International Journal of Innovative Computing, Information and Control
Volume **8**, Number **11**, November **2012**

ICIC International ©2012 ISSN 1349-4198
pp. 7747–7760

BACKSTEPPING CONTROL DESIGN FOR A CONTINUOUS-STIRRED TANK

Saleh Alshamali and Mohamed Zribi

Electrical Engineering Department
Kuwait University
P.O. Box 5969, Safat 13060, Kuwait
{ s.alshamali; m.zribi }@ku.edu.kw

Received July 2011; revised March 2012

Abstract. *This paper deals with the nonlinear control of a three-state continuous-stirred tank system (CSTR) using the backstepping technique. A nonlinear state transformation is introduced so that the backstepping technique could be utilized on the transformed system. The backstepping controller is first designed for the nominal CSTR system. Then an augmented backstepping control law is designed for the uncertain CSTR model where a high gain switching term, based on sliding mode control theory, is utilized to add robustness to the proposed controller. The proposed controllers are shown to drive the output trajectories to their respective desired values, hence, ensuring the stability of the closed-loop system. Finally, several simulation studies are presented to illustrate the effectiveness of the proposed control laws.*

Keywords: Nonlinear systems, CSTR, Backstepping control, Stability

1. **Introduction.** Continuous-stirred tank rectors (CSTRs) are essential components in chemical industries, where the tanks are continuously stirred to reach specific outputs. The different mathematical models derived for CSTRs are generally highly nonlinear which poses a challenge to the design of controllers for these systems. It is common to linearize the CSTR system about some operating points in order to simplify the control design. Several control strategies were proposed for the stabilization and tracking of CSTRs. Some of these strategies are reviewed below.

In [1], a linearizing state-feedback controller transforms the nonlinear dynamics of a CSTR into a first order linear model. Then, a predictive functional controller based on neural networks is used to control the linearized model of the system. The work of Colantonio et al. [2] uses an input-output feedback linearization controller and a second order sliding-mode controller for the disturbance rejection of a CSTR system. An adaptive control approach to the CSTR problem with a neural network model is introduced in [3]; stability of the closed-loop system is analysed using Lyapunov techniques. In [4], a model-based predictive control algorithm is presented for a nonlinear time-delay system with input constraints. The proposed algorithm showed a satisfactory performance when implemented on a constrained CSTR system. In [5, 6], a model predictive controller is used for the disturbance rejection in a CSTR model; sufficient conditions for offset-free control are given. In [7], a control law based on quantitative feedback theory (QFT) is designed for the temperature control of a CSTR model with exothermic reactions. The linear controllers are derived based on the linearization of the nonlinear model about some operating points. In [8], adaptive control of the CSTR is introduced where an external linear model is used to approximate the process. In [9], output tracking of a constrained CSTR using LPV description of the nonlinear process along with admissible set theory is addressed. Backstepping control is a well-known technique that is used for the control design of systems with special structures [10]. Several types of backstepping control strategies exist in the literature where it is combined with other control strategies such as sliding mode control (SMC) and fuzzy control, to improve the overall system performance. A backstepping control law is derived for the CSTR with uncertainty in [11]; global uniform boundness results of the closed-loop dynamics are given. In [12], an adaptive backstepping controller is designed for the temperature control of a CSTR; a fuzzy estimator is utilized to estimate the unknown terms. An adaptive backstepping controller for the tracking control of a CSTR is presented in [13]; a transformation of the original system variables is introduced in order to be able to use the backstepping technique.

In this work, a backstepping-based control design of a CSTR with non-constant tank level is proposed. The CSTR model considered is given in [5, 6]. The objective of the control design is to steer the controlled variables (tank level and mass concentration) to their desired values. The key step in the design is the introduction of a nonlinear transformation which changes the dynamics of the system into a form that is suitable for backstepping design.

The paper is organized as follows. Section 2 of the paper presents the CSTR model along with the transformation of the model into the new form. Section 3 proposes a backstepping control design for the nominal (unperturbed) model of the CSTR. A robust backstepping controller for the uncertain CSTR is proposed in Section 4, where external disturbances as well as parametric uncertainties are introduced into the model. Simulation results verifying the effectiveness of the control design are given. Moreover, this controller is compared with the standard PI controller. Finally, some concluding remarks are given in Section 5.

Notation: For convenience, the arguments of a function are sometimes omitted in the analysis when no confusion can arise.

2. The Model of the Tank System.

This work considers a continuously stirred tank reactor, depicted in Figure 1, where an irreversible exothermic reaction $A \to B$ takes place. The reactor utilises an external cooling unit to regulate the temperature. The CSTR model is adopted from the work in [5], and is given by

$$\dot{h}(t) = \frac{F_0 - F}{\pi r^2} \tag{1}$$

$$\dot{c}(t) = \frac{F_0(c_0 - c)}{\pi r^2 h} - k_0 c e^{-\frac{E/R}{T}} \tag{2}$$

$$\dot{T}(t) = \frac{F_0(T_0 - T)}{\pi r^2 h} + \frac{-\Delta H}{\rho C_p} k_0 c e^{-\frac{E/R}{T}} + \frac{2U}{r\rho\, C_p}(T_c - T) \tag{3}$$

$$y(t) = [h(t) \;\; c(t)]^T \tag{4}$$

where the level of the tank, $h(t)$, the molar concentration of species A, $c(t)$, and the reactor temperature, $T(t)$, are the state variables of the reactor system. The control variables are the outlet flow rate $F(t)$, and the coolant liquid temperature, $T_c(t)$. The output of the process is $y(t)$. The definitions and the values of the parameters in Equations (1)-(3) are given in Table 1.

The objective of controlling the CSTR system is to steer the tank level, $h(t)$, and the molar concentration, $c(t)$, to their desired values.

