90	82.53	64084
Nanjing-Shanghai	4020	73.38	82.50	30061

TABLE 4. The adjusted timetable of G11

	Trip time (s)	Average speed (m/s)	Top speed (m/s)	Energy (J)
Beijing-Jinan	6111	66.44	71.08	31370
Jinan-Nanjing	8792	70.18	73.60	51267
Nanjing-Shanghai	4432	66.56	73.31	24049

However, since the conventional energy-efficient operation models formulate energy as an optimization objective instead of constraint condition, they are incapable of controlling the amount of energy consumption accurately to achieve the energy reduction target.

7. **Conclusions.** This research proposed a novel formulation of train energy-constraint operation problem that provides a new tradeoff strategy between trip time and energy consumption. This research is also capable of developing an energy-constraint timetable optimization model by optimizing distributions of the energy level for an entire route or any part of railway systems. We solved the optimal speed profile by the proposed analytical approach and proved the theorems on the optimal operation strategy. Based on the analytical solution approach, we designed a dichotomy algorithm to solve the energy-efficient speed profile. Due to its analytical origin, the algorithm has inherent accuracy

and efficiency on computation time, which makes it possible to apply the algorithm to the automatic train operation system for calculating the reference speed profile onboard.

Further study for the energy-constraint operation problem can be done with variable gradients and speed limits. As the quick development and wide application of regenerative braking technique, we should also consider the recovery energy in the future research, which can effect the optimal switching strategy significantly.

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REFERENCES

[1] I. A. Asnis, A. V. Dmitruk and N. P. Osmolovskii, Solution of the problem of the energetically optimal control of the motion of a train by the maximum principle, *USSR Computational Mathematics and Mathematical Physics*, vol.25, no.6, pp.37-44, 1985.

[2] B. R. Benjamin, I. P. Milroy and P. J. Pudney, Energy-efficient operation of long-haul trains, *Proc. of the 4th International Heavy Haul Railway Conference*, Brisbane, pp.369-372, 1989.

[3] C. S. Chang and S. S. Sim, Optimising train movements through coast control using genetic algorithms, *IEE Proc. of Electric Power Applications*, vol.144, no.1, pp.65-73, 1997.

[4] J. X. Cheng and P. G. Howlett, Application of critical velocities to the minimisation of fuel consumption in the control of trains, *Automatica*, vol.28, no.1, pp.165-169, 1992.

[5] J. X. Cheng and P. G. Howlett, A note on the calculation of optimal strategies for the minimization of fuel consumption in the control of trains, *IEEE Transactions on Automatic Control*, vol.38, no.11, pp.1730-1734, 1993.

[6] J. X. Cheng, Y. Davydova, P. G. Howlett and P. J. Pudney, Optimal driving strategies for a train journey with non-zero track gradient and speed limits, *IMA Journal of Mathematics Applied in Business and Industry*, vol.10, pp.89-115, 1999.

[7] K. Ghoseiri, F. Szidarovszky and M. J. Asgharpour, A multi-objective train scheduling model and solution, *Transportation Research Part B*, vol.38, pp.927-952, 2004.

[8] P. G. Howlett, An optimal strategy for the control of a train, *The ANZIAM Journal*, vol.31, pp.454-471, 1990.

[9] P. G. Howlett, Optimal strategies for the control of a train, *Automatica*, vol.32, no.4, pp.519-532, 1996.

[10] P. G. Howlett, The optimal control of a train, *Annals of Operations Research*, vol.98, pp.65-87, 2000.

[11] P. G. Howlett, P. J. Pudney and X. Vu, Local energy minimization in optimal train control, *Automatica*, vol.45, pp.2692-2698, 2009.

[12] J. C. Jong and S. Chang, Algorithms for generating train speed profiles, *Journal of the Eastern Asia Society for Transportation Studies*, vol.6, pp.356-371, 2005.

[13] B. R. Ke and N. Chen, Signalling blocklayout and strategy of train operation for saving energy in mass rapid transit systems, *IEE Proc. of Electric Power Applications*, vol.152, no.2, pp.129-140, 2005.

[14] B. R. Ke, M. C. Chen and C. L. Lin, Block-layout design using max-min ant system for saving energy on mass rapid transit systems, *IEEE Transactions on Intelligent Transportation Systems*, vol.10, no.2, pp.226-235, 2009.

[15] E. Khmelnitsky, On an optimal control problem of train operation, *IEEE Transactions on Automatic Control*, vol.45, no.7, pp.1257-1266, 2000.

[16] P. Kokotović and G. Singh, Minimum-energy control of a traction motor, *IEEE Transactions on Automatic Control*, vol.17, no.1, pp.92-95, 1972.

[17] D. H. Lee, I. P. Milroy and K. Tyler, Application of Pontryagin's maximum principle to the semi-automatic control of rail vehicles, *Proc. of the 2nd Conference on Control Engineering*, Newcastle, pp.233-236, 1982.
[18] R. Liu and I. Golovitcher, Energy-efficient operation of rail vehicles, *Transportation Research Part A*, vol.37, pp.917-932, 2003.
[19] I. P. Milroy, *Aspects of Automatic Train Control*, Ph.D. Thesis, Loughborough University, 1980.
[20] B. H. Park, C.-S. Kim, T. Lim and H.-L. Rho, A new railway line planning model considering multinomial LOGIT-based traffic assignment, *ICIC Express Letters*, vol.5, no.8(B), pp.2919-2926, 2011.
[21] C. Rémy, M. Grégory, V. Bogdan and R. Joaquín, Multi-objective evolutionary algorithm for speed tuning optimization with energy saving in railway: Application and case study, *RailRome*, 2011.
[22] L. Yang and K. Li, The railway transportation planning problem and its genetic algorithm based tabu search algorithm, *ICIC Express Letters*, vol.3, no.3(A), pp.361-366, 2009.