International Journal of Innovative Computing, Information and Control
Volume **8**, Number **10(B)**, October **2012**

pp. 7289–7305

LINEAR OPTIMAL STATE AND INPUT ESTIMATORS FOR NETWORKED CONTROL SYSTEMS WITH MULTIPLE PACKET DROPOUTS

Shuli Sun

Department of Automation
Heilongjiang University
No. 74, Xuefu Road, Nangang District, Harbin 150080, P. R. China
sunsl@hlju.edu.cn

Received August 2011; revised January 2012

Abstract. *This paper is concerned with the linear optimal full-order estimation problem for networked control systems with multiple packet dropouts of both sides from a sensor to an estimator and from a controller to an actuator. The phenomena of packet dropouts are described by two mutually uncorrelated random variables satisfying Bernoulli distributions. A full-order linear optimal state filter and an input filter in the least mean square sense are designed via completing square approach. They employ the received measurements at the present and last moments. The proposed filters are recursively computed in terms of the solutions of a Riccati equation, a Lyapunov equation and a simple difference equation. Further, the full-order linear optimal predictors and smoothers are also derived for the state and input, respectively. The stability of the proposed linear optimal filters is analyzed and the steady-state property is also investigated. The sufficient conditions for the stability of the linear optimal filters and the existence of the steady-state filters are given, respectively. Two simulation examples show the effectiveness of the proposed estimators.*
Keywords: Linear optimal estimator, Packet dropout, Completing square, Steady state, Networked control system

1. **Introduction.** In recent years, the research on control and estimation problems for systems with packet dropouts has gained lots of attention due to wide applications in networked control systems (NCSs) and sensor networks (SNs) [1-3]. In NCSs and SNs, random delays and packet dropouts almost exist in data transmissions from sensors to estimators and from controllers to actuators through the communication channels. So, the data available for the state estimation and control may not be up to date. This imposes significant challenges in estimation and control over networks [4].

The research on NCSs is focused on random delays, packet dropouts or missing measurements over the last few years. For the control problems of NCSs, some results have been developed, such as the finite- or infinite-horizon LQG control for a partially observed system [5], H_∞ control with missing measurements and delays [6]. For the estimation problems of NCSs, many algorithms about random delays have been proposed [7-10]. Yaz et al. study the estimation for systems with stochastic parameters via a linear matrix inequality (LMI) and apply to deal with random sensor delays or packet dropouts [11]. Wang et al. study the robust filter with missing measurements via LMI [12]. [13,14] study the polynomial state estimation for systems with stochastic multiplicative state noises and non-Gaussian noises, respectively. However, the presented filters are nonlinear and have the expensive computational cost by the high-dimension state augmentation. So, it is not realistic for application in real time. Xiao et al. study the peak covariance stability of time-varying Kalman filter for systems with bounded Markovian packet dropouts [15]. Furthermore, it is noteworthy that the stability analysis for some kinds of systems with random delays and packet dropouts, or stochastic parameters has also been developed [16,17].

In [18], an H_2 filter with multiple packet dropouts from a sensor to an estimator is designed via LMI. Based on a similar model to [18], the optimal and steady-state linear estimators are presented in the linear minimum variance sense via an innovation analysis approach [19], and full- and reduced-order estimators are also developed [20]. For systems with the finite consecutive packet dropouts, a full-order suboptimal filter is designed by assuming the filter to be the structure like Kalman recursive form [21]. However, only packet dropouts of single side from the sensor to the estimator are taken into account in the above literature. Recently, the augmented H_2 and H_∞ prior filters are designed for NCSs with packet dropouts of both sides from the sensor to the estimator and from the controller to the actuator via LMI [22,23]. However, the LMI approach is not applicable in real time for time-varying systems since it will cost much time to compute the filtering gain matrices at each time. The augmented optimal and steady-state linear filters for the augmented state are presented in the linear minimum variance sense via the projection theory [24], where, however, the augmented filters derived are complicated and expensive in the computational cost, and moreover, the predictor and smoother are not taken into account. So far, the full-order filters for systems with packet dropouts of both sides have not been studied, which have the reduced online computational cost compared with the augmented filters [22-24] since the additional computation is avoided. Recently, the distributed fusion estimators for systems with multiple sensors of packet dropouts and delays are also investigated [25,26]. For the input estimation problem, many results are mainly focused on the non stochastic input or disturbance [27,28]. Few results are reported on the stochastic input.