For convenience, the state vector $x(t)$ and the inputs $u_1(t)$ and $u_2(t)$ are defined such that: $x(t) = [x_1(t) \; x_2(t) \; x_3(t)]^T = [h(t) \; c(t) \; T(t)]^T$, $u_1(t) = F(t)$ and $u_2(t) = T_c(t)$.

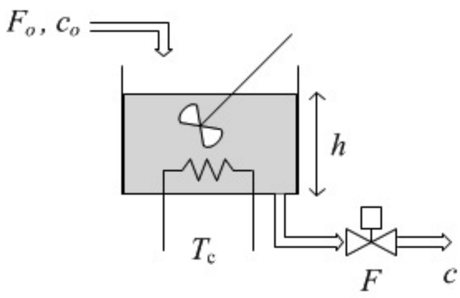

FIGURE 1. Schematic of the continuous-stirred tank reactor

TABLE 1. Parameters of the CSTR system

Parameter	Definition of the parameter	Nominal value
F_0	Volumetric flow rate	0.1 m^3/min
T_0	Feed Temperature	350 K
c_0	Feed concentration	1 kmol/m^3
r	Radius of the tank	0.219 m
k_0	Reaction velocity/frequency factor	7.2×10^{10} min^{-1}
E/R	Ratio of Arrehenius activation energy to the gas constant	8,750 K
U	Molar hold up	54.92 W/m^2.K
ρ	Molar density	1000 kg/m^3
C_p	Heat capacity	0.239 kJ/kg.K
ΔH	Heat of the reaction	-5×10^4 kJ/kmol

Therefore, the dynamic model, given by (1)-(4) can be written as

$$\dot{x}_1(t) = a_0 + a_1\ u_1(t) \tag{5}$$

$$\dot{x}_2(t) = a_2 \frac{1}{x_1(t)} + a_3 \frac{x_2(t)}{x_1(t)} + a_4 x_2(t) e^{-\frac{a_5}{x_3(t)}} \tag{6}$$

$$\dot{x}_3(t) = a_6 \frac{1}{x_1(t)} + a_3 \frac{x_3(t)}{x_1(t)} + a_4 a_7 x_2(t) e^{-\frac{a_5}{x_3(t)}} + a_8\ x_3(t) + a_9\ u_2(t) \tag{7}$$

$$y(t) = [x_1(t) \ \ x_2(t)]^T \tag{8}$$

where the parameters a_i $(i = 0, \ldots, 9)$ are defined in terms of the original system parameters as given in Table 2.

Clearly, the model given by (5)-(8) is highly nonlinear. Therefore, a nonlinear transformation that transforms the system dynamics given in (5)-(8) into a form suitable for backstepping design, is introduced. To this end, define the transformation:

$$z(t) = [z_1(t)\ z_2(t)\ z_3(t)]^T$$

such that,

$$\begin{aligned} z_1(t) &= x_1(t) \\ z_2(t) &= x_2(t) \\ z_3(t) &= \bar{a}_4 e^{-\frac{a_5}{x_3(t)}} \end{aligned} \tag{9}$$

Table 2. New parameters of the CSTR

Parameter	Definition of the parameter
a_0	$\frac{F_0}{\pi r^2}$
a_1	$-\frac{1}{\pi r^2}$
a_2	$\frac{F_0 c_0}{\pi r^2}$
a_3	$-\frac{F_0}{\pi r^2}$
a_4	$-k_0$
a_5	$\frac{E}{R}$
a_6	$\frac{F_0 T_0}{\pi r^2}$
a_7	$\frac{\Delta H}{\rho C_p}$
a_8	$-\frac{2U}{r\rho C_p}$
a_9	$\frac{2U}{r\rho C_p}$

where $\bar{a}_4 = -a_4$. It can be easily checked that

$$\begin{aligned} \dot{z}_3 &= \frac{\bar{a}_4 a_5}{x_3^2} e^{-\frac{a_5}{x_3}} \dot{x}_3 \\ &= \frac{z_3 \ln^2(z_3/\bar{a}_4)}{a_5} \dot{x}_3 \end{aligned}$$

Taking the time-derivative of $z(t)$ and invoking expression (9), the dynamics of the CSTR system in terms of the new coordinates can be written as

$$\dot{z}_1(t) = a_0 + a_1 \ u_1(t) \tag{10}$$

$$\dot{z}_2(t) = a_2 \frac{1}{z_1} + a_3 \frac{z_2}{z_1} - z_2 z_3 \tag{11}$$

$$\dot{z}_3(t) = \Gamma(z) + \frac{a_9 z_3 \ln^2(z_3/\bar{a}_4)}{a_5} u_2(t) \tag{12}$$

$$y(t) = [z_1(t) \ \ z_2(t)]^T \tag{13}$$

where

$$\Gamma(z) = \frac{z_3 \ln^2(z_3/\bar{a}_4)}{a_5} \left(\frac{a_6}{z_1} - \frac{a_3 a_5}{z_1 \ln(z_3/\bar{a}_4)} - a_7 z_2 z_3 - \frac{a_5 a_8}{\ln(z_3/\bar{a}_4)} \right).$$

Remark 2.1. *Notice that the state $x_1 = z_1$ is a non-zero quantity since it represents the level of the tank.*

The system Equations (10)-(12) can be thought of as two sub-systems: sub-system (I) which consists of Equation (10), and sub-system (II) which consists of Equations (11) and (12). The two sub-systems will be treated separately. First, a classic feedback control design is utilised for the stabilisation of sub-system (I). Then, a backstepping control design is proposed for sub-system (II).

3. Design of a Backstepping Controller.

3.1. Control design for sub-system (I).

This sub-section considers a feedback control design for Equation (10). Let α_1 be a positive scalar, and define the error signal e_1 such that

$$e_1 = z_1 - z_{1d} = y_1 - y_{1d} = x_1 - x_{1d} \tag{14}$$

where $z_{1d} = y_{1d} = x_{1d}$ is the desired constant value of $y_1(t) = x_1(t)$.

Proposition 3.1. *Sub-system (I) is asymptotically stable under the control law*

$$u_1(t) = -\frac{1}{a_1}(a_0 + \alpha_1 e_1) \tag{15}$$

Proof: Let $V_1(t) = \frac{1}{2}e_1^2$ be a Lyapunov function candidate for sub-system (I). Taking the time-derivative of V_1 along the trajectories of (10) and substituting for the control law (15) gives

$$\begin{aligned} \dot{V}_1 &= e_1\dot{e}_1 \\ &= e_1(a_0 + a_1 u_1) \\ &= -\alpha_1 e_1^2 \end{aligned}$$

Since $\alpha_1 > 0$, then $\dot{V}_1$ is negative definite, and e_1 tends to zero as $t \to \infty$. Therefore, it follows from Equation (14) that $y_1 \to y_{1d}$ and $z_1 \to z_{1d}$ as $t \to \infty$. Therefore, x_1 converges to its desired value x_{1d} as t tends to infinity. Thus, the steady-state value of the level of the tank is achieved.

3.2. Backsteping control design of sub-system (II).

In this sub-section, a control design based on the backstepping methodology is proposed for sub-system (II). Recall that the dynamics of sub-system (II) are such that

$$\dot{z}_2(t) = a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3 \tag{16}$$

$$\dot{z}_3(t) = \Gamma(z) + \frac{a_9 z_3 \ln^2(z_3/\bar{a}_4)}{a_5}u_2(t) \tag{17}$$

Now, let

$$u_2(t) = -\frac{a_5}{a_9 z_3 \ln^2(z_3/\bar{a}_4)}\left(\Gamma(z) - u_{bs}(t)\right) \tag{18}$$

where $u_{bs}(t)$ is the backstepping controller to be designed. Substituting for the control law (18) into (17) renders the dynamics of sub-system (II) as follows:

$$\dot{z}_2(t) = a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3 \tag{19}$$

$$\dot{z}_3(t) = u_{bs}(t) \tag{20}$$

Viewing z_3 as a virtual input to the ordinary differential equation given by Equation (19), the control law $z_3 = \phi(z_2)$ is designed to stabilize the dynamics. Define the error signal e_2 such that

$$e_2 = z_2 - z_{2d} = y_2 - y_{2d} = x_2 - x_{2d} \tag{21}$$

where $z_{2d} = y_{2d} = x_{2d}$ is the desired value of $y_2(t)$. Let α_2 be a positive scalar. It can be shown that the control law

$$\begin{aligned} z_3 &= \phi(z_2) \\ &= \frac{1}{z_2}\left[\frac{a_2}{z_1} + \frac{a_3 z_2}{z_1} + \alpha_2 e_2\right] \end{aligned} \tag{22}$$

allows for asymptotic convergence of Equation (19) to its desired value. To show the convergence of the output y_2 to its desired value, let $V_2(t) = \frac{1}{2}e_2^2(t)$ be a Lyapunov function candidate. Taking the time derivative of $V_2(t)$ along the trajectories of (19), and substituting for z_3 from (22) gives

$$\begin{aligned} \dot{V}_2 &= e_2\ \dot{e}_2 = e_2\dot{z}_2 \\ &= e_2\left(\frac{a_2}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3\right) \\ &= e_2(-\alpha_2\ e_2) \\ &= -\alpha_2 e_2^2 \end{aligned}$$

Since $\alpha_2 > 0$, then $\dot{V}_2 < 0$, and e_2 tends to zero as $t \to \infty$. Therefore, from Equation (21), it can be shown that y_2 tends to y_{2d} as $t \to \infty$. Hence, x_2 converges to its desired value, x_{2d}, as $t \to \infty$ since $x_2 = z_2$. Therefore, the species concentration reaches the steady-state value.

In the final step of the backstepping design, we define the signal q such that

$$q = z_3 - \phi(z_2) \tag{23}$$

Using (23), the system of Equations (19) and (20) can now be written as

$$\dot{z}_2(t) = a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2\phi(z_2) - z_2 q \tag{24}$$

$$\dot{q}(t) = v \tag{25}$$

where $v = u_{bs}(t) - \dot{\phi}(z_2)$. To design the controller v, we will consider the Lyapunov function candidate $V_3 = V_2 + \frac{1}{2}q^2$. The time-derivative of V_3 along the trajectories of (24) and (25) is such that

$$\begin{aligned} \dot{V}_3 &= \frac{\partial V_2}{\partial e_2}\dot{e}_2 + q\dot{q} \\ &= \frac{\partial V_2}{\partial e_2}\left[a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2\phi(z_2)\right] - \frac{\partial V_2}{\partial e_2}z_2 q + qv \end{aligned}$$

The choice of v such that $v = \frac{\partial V_2}{\partial e_2}z_2 - \alpha_3 q$, where α_3 is a positive scalar, results in $\dot{V}_3$ being negative definite such that

$$\dot{V}_3 \le -\alpha_2 e_2^2 - \alpha_3 q^2.$$

Therefore, it can be concluded that e_2 and q converge to zero as t tends to infinity. Finally, the backstepping control law can be written as

$$\begin{aligned} u_{bs}(t) &= \dot{\phi}(z_2) + v \\ &= \frac{\partial\phi(z_2)}{\partial z_2}\left[a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3\right] + \frac{\partial V_2}{\partial e_2}\ z_2 - \alpha_3 q \\ &= \xi(z_2)\left[a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3\right] + z_2 e_2 - \alpha_3(z_3 - \phi(z_2)) \end{aligned}$$

where
$$\xi(z_2) = \frac{\partial \phi(z_2)}{\partial z_2} = -\frac{a_2}{z_1 z_2^2} + \frac{\alpha_2 z_{2d}}{z_2^2}.$$
The previous development leads us to the following proposition.