In this paper, the results in [20] with packet dropouts of single side from the sensor to the estimator are generalized to those with packet dropouts of both sides from the sensor to the estimator and from the controller to the actuator. The full-order linear optimal state and input estimators in the least mean square sense are developed via completing square approach, which, however, cannot be obtained by simple extension from [20] like the standard Kalman filter with deterministic input since the control input here is a stochastic variable. The linear optimal full-order filter, predictor and smoother are given in terms of a Riccati equation, a Lyapunov equation and a simple difference equation. The proposed full-order filter has the reduced online computational cost compared with the augmented filters in [22-24]. Differently from the general estimators [29], the estimation error covariance and gain matrices are affected by the control input since there are the stochastic packet dropouts. So, the steady-state estimators do not exist generally. The stability and steady-state property of the proposed estimators are analyzed. A sufficient condition for the stability of the optimal estimators is given. Moreover, a sufficient condition for the existence of the steady-state estimators is also given. In the absence of packet dropouts, the proposed linear optimal estimators are reduced to the standard Kalman estimators [29].

2. Problem Formulation and Preliminary Lemma.

2.1. Problem formulation.

Consider the following linear discrete-time stochastic system with multiple packet dropouts:

$$x(t+1) = \Phi x(t) + B\tilde{u}(t) + \Gamma w(t) \tag{1}$$

$$z(t) = Hx(t) + v(t) \tag{2}$$

$$y(t) = \xi(t)z(t) + (1 - \xi(t))y(t-1) \tag{3}$$

$$\tilde{u}(t) = \gamma(t)u(t) + (1 - \gamma(t))\tilde{u}(t-1) \tag{4}$$

where $x(t) \in R^n$ is the state, $z(t) \in R^m$ is the measured output, $u(t) \in R^r$ is the known control input, $y(t) \in R^m$ is the measurement received by the estimator to be designed, $\tilde{u}(t) \in R^r$ is the control input received by the actuator, $w(t) \in R^h$ and $v(t) \in R^m$ are white noises, Φ, B, Γ and H are constant matrices of suitable dimensions, and $\xi(t)$ and $\gamma(t)$ are mutually uncorrelated random variables that satisfy the Bernoulli distributions with the known probabilities $\mathrm{Prob}\{\xi(t) = 1\} = \alpha$, $\mathrm{Prob}\{\xi(t) = 0\} = 1 - \alpha$, $\mathrm{Prob}\{\gamma(t) = 1\} = \beta$ and $\mathrm{Prob}\{\gamma(t) = 0\} = 1 - \beta$, $0 \leq \alpha, \beta \leq 1$, and are uncorrelated with other random variables. For the brevity of notations, only linear time-invariant systems are taken into account. However, the results derived in this paper can be easily extended to linear time-varying systems.

A similar model to (1)-(4) is introduced in [22] to describe multiple packet dropouts of both sides from a sensor to an estimator and from a controller to an actuator in NCSs. The models (3) and (4) show that the latest data received previously will be used if the present measurement or control input is lost during data transmissions.

In this paper, the mathematical expectation E operates on $\xi(t)$, $\gamma(t)$, $w(t)$ and $v(t)$. 0 is a zero matrix of suitable dimension. The function diag(.) denotes a block diagonal matrix. The superscript T denotes the transpose. δ_{tk} is the Kronecker delta function. The following assumptions are used in the paper.

Assumption 2.1. *$w(t)$ and $v(t)$ are uncorrelated white noises with zero means and variances Q_w and Q_v.*

Assumption 2.2. *The initial state $x(0)$ with mean μ_0 and covariance P_0 is uncorrelated with $\xi(t), \gamma(t), w(t)$ and $v(t)$.*

Our aim is to find the linear optimal full-order state estimator $\hat{x}(t+N|t)$ and input estimator $\hat{\tilde{u}}(t+N|t)$ in the least mean square sense based on the received measurements $(y(t), y(t-1), \ldots, y(0))$. They are filters if $N = 0$, predictors if $N > 0$ and smoothers if $N < 0$.