Proposition 3.2. *Sub-system (II) is asymptotically stable under the following control law*
$$u_{bs}(t) = \xi(z_2)\left[a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} - z_2 z_3\right] + z_2 e_2 - \alpha_3(z_3 - \phi(z_2)) \tag{26}$$
with
$$\xi(z_2) = -\frac{a_2}{z_1 z_2^2} + \frac{\alpha_2 z_{2d}}{z_2^2} \tag{27}$$
$$\phi(z_2) = \frac{1}{z_2}\left[\frac{a_2}{z_1} + \frac{a_3 z_2}{z_1} + \alpha_2 e_2\right] \tag{28}$$

3.3. **Simulation studies.** A simulation study of the proposed backstepping controller is presented in this sub-section. Consider the CSTR model given by (1)-(4) and re-defined in (5)-(8), where the system parameters are given in Tables 1 and 2. The open-loop steady-state values for the state and input variables, respectively, are given such that $h_{ss} = 0.659$ m, $c_{ss} = 0.877$ kmol/m^3, $T_{ss} = 324.5$ K, $F_{ss} = 0.1$ m^3/min and $T_{css} = 300$ K [5]. Therefore, the desired values of the states are taken such that $h_d = 0.659$, $c_d = 0.877$ and $T_d = 324.5$. The initial conditions of the tank level, $h(t)$, and the molar concentration, $c(t)$, are chosen to be 10% off the steady-state conditions. The controller parameters are chosen to be $\alpha_1 = 3$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

The performance of the system given by (1)-(4) under the control law (15), (18) and (26)-(28) is simulated using the Matlab/Simulink software. The simulation results are presented in Figures 2 and 3. Figure 2 shows the evolution versus time of the CSTR output trajectories [$h(t)$ and $c(t)$] to their corresponding steady-state values. It can be seen that the tank level $h(t)$, and the molar concentration $c(t)$ converge to their desired values.

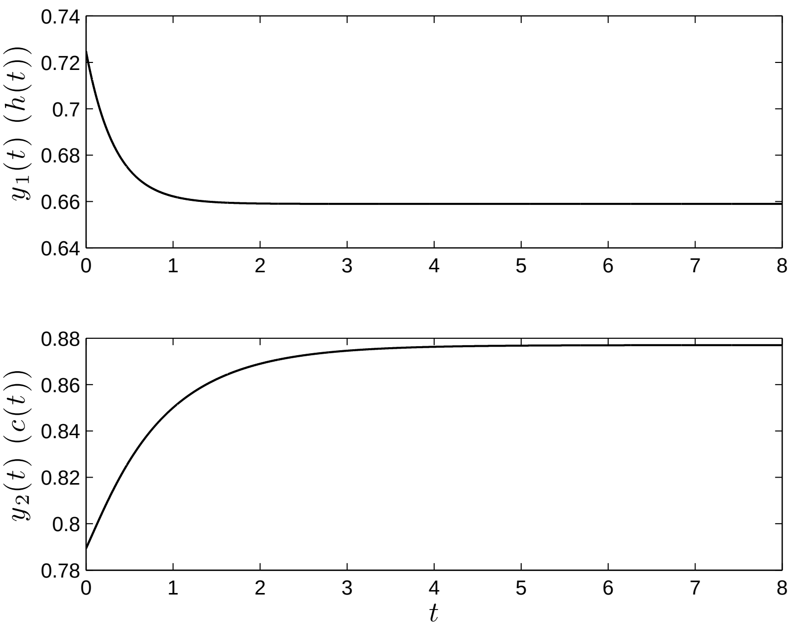

FIGURE 2. The tank level $h(t)$, and the molar concentration $c(t)$

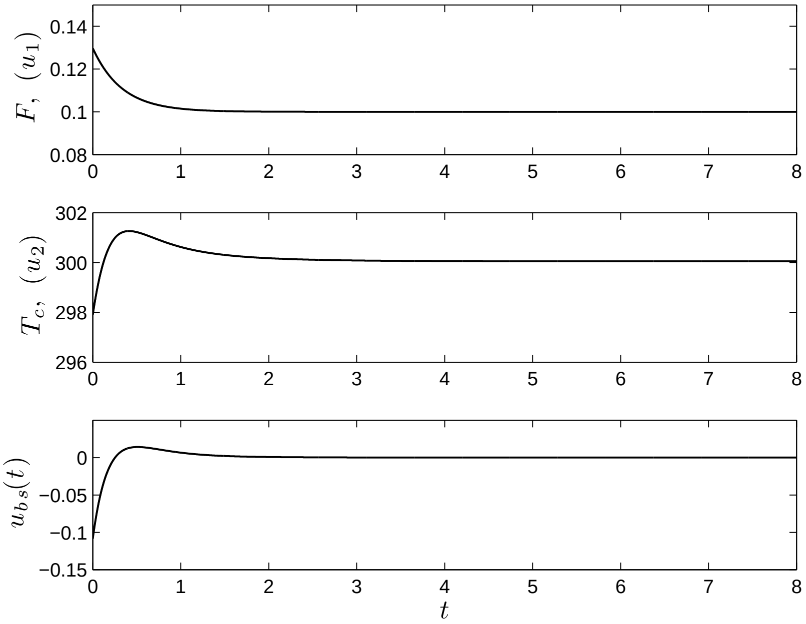

Figure 3. The CSTR inputs u_1 (F) and u_2 (T_c), and the backstepping control action $u_{bs}(t)$

Figure 3 depicts the state feedback control action, $u_1(t)$, the control input $u_2(t)$, and the backstepping control action $u_{bs}(t)$. The outlet flow rate control action F shows almost a steady action. The coolant liquid temperature control action T_c converges smoothly to its steady-state value, 300 K. Finally, the backstepping control law decreases from its initial value, and reaches the value of zero in finite time. Therefore, it can be concluded that the designed backstepping controller for the CSTR model shows a satisfactory performance, and hence it works well.