2.2. **Preliminary lemma.** First, system (1)-(4) can be rewritten as a compact form [22]:

$$X(t+1) = \tilde{\Phi}(t)X(t) + \tilde{B}(t)u(t) + \tilde{\Gamma}(t)W(t) \tag{5}$$

$$y(t) = \tilde{H}(t)X(t) + \xi(t)v(t) \tag{6}$$

where

$$X(t) = \begin{bmatrix} x(t) \\ y(t-1) \\ \tilde{u}(t-1) \end{bmatrix}, \; W(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \; \tilde{\Phi}(t) = \begin{bmatrix} \Phi & 0 & (1-\gamma(t))B \\ \xi(t)H & (1-\xi(t))I_m & 0 \\ 0 & 0 & (1-\gamma(t))I_r \end{bmatrix},$$

$$\tilde{B}(t) = \begin{bmatrix} \gamma(t)B \\ 0 \\ \gamma(t)I_r \end{bmatrix}, \; \tilde{\Gamma}(t) = \begin{bmatrix} \Gamma & 0 \\ 0 & \xi(t)I_m \\ 0 & 0 \end{bmatrix},$$

$$\tilde{H}(t) = \begin{bmatrix} \xi(t)H & (1-\xi(t))I_m & 0 \end{bmatrix} \tag{7}$$

From Assumption 2.1, we have $\mathrm{E}[W(t)W^{\mathrm{T}}(k)] = \mathrm{diag}[\ Q_w \quad Q_v\ ]\delta_{tk}$, $\mathrm{E}[W(t)v^{\mathrm{T}}(k)] = [\ 0 \quad Q_v\ ]^{\mathrm{T}}\delta_{tk}$, and

$$\begin{aligned} \bar{\Phi} &= \mathrm{E}[\tilde{\Phi}(t)] = \Phi_0 + \alpha\Phi_1 + \beta\Phi_2, & \bar{B} &= \mathrm{E}[\tilde{B}(t)] = \beta B_1, \\ \bar{H} &= \mathrm{E}[\tilde{H}(t)] = H_0 + \alpha H_1, & \bar{\Gamma} &= \mathrm{E}[\tilde{\Gamma}(t)] = \Gamma_0 + \alpha\Gamma_1 \end{aligned} \tag{8}$$

where

$$\Phi_0 = \begin{bmatrix} \Phi & 0 & B \\ 0 & I_m & 0 \\ 0 & 0 & I_r \end{bmatrix}, \Phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ H & -I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 0 & -B \\ 0 & 0 & 0 \\ 0 & 0 & -I_r \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B \\ 0 \\ I_r \end{bmatrix},$$

$$H_0 = [\ 0 \quad I_m \quad 0\ ], \quad H_1 = [\ H \quad -I_m \quad 0\ ] \tag{9}$$

Lemma 2.1. *Under Assumptions 2.1 and 2.2, the state covariance matrix* $q(t) = \mathrm{E}[X(t)X^{\mathrm{T}}(t)]$ *of system (5) satisfies the following recursive Lyapunov equation:*

$$\begin{aligned} q(t+1) \ = \ & \Phi_0 q(t)\Phi_0^{\mathrm{T}} + \alpha\Phi_1 q(t)\Phi_1^{\mathrm{T}} + \beta\Phi_2 q(t)\Phi_2^{\mathrm{T}} + \alpha\Phi_0 q(t)\Phi_1^{\mathrm{T}} + \beta\Phi_0 q(t)\Phi_2^{\mathrm{T}} \\ & +\alpha\Phi_1 q(t)\Phi_0^{\mathrm{T}} + \alpha\beta\Phi_1 q(t)\Phi_2^{\mathrm{T}} + \beta\Phi_2 q(t)\Phi_0^{\mathrm{T}} + \alpha\beta\Phi_2 q(t)\Phi_1^{\mathrm{T}} \\ & +\beta B_1 u(t)u^{\mathrm{T}}(t)B_1^{\mathrm{T}} + \beta[\Phi_0 + \alpha\Phi_1 + \Phi_2]\bar{X}(t)u^{\mathrm{T}}(t)B_1^{\mathrm{T}} \\ & +\beta B_1 u(t)\bar{X}^{\mathrm{T}}(t)[\Phi_0 + \alpha\Phi_1 + \Phi_2]^{\mathrm{T}} + Q_0 \end{aligned} \tag{10}$$

with the initial value $q(0) = \mathrm{diag}(\ P_0 + \mu_0\mu_0^{\mathrm{T}} \quad 0\ )$*, and* $Q_0 = \mathrm{diag}(\ \Gamma Q_w \Gamma^{\mathrm{T}} \quad \alpha Q_v \quad 0\ )$*.*

The mean $\bar{X}(t) = \mathrm{E}[X(t)]$ *of the state* $X(t)$ *of system (5) satisfies the difference equation:*

$$\bar{X}(t+1) = \bar{\Phi}\bar{X}(t) + \bar{B}u(t) \tag{11}$$

with the initial value $\bar{X}(0) = \begin{bmatrix} \mu_0^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}$*.*

Proof: It follows from (5) that

$$X(t+1) = [\Phi_0 + \xi(t)\Phi_1 + \gamma(t)\Phi_2]X(t) + \gamma(t)B_1 u(t) + [\Gamma_0 + \xi(t)\Gamma_1]W(t) \tag{12}$$

So, we can obtain (10) by computing $q(t+1) = \mathrm{E}[X(t+1)X^{\mathrm{T}}(t+1)]$ and the matrix $Q_0 = \mathrm{E}[(\Gamma_0+\xi(t)\Gamma_1)W(t)W^{\mathrm{T}}(t)(\Gamma_0+\xi(t)\Gamma_1)^{\mathrm{T}}]$, respectively. (11) can be obtained directly by taking expectation on both sides of (5).

3. **Linear Optimal Full-Order Filters for the State and Input.** In this section, we shall present our main results on the linear optimal full-order filters for the state and input.

From projection property [29], the linear optimal filter of the state for augmented system (5)-(6) is given by $\hat{X}(t|t) = \hat{X}(t|t-1) + K_X(t)[y(t) - \bar{H}\hat{X}(t|t-1)]$ and that of the control input is given by $\hat{\tilde{u}}(t|t) = \hat{\tilde{u}}(t|t-1) + K_{\tilde{u}}(t)[y(t) - \bar{H}\hat{X}(t|t-1)]$ where $K_X(t)$ and $K_{\tilde{u}}(t)$ are the filtering gains. Note that $\bar{H} = \begin{bmatrix} \alpha H & (1-\alpha)I_m & 0 \end{bmatrix}$ and $\hat{X}(t|t-1) = \begin{bmatrix} \hat{x}(t|t-1)^{\mathrm{T}} & y^{\mathrm{T}}(t-1) & \hat{\tilde{u}}^{\mathrm{T}}(t-1|t-1) \end{bmatrix}^{\mathrm{T}}$, so the linear optimal full-order state filter and input filter can be designed in Theorem 3.1.

Theorem 3.1. *For system (1)-(4) with Assumptions 2.1 and 2.2, the linear optimal full-order state filter and input filter are given by*

$$\hat{x}(t|t) = (I_n - \alpha K_x(t)H)\hat{x}(t|t-1) + K_x(t)y(t) - (1-\alpha)K_x(t)y(t-1) \tag{13}$$

$$\hat{x}(t+1|t) = \Phi\hat{x}(t|t) + B\hat{\tilde{u}}(t|t) \tag{14}$$

$$\hat{\tilde{u}}(t|t) = \hat{\tilde{u}}(t|t-1) - \alpha K_{\tilde{u}}(t)H\hat{x}(t|t-1) + K_{\tilde{u}}(t)y(t) - (1-\alpha)K_{\tilde{u}}(t)y(t-1) \tag{15}$$

$$\hat{\tilde{u}}(t+1|t) = \beta u(t+1) + (1-\beta)\hat{\tilde{u}}(t|t) \tag{16}$$

The filtering gain matrices $K_x(t)$ *for the state and* $K_{\tilde{u}}(t)$ *for the input are computed by*

$$K_x(t) = P_x(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t) \tag{17}$$

$$K_{\tilde{u}}(t) = P_{x\tilde{u}}^{\mathrm{T}}(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t) \tag{18}$$

$$\Lambda(t) = (1-\alpha)H_1 q(t) H_1^{\mathrm{T}} + \alpha H P_x(t|t-1) H^{\mathrm{T}} + Q_v \tag{19}$$