4. **Design of a Robust Backsteping Controller.** In the previous section, a backstepping design for the CSTR was considered assuming the knowledge of the exact values of the parameters and assuming no external disturbances acting on the system. In this section, a modified CSTR dynamics are considered; where additional terms that represent external disturbances and uncertainties in the parameters of the system are incorporated into the original model of the system. As a consequence, a robust controller ensuring closed-loop stability is necessary. A new backstepping design, utilizing high frequency terms similar to those used in the well-known sliding mode control (SMC) scheme, is proposed.

4.1. **The model of the CSTR system with external disturbances.** Consider the following CSTR model:

$$\dot{x}_1(t) = a_0 + a_1 u_1(t) + d_1 \tag{29}$$

$$\dot{x}_2(t) = a_2 \frac{1}{x_1(t)} + a_3 \frac{x_2(t)}{x_1(t)} + a_4 x_2(t) e^{-\frac{a_5}{x_3(t)}} + d_2 \tag{30}$$

$$\dot{x}_3(t) = a_6 \frac{1}{x_1(t)} + a_3 \frac{x_3(t)}{x_1(t)} + a_4 a_7 x_2(t) e^{-\frac{a_5}{x_3(t)}} + a_8\, x_3(t) + a_9\, u_2(t) + d_3 \tag{31}$$

$$y(t) = [x_1(t) \;\; x_2(t)]^T \tag{32}$$

where the terms d_1, d_2 and d_3 are disturbances on the tank's height, the inlet's concentration, and the inlet's temperature equations, respectively. The uncertainties are assumed to be unknown but with known maximum bounds such that $0 \leq |d_1| \leq \bar{d}_1$, $0 \leq |d_2| \leq \bar{d}_2$ and $0 \leq |d_3| \leq \bar{d}_3$, where $\bar{d}_1$, $\bar{d}_2$ and $\bar{d}_3$ are positive known upper bounds on d_1, d_2 and d_3, respectively. Applying the transformation given by (9), the dynamic model of the CSTR in terms of the transformed variables is given by

$$\dot{z}_1(t) = a_0 + a_1 u_1(t) + d_1 \tag{33}$$

$$\dot{z}_2(t) = a_2 \frac{1}{z_1} + a_3 \frac{z_2}{z_1} - z_2 z_3 + d_2 \tag{34}$$

$$\dot{z}_3(t) = \Gamma(z) + d_3 \frac{\ln^2(z_3/\bar{a}_4)}{a_5} z_3 + \frac{a_9 z_3 \ln^2(z_3/\bar{a}_4)}{a_5} u_2(t) \tag{35}$$

$$y(t) = [z_1(t) \;\; z_2(t)]^T \tag{36}$$

where

$$\Gamma(z) = \frac{z_3 \ln^2(z_3/\bar{a}_4)}{a_5} \left(\frac{a_6}{z_1} - \frac{a_3 a_5}{z_1 \ln(z_3/\bar{a}_4)} - a_7 z_2 z_3 - \frac{a_5 a_8}{\ln(z_3/\bar{a}_4)} \right)$$

Now, let

$$u_2(t) = -\frac{a_5}{a_9 z_3 \ln^2(z_3/\bar{a}_4)} \left(\Gamma(z) - u_{bs}(t) \right)$$

where $u_{bs}(t)$ is the backstepping controller to be designed. Then, Equation (35) becomes

$$\dot{z}_3(t) = u_{bs}(t) + d_3 \frac{z_3 \ln^2(z_3/\bar{a}_4)}{a_5} \tag{37}$$

4.2. **Design of the controller.** Let α_1, α_2, W_1, W_2, ζ_1, ζ_2, ζ_3, ζ_4 and ζ_5 be positive scalars. Also, let $sign(\sigma)$ denote the *signum* function such that

$$sign(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma = 0 \\ -1 & \text{if } \sigma < 0 \end{cases}$$

Proposition 4.1. *The following control laws*

$$u_1(t) = -\frac{1}{a_1} \left(a_0 + \alpha_1 e_1 + W_1 sign(e_1) \right) \tag{38}$$

$$u_2(t) = -\frac{a_5}{a_9 z_3 \ln^2(z_3/\bar{a}_4)} \left(\Gamma(z) - u_{bs}(t) \right) \tag{39}$$

$$u_{bs}(t) = \xi(z_2) \left[a_2 \frac{1}{z_1} + a_3 \frac{z_2}{z_1} + \hat{d}_2 - z_2 z_3 \right] + z_2 e_2 - W_4(z_3 - \phi(z_2)) - W_3 sign(q) \tag{40}$$

with

$$W_1 = \bar{d}_1 + \zeta_1 \tag{41}$$

$$W_2 = \bar{d}_2 + \zeta_2 \tag{42}$$

$$W_3 = \bar{d}_3 \frac{\ln^2(z_3/\bar{a}_4)}{a_5} |\phi(z_2)| + \zeta_3 \tag{43}$$

$$W_4 = \bar{d}_3 \frac{\ln^2(z_3/\bar{a}_4)}{a_5} + \zeta_4 \tag{44}$$

$$\phi(z_2) = \frac{1}{z_2}\left[\frac{a_2}{z_1} + \frac{a_3 z_2}{z_1} + \alpha_2 e_2 + W_2 sign(e_2)\right] \quad (45)$$

$$\xi(z_2) = \frac{\partial \phi(z_2)}{\partial z_2} \quad (46)$$

when applied to the system Equations (33)-(35) results in the asymptotic stability of the closed loop system.