The filtering error covariance matrices $P_x(t|t), P_{\tilde{u}}(t|t), P_{x\tilde{u}}(t|t), P_x(t|t-1), P_{\tilde{u}}(t|t-1)$ *and* $P_{x\tilde{u}}(t|t-1)$ *are computed by*

$$P_x(t|t) = P_x(t|t-1) - \alpha K_x(t)\Lambda(t)K_x^{\mathrm{T}}(t) \tag{20}$$

$$P_x(t+1|t) = \Phi P_x(t|t)\Phi^{\mathrm{T}} + BP_{\tilde{u}}(t|t)B^{\mathrm{T}} + \Phi P_{x\tilde{u}}(t|t)B^{\mathrm{T}} + BP_{x\tilde{u}}^{\mathrm{T}}(t|t)\Phi^{\mathrm{T}} + \Gamma Q_w \Gamma^{\mathrm{T}} \tag{21}$$

$$P_{\tilde{u}}(t|t) = P_{\tilde{u}}(t|t-1) - \alpha K_{\tilde{u}}(t)\Lambda(t)K_{\tilde{u}}^{\mathrm{T}}(t) \tag{22}$$

$$\begin{aligned} P_{\tilde{u}}(t+1|t) \;=\; & (1-\beta)[(1-\beta)P_{\tilde{u}}(t|t) + \beta u(t+1)u^{\mathrm{T}}(t+1) + \beta C q(t+1) C^{\mathrm{T}} \\ & -\beta u(t+1)\bar{X}^{\mathrm{T}}(t+1)C^{\mathrm{T}} - \beta C\bar{X}(t+1)u^{\mathrm{T}}(t+1)] \end{aligned} \tag{23}$$

$$P_{x\tilde{u}}(t|t) = P_{x\tilde{u}}(t|t-1) - \alpha K_x(t)\Lambda(t)K_{\tilde{u}}^{\mathrm{T}}(t) \tag{24}$$

$$P_{x\tilde{u}}(t+1|t) = (1-\beta)[\Phi P_{x\tilde{u}}(t|t) + BP_{\tilde{u}}(t|t)] \tag{25}$$

with $C = [0, 0, I_r]$ *and the initial values* $\hat{x}(0|-1) = \mu_0, \hat{\tilde{u}}(0|-1) = \beta u(0), P_x(0|-1) = P_0, P_{x\tilde{u}}(0|-1) = 0$ *and* $P_{\tilde{u}}(0|-1) = \beta u(0)u^{\mathrm{T}}(0)$*.*

Proof: From the analysis before Theorem 3.1, it is known that the filters for the state and input have the forms of (13) and (15). Furthermore, (14) and (16) can be obtained readily from (1) and (4). Using (13), (3) and (4), we have the filtering error equation for the state:

$$\tilde{x}(t|t) = (I_n - \alpha K_x(t)H)\tilde{x}(t|t-1) - (\xi(t)-\alpha)K_x(t)H_1 X(t) - \xi(t)K_x(t)v(t) \tag{26}$$

with the estimation error $\tilde{x}(t|t) = x(t) - \hat{x}(t|t)$. A similar definition can be applied to $\tilde{u}(t)$. Then, we can derive the filtering error covariance matrix as:

$$\begin{aligned} P_x(t|t) \;&=\; \mathrm{E}[\tilde{x}(t|t)\tilde{x}^{\mathrm{T}}(t|t)] \\ &=\; (I_n - \alpha K_x(t)H)P_x(t|t-1)(I_n - \alpha K_x(t)H)^{\mathrm{T}} \\ &\quad +\alpha(1-\alpha)K_x(t)H_1 q(t)H_1^{\mathrm{T}}K_x^{\mathrm{T}}(t) + \alpha K_x(t)Q_v K_x^{\mathrm{T}}(t) \end{aligned} \tag{27}$$

By arranging and completing the square, we can rewrite (27) as:

$$\begin{aligned} P_x(t|t) \;=\; & \alpha\{K_x(t) - P_x(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)\}\Lambda(t)\{K_x(t) - P_x(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)\}^{\mathrm{T}} \\ & +P_x(t|t-1) - \alpha P_x(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)HP_x^{\mathrm{T}}(t|t-1) \end{aligned} \tag{28}$$

where $\Lambda(t)$ is defined by (19). From (28), we see that the first term in the right hand side of (28) must be equal to zero to minimize $P_x(t|t)$, which yields the gain matrix $K_x(t)$ defined by (17) and the minimal variance $P_x(t|t)$ defined by (20).