Proof: Let $V_4(t) = \frac{1}{2}e_1^2$ be a Lyapunov function candidate where $e_1 = z_1 - z_{1d}$. Taking the time-derivative of V_4 along the trajectories of (33) and substituting for the control law (38) gives

$$\begin{aligned}\dot{V}_4 &= e_1(-\alpha_1 e_1 - W_1 sign(e_1) + d_1)\\ &\le -e_1^2 - W_1|e_1| + \bar{d}_1|e_1|\\ &= -\alpha_1 e_1^2 - \zeta_1|e_1|\end{aligned}$$

where the gain (41) has been utilised. Since $\alpha_1 > 0$ and $\zeta_1 > 0$, then $\dot{V}_4$ is negative definite, hence, e_1 tends to zero as $t \to \infty$.

Proceeding in a similar fashion as that of the previous section, the virtual control law for Equation (34) is chosen to be

$$z_3 = \phi(z_2) = \frac{1}{z_2}\left[\frac{a_2}{z_1} + \frac{a_3 z_2}{z_1} + \alpha_2 e_2 + W_2 sign(e_2)\right] \quad (47)$$

To show that this virtual controller stabilizes (34), let $V_5 = \frac{1}{2}e_2^2$, where $e_2 = z_2 - z_{2d}$. Taking the time-derivative of V_5 along the trajectories of (34) and substituting for the virtual control law (47) gives

$$\begin{aligned}\dot{V}_5 &= e_2(-\alpha_2 e_2 - W_2 sign(e_2) + d_2)\\ &\le -\alpha_2 e_2^2 - W_2|e_2| + \bar{d}_2|e_2|\end{aligned}$$

which after using (42) gives

$$\dot{V}_5 \le -\alpha_2 e_2^2 - \zeta_2|e_2|.$$

For the final step of the backstepping design, let

$$q = z_3 - \phi(z_2) \quad (48)$$

The system Equations (34) and (37) can now be written as

$$\dot{z}_2(t) = a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} + d_2 - z_2\phi(z_2) - q z_2 \quad (49)$$

$$\dot{q}(t) = v + d_3\frac{\ln^2(z_3/\bar{a}_4)}{a_5}z_3 \quad (50)$$

where $v = u_{bs}(t) - \dot{\phi}(z_2)$. Now, consider the Lyapunov function candidate $V_6 = V_5 + \frac{1}{2}q^2$. Then the time-derivative of V_6 along the trajectories of (49) and (50) is such

$$\begin{aligned}\dot{V}_6 &= \frac{\partial V_5}{\partial e_2}\dot{e}_2 + q\dot{q}\\ &= e_2\left[a_2\frac{1}{z_1} + a_3\frac{z_2}{z_1} + d_2 - z_2\phi(z_2)\right] - q z_2 e_2 + q\left(v + d_3\frac{\ln^2(z_3/\bar{a}_4)}{a_5}z_3\right) \quad (51)\end{aligned}$$

Let

$$v = e_2 z_2 - W_4 q - W_3 sign(q) \quad (52)$$

where W_3 and W_4 are given by (43) and (44), respectively. Substituting Equation (52) into (51) and invoking expressions (43) and (44) gives

$$\dot{V}_6 \leq -\alpha_2 e_2^2 - \zeta_3 q^2 - \zeta_3 |q|$$

Hence, $\dot{V}_6$ is negative definite. Therefore, e_2 and q converge to zero as $t \to \infty$. Finally, the backstepping control law, $u_{bs}(t)$, is expressed as

$$\begin{aligned} u_{bs}(t) &= \dot{\phi}(z_2) + v \\ &= \xi(z_2)\left[a_2 \frac{1}{z_1} + a_3 \frac{z_2}{z_1} + \hat{d}_2 - z_2 z_3\right] + z_2 e_2 - W_4(z_3 - \phi(z_2)) - W_3 sign(q) \end{aligned}$$

where $\xi(z_2)$ is given by (46).

Remark 4.1. *Notice that due to the discontinuous nature of the signum function, a continuous approximation of this function is used. In this case, the signum function* $sign(e_2)$ *is replaced by the hyperbolic tangent,* $\tanh\left(\frac{e_2}{\epsilon}\right)$, *where* ϵ *is a small number.*

Remark 4.2. *Due to the fact that the actual disturbance* d_2 *is unknown, in the expression for* $u_{bs}(t)$, *the value of* d_2 *is substituted for by* $\hat{d}_2$. *In the simulations, the two extreme values of* $\hat{d}_2$, *namely,* $\hat{d}_2 = 0$ *and* $\hat{d}_2 = \bar{d}_2$ *are used.*

4.3. **Simulation results.** Consider the CSTR model (29)-(32) where the external signals and their corresponding amplitude and duration are given in Table 3. The initial conditions are chosen to be $h(0) = 0.7249$, $c(0) = 0.7893$ and $T(0) = 324.50$. The desired values of the tank level and the molar concentration are 0.6590 and 0.8770, respectively. The controller gains are chosen to be $\alpha_1 = 1$, $\alpha_2 = 1$, $\zeta_1 = 0.75$, $\zeta_2 = 0.15$, $\zeta_3 = 0.15$, $\zeta_4 = 1.5$. The maximum values of the external disturbances are $\bar{d}_{1z} = 0.3$, $\bar{d}_{2z} = 0.2$ and $\bar{d}_{3z} = 3$. In the simulation, the values zero and $\bar{d}_{2z}$ were used for the term $\hat{d}_2$ given in Equation (40). The difference in the simulations was insignificant. Finally, the approximation of the signum function used in the simulation is $\tanh(q/\epsilon)$ where $\epsilon = 0.02$. The simulation results are presented in Figure 4.

Figure 4 depicts the output trajectories of the perturbed CSTR model. The proposed backstepping controller is shown to steer the outputs to their corresponding desired values. It can be seen that the tank level $h(t)$ exhibits a fast convergence towards its desired value. The trajectory of the molar concentration $c(t)$ displays little oscillations before converging to its desired value. Also, notice the effect of the disturbance d_2 applied at $4 \leq t \leq 5$ on the response of $c(t)$.

4.4. **Comparison with a PI controller.** The performance of the proposed robust sliding mode controller is compared to a PI controller. Consider the CSTR system (33)-(36), and the error signals $e_1 = z_1 - z_{1d}$, $e_2 = z_2 - z_{2d}$ and $e_3 = z_3 - z_{3d}$. The error dynamics

Table 3. External disturbance signals

Disturbance	Amplitude	Duration
d_1	0.05	[2 3]
d_2	0.1	[4 5]
d_3	2	[2.5 3.5]

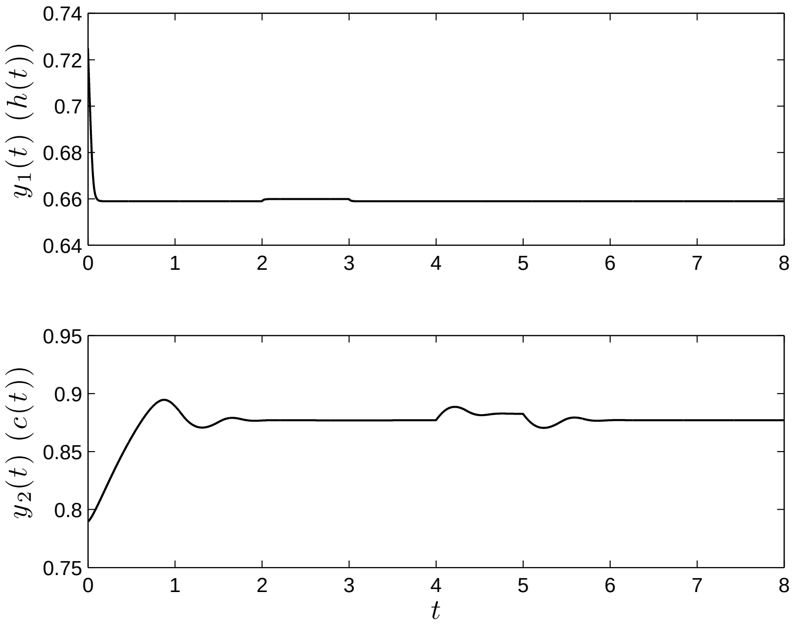

FIGURE 4. The tank level $h(t)$, and the molar concentration $c(t)$ (with disturbances)

are given by

$$\dot{e}_1(t) = a_0 + a_1\ u_1(t) + d_1 \tag{53}$$

$$\dot{e}_2(t) = a_2 \frac{1}{e_1 + z_{1d}} + a_3 \frac{e_2}{e_1 + z_1 d} - (e_2 + z_{2d})(e_3 + z_{3d}) + d_2 \tag{54}$$

$$\dot{e}_3(t) = \Gamma(e) + \frac{\ln^2((e_3 + z_{3d})/\bar{a}_4)}{a_5}(e_3 + z_{3d})[d_3 + a_9\ u_2(t)] \tag{55}$$

Notice that Equation (53) and (55) can be written as

$$\dot{e}_1(t) = f_1 + g_1 u_1(t)$$
$$\dot{e}_3(t) = f_3 + g_3 u_2(t)$$

where f_1, f_3, g_1 and g_3 can be easily determined from (53) and (55). Let the PI controller be such that

$$u_{1pi}(t) = -\frac{1}{g_1}\left(f_1 + k_{1p}e_1 + k_{1i}\int e_1\right)$$
$$u_{2pi}(t) = -\frac{1}{g_3}\left(f_3 + k_{3p}e_3 + k_{3i}\int e_3\right)$$

where k_{1p}, k_{1i}, k_{3p} and k_{3i} are the proportional and integral gains for $u_{1pi}(t)$ and $u_{2pi}(t)$, respectively. Figure 5 shows the performance of the PI controller for the nominal case (i.e., $d_i = 0$). Figure 6 shows the performance of the system when the PI controller is used for the perturbed system. The controller gains are chosen as $k_{1p} = 3$, $k_{1i} = 2$, $k_{3p} = 0.9$ and $k_{3i} = 0.08$. In Figure 5, the tank level response shows a fast convergence to the desired value, however, the evolution of the concentration level takes a long time before reaching the steady-state value. Figure 6 depicts the case where the disturbances given in Table 3 are acting on the system. The tank level shows a sluggish performance before reaching the steady-state value. The concentration level converges slowly to the

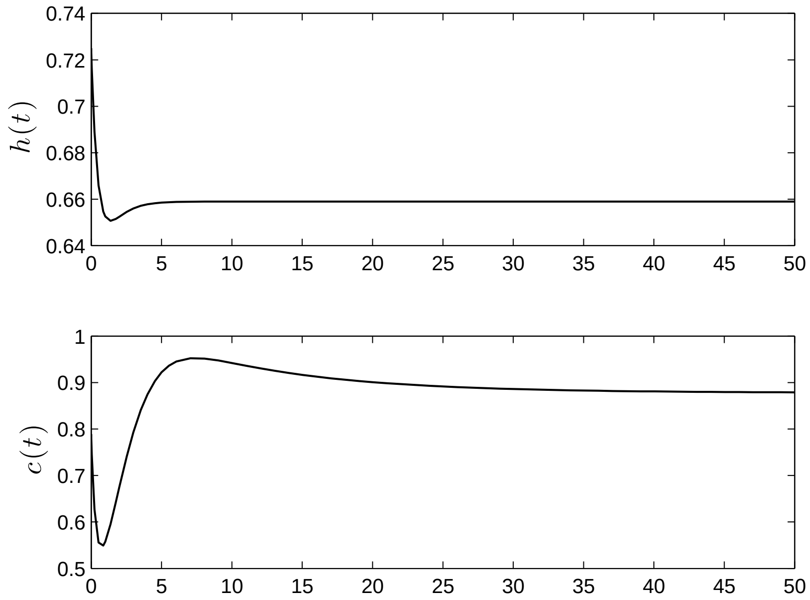

FIGURE 5. The tank level $h(t)$, and the molar concentration $c(t)$ when the PI controller is used (nominal system)