Similarly, from (1), (4) and (14)-(16), we have the estimation error equations:

$$\tilde{x}(t+1|t) = \Phi\tilde{x}(t|t) + B\tilde{\tilde{u}}(t|t) + \Gamma w(t) \tag{29}$$

$$\tilde{\tilde{u}}(t+1|t) = (\gamma(t+1)-\beta)u(t+1) - (\gamma(t+1)-\beta)CX(t+1) + (1-\beta)\tilde{\tilde{u}}(t|t) \tag{30}$$

$$\tilde{\tilde{u}}(t|t) = \tilde{\tilde{u}}(t|t-1) - \alpha K_{\tilde{u}}(t)H\tilde{x}(t|t-1) - (\xi(t)-\alpha)K_{\tilde{u}}(t)H_1X(t) - \xi(t)K_{\tilde{u}}(t)v(t) \tag{31}$$

Then, (21) and (23) can be obtained by computing $P_x(t+1|t) = \mathrm{E}[\tilde{x}(t+1|t)\tilde{x}^{\mathrm{T}}(t+1|t)]$ and $P_{\tilde{u}}(t+1|t) = \mathrm{E}[\tilde{\tilde{u}}(t+1|t)\tilde{\tilde{u}}^{\mathrm{T}}(t+1|t)]$. Similarly to the derivation of (28), we can obtain the filtering error covariance matrix $P_{\tilde{u}}(t|t) = \mathrm{E}[\tilde{\tilde{u}}(t|t)\tilde{\tilde{u}}^{\mathrm{T}}(t|t)]$ from (31) as follows:

$$\begin{aligned} P_{\tilde{u}}(t|t) \;=\; & \alpha\{K_{\tilde{u}}(t) - P_{x\tilde{u}}^{\mathrm{T}}(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)\}\Lambda(t)\{K_{\tilde{u}}(t) - P_{x\tilde{u}}^{\mathrm{T}}(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)\}^{\mathrm{T}} \\ & +P_{\tilde{u}}(t|t-1) - \alpha P_{x\tilde{u}}^{\mathrm{T}}(t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)HP_{x\tilde{u}}(t|t-1) \end{aligned} \tag{32}$$

To minimize $P_{\tilde{u}}(t|t)$, the first term in the right hand side of (32) must be equal to zero, which yields the gain matrix $K_{\tilde{u}}(t)$ defined by (18) and the minimal variance $P_{\tilde{u}}(t|t)$ defined by (22). On the other hand, (24) and (25) can be obtained readily by computing $P_{x\tilde{u}}(t|t) = \mathrm{E}[\tilde{x}(t|t)\tilde{\tilde{u}}^{\mathrm{T}}(t|t)]$ and $P_{x\tilde{u}}(t+1|t) = \mathrm{E}[\tilde{x}(t+1|t)\tilde{\tilde{u}}^{\mathrm{T}}(t+1|t)]$ from (26), (29)-(31).

Remark 3.1. *Theorem 3.1 gives the linear optimal full-order filters for the state and input by completing square approach. We see that they have the simple computational formulas and derivations than the augmented filters for augmented systems in [22-24]. Without loss of generality, we assume $n \geq m$ and $n \geq p$. The computational cost of gain and covariance matrices is not taken into account since they can be computed offline. Then, the proposed filters have the online computational order of magnitude $O(n^2)$, so they have the reduced online computational cost than the augmented filters in [22-24] which have the online computational order of magnitude $O((n+m+p)^2)$.*

Remark 3.2. *From Theorem 3.1, we see that the gain and covariance matrices of the proposed filters are affected by the control input $u(t)$, which cannot be obtained by simple extension from [20] without the control input. It is different from the standard Kalman filter with deterministic input [29] where gain and covariance matrices are not affected by the input. The reason is that the concerned system is affected by the stochastic input $\tilde{u}(t)$.*

Remark 3.3. *The optimal linear full-order filters designed only depend on the knowledge of the distribution α but do not depend on the values of $(\xi(0), \xi(1), \ldots, \xi(t))$, which implies that the proposed filters can be computed offline and the steady-state filters can be obtained (see the later Section 5). They are different from the filter given in [3] where the intermittent Kalman filter depends on the values of $(\xi(0), \xi(1), \ldots, \xi(t))$, which implies that the filter in [3] must be computed online and the steady-state filter cannot be obtained.*

4. **Linear Optimal Full-Order Predictors and Smoothers.** In this section, we will derive the linear optimal full-order predictors and smoothers for the state and input based on Theorem 3.1. The following Theorem 4.1 and Theorem 4.2 give the results.