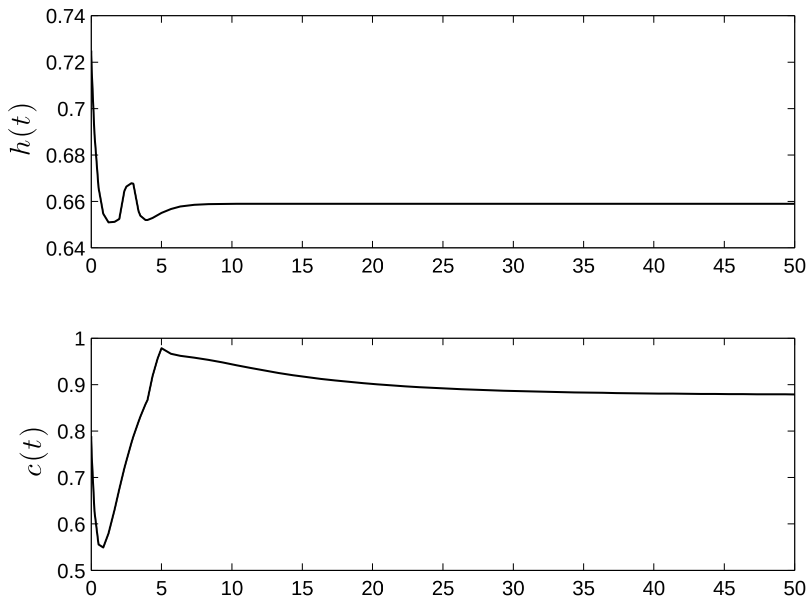

FIGURE 6. The tank level $h(t)$, and the molar concentration $c(t)$ when the PI controller is used (perturbed system)

desired value. It is, therefore, concluded that the proposed robust SMC outperforms the PI-controller in terms of performance of the closed-loop system.

5. **Conclusions.** Nonlinear controllers based on the backstepping methodology are proposed for the stabilization of a three-state CSTR system. An important step in the design is the introduction of a nonlinear transformation that puts the CSTR dynamics into a

form that is suitable for backstepping control design. At first, a control scheme is proposed for the nominal model of the CSTR system. The design approach treats the model as two sub-systems: a tank level sub-system and a molar concentration sub-system. The tank level sub-system is controlled via a feedback control law, while the molar concentration sub-system is controlled through a backstopping control law. Then, a backstopping controller is designed for the perturbed system. A switching term is incorporated into the backstepping control law to ensure the closed-loop stability and improve the robustness. It is shown that the proposed controllers drive the output trajectories to their corresponding desired values. Simulation results have shown the effectiveness of the proposed control schemes. Moreover, the performance of the proposed controller is compared with the performance of the standard PI controller. The simulation results show that the proposed controller outperforms the PI controller.

REFERENCES

[1] P. Guo, Nonlinear predictive functional control based on hopfield network and its application in CSTR, *Proc. of the 5th International Conference on Machine Learning and Cybernetics*, Dalian, China, pp.3036-3039, 2006.
[2] M. C. Colantonio, A. C. Desages, J. A. Romagnoli and A. Palazoglu, Nonlinear control of a CSTR: Disturbance rejection using sliding mode control, *Ind. Eng. Chem. Res.*, vol.34, pp.2383-2392, 1995.
[3] T. D. Knapp, H. M. Budman and G. Broderick, Adaptive control of a CSTR with neural network model, *Journal of Process Control*, vol.11, pp.53-68, 2001.
[4] M. Lu, C. Jin and H. Shao, An improved fuzzy predictive control algorithm and its application to an industrial CSTR process, *Journal of Chemical Engineering*, vol.17, no.1, pp.100-107, 2009.
[5] G. Pannocchia and J. B. Rawlings, Disturbance models for offset-free model-predictive control, *AIChE*, vol.49, no.2, pp.426-437, 2003.
[6] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, Nob Hill Pub., LLC, 2009.
[7] V. Olesen, C. Breitholtz and T. Wik, Tank reactor temperature control using quantitative feedback theory, *Proc. of the 17th World Congress: The International Federation of Automatic Control*, Seoul, Korea, pp.4970-4975, 2008.
[8] P. Dostal, M. Bakosova and V. Bobal, An approach to adaptive control of a CSTR, *Chem. Pap.*, vol.58, no.3, pp.184-190, 2004.
[9] L. Chisci, P. Falugi and G. Zappa, Set-point tracking for a class of constrained nonlinear systems with application to a CSTR, *Proc. of the 42nd Conference on Decision and Control*, Maui, HI, USA, vol.4, pp.3930-3935, 2003.
[10] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, 2nd Edition, 1996.
[11] J. She, H. Odajima, H. Hashimoto and M. Higashiguchi, Flow and temperature control of a tank system by backstepping method, *The 4th Int. Conf. on Motion and Vibration Control*, ETH Zurich, Switzerland, pp.449-454, 1998.
[12] S. Salehi and M. Shahrokhi, Adaptive fuzzy backstepping approach for temperature control of continuous stirred tank reactors, *Fuzzy Sets and Systems*, vol.160, pp.1804-1818, 2009.
[13] R. B. Gopaluni, I. Mizumoto and S. L. Shah, A robust nonlinear adaptive backstepping controller for a CSTR, *Ind. Eng. Chem. Res.*, vol.42, pp.4628-4644, 2003.