Theorem 4.1. *For system (1)-(4) with Assumptions 2.1 and 2.2, the linear optimal full-order N-step ($N > 1$) predictors for the state and input are given by*

$$\hat{x}(t+N|t) = \Phi\hat{x}(t+N-1|t) + B\hat{\tilde{u}}(t+N-1|t) \tag{33}$$

$$\hat{\tilde{u}}(t+N|t) = \beta u(t+N) + (1-\beta)\hat{\tilde{u}}(t+N-1|t) \tag{34}$$

where the initial values $\hat{x}(t|t)$ and $\hat{\tilde{u}}(t|t)$ are computed by Theorem 3.1. The prediction error covariance matrices are given by

$$\begin{aligned} P_x(t+N|t) \; = \; & \Phi P_x(t+N-1|t)\Phi^{\mathrm{T}} \\ & +BP_{\tilde{u}}(t+N-1|t)B^{\mathrm{T}} + \Phi P_{x\tilde{u}}(t+N-1|t)B^{\mathrm{T}} \\ & +BP_{x\tilde{u}}^{\mathrm{T}}(t+N-1|t)\Phi^{\mathrm{T}} + \Gamma Q_w\Gamma^{\mathrm{T}} \end{aligned} \tag{35}$$

$$\begin{aligned} P_{\tilde{u}}(t+N|t) \; = \; & \beta(1-\beta)u(t+N)u^{\mathrm{T}}(t+N) + \beta(1-\beta)Cq(t+N)C^{\mathrm{T}} \\ & +(1-\beta)^2P_{\tilde{u}}(t+N-1|t) - \beta(1-\beta)u(t+N)\bar{X}^{\mathrm{T}}(t+N)C^{\mathrm{T}} \\ & -\beta(1-\beta)C\bar{X}(t+N)u^{\mathrm{T}}(t+N) \end{aligned} \tag{36}$$

$$P_{x\tilde{u}}(t+N|t) = (1-\beta)\Phi P_{x\tilde{u}}(t+N-1|t) + (1-\beta)BP_{\tilde{u}}(t+N-1|t) \tag{37}$$

where the initial values $P_x(t+1|t), P_{\tilde{u}}(t+1|t)$ and $P_{x\tilde{u}}(t+1|t)$ are computed by (21), (23) and (25), and $q(t+N)$ and $\bar{X}(t+N)$ are computed by (10) and (11).

Proof: (33) and (34) directly follow from (1) and (4). Further, we have the prediction error equations

$$\tilde{x}(t+N|t) = \Phi\tilde{x}(t+N-1|t) + B\tilde{\tilde{u}}(t+N-1|t) + \Gamma w(t+N-1) \tag{38}$$

$$\tilde{\tilde{u}}(t+N|t) = (\gamma(t+N)-\beta)u(t+N) - (\gamma(t+N)-\beta)CX(t+N) + (1-\beta)\tilde{\tilde{u}}(t+N-1|t) \tag{39}$$

with the prediction errors $\tilde{x}(t+N|t) = x(t+N) - \hat{x}(t+N|t)$ and $\tilde{\tilde{u}}(t+N|t) = \tilde{u}(t+N) - \hat{\tilde{u}}(t+N|t)$. Then, the covariance matrices (35)-(37) can be obtained readily from (38) and (39) by computing $P_x(t+N|t) = \mathrm{E}[\tilde{x}(t+N|t)\tilde{x}^{\mathrm{T}}(t+N|t)]$, $P_{\tilde{u}}(t+N|t) = \mathrm{E}[\tilde{\tilde{u}}(t+N|t)\tilde{\tilde{u}}^{\mathrm{T}}(t+N|t)]$ and $P_{x\tilde{u}}(t+N|t) = \mathrm{E}[\tilde{x}(t+N|t)\tilde{\tilde{u}}^{\mathrm{T}}(t+N|t)]$.

Theorem 4.2. *For system (1)-(4) with Assumptions 2.1 and 2.2, the linear optimal full-order fixed-lag N-step ($N < 0$) smoothers for the state and input are given by*

$$\hat{x}(t+N|t) = \hat{x}(t+N|t-1) + M_x(t+N|t)[y(t) - \alpha H\hat{x}(t|t-1) - (1-\alpha)y(t-1)] \tag{40}$$
$$\hat{\tilde{u}}(t+N|t) = \hat{\tilde{u}}(t+N|t-1) + M_{\tilde{u}}(t+N|t)[y(t) - \alpha H\hat{x}(t|t-1) - (1-\alpha)y(t-1)] \tag{41}$$

where the initial values $\hat{x}(t+N|t+N)$ and $\hat{\tilde{u}}(t+N|t+N)$ are computed by Theorem 3.1. The smoothing ($N < 0$) gain matrices $M_x(t+N|t)$ and $M_{\tilde{u}}(t+N|t)$ are computed by

$$M_x(t+N|t) = \Delta_x^{(1)}(t+N,t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t), \quad M_{\tilde{u}}(t+N|t) = \Delta_{\tilde{u}}^{(1)}(t+N,t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t) \tag{42}$$

where

$$\begin{aligned} \Delta_x^{(1)}(t+N,t|t-1) &= \Delta_x^{(1)}(t+N,t-1|t-2)[\Phi - \alpha(\Phi K_x(t-1) \\ &\quad + BK_{\tilde{u}}(t-1))H]^{\mathrm{T}} + \Delta_x^{(2)}(t+N,t-1|t-2)B^{\mathrm{T}}, \\ \Delta_x^{(2)}(t+N,t|t-1) &= (1-\beta)[\Delta_x^{(2)}(t+N,t-1|t-2) \\ &\quad -\alpha\Delta_x^{(1)}(t+N,t-1|t-2)H^{\mathrm{T}}K_{\tilde{u}}^{\mathrm{T}}(t-1)] \end{aligned} \tag{43}$$

$$\begin{aligned} \Delta_{\tilde{u}}^{(1)}(t+N,t|t-1) &= \Delta_{\tilde{u}}^{(1)}(t+N,t-1|t-2)[\Phi - \alpha(\Phi K_x(t-1) \\ &\quad + BK_{\tilde{u}}(t-1))H]^{\mathrm{T}} + \Delta_{\tilde{u}}^{(2)}(t+N,t-1|t-2)B^{\mathrm{T}}, \\ \Delta_{\tilde{u}}^{(2)}(t+N,t|t-1) &= (1-\beta)[\Delta_{\tilde{u}}^{(2)}(t+N,t-1|t-2) \\ &\quad -\alpha\Delta_{\tilde{u}}^{(1)}(t+N,t-1|t-2)H^{\mathrm{T}}K_{\tilde{u}}^{\mathrm{T}}(t-1)] \end{aligned} \tag{44}$$

where the initial values $\Delta_x^{(1)}(t+N,t+N|t+N-1) = P_x(t+N|t+N-1)$, $\Delta_x^{(2)}(t+N,t+N|t+N-1) = P_{x\tilde{u}}(t+N|t+N-1)$, $\Delta_{\tilde{u}}^{(1)}(t+N,t+N|t+N-1) = P_{x\tilde{u}}^{\mathrm{T}}(t+N|t+N-1)$ and $\Delta_{\tilde{u}}^{(2)}(t+N,t+N|t+N-1) = P_{\tilde{u}}(t+N|t+N-1)$ are computed by Theorem 3.1. The smoothing error covariance matrices are computed by

$$P_x(t+N|t) = P_x(t+N|t-1) - \alpha M_x(t+N|t)\Lambda(t)M_x^{\mathrm{T}}(t+N|t) \tag{45}$$
$$P_{\tilde{u}}(t+N|t) = P_{\tilde{u}}(t+N|t-1) - \alpha M_{\tilde{u}}(t+N|t)\Lambda(t)M_{\tilde{u}}^{\mathrm{T}}(t+N|t) \tag{46}$$

where the initial values $P_x(t+N|t+N)$ and $P_{\tilde{u}}(t+N|t+N)$ are computed by Theorem 3.1.

Proof: Here, the proof of the state smoother is only given. The proof of the input smoother is similar. From Kalman filtering approach, we design the following unbiased smoother for the state

$$\hat{x}(t+N|t) = \hat{x}(t+N|t-1) + M_x(t+N|t)[y(t) - \bar{H}\hat{X}(t|t-1)] \tag{47}$$

Note that $\bar{H}\hat{X}(t|t-1) = \alpha H\hat{x}(t|t-1) + (1-\alpha)y(t-1)$ from (7) and (8), which together with (47) yields (40). Then smoothing error equation is given by

$$\begin{aligned} \tilde{x}(t+N|t) &= \tilde{x}(t+N|t-1) - (\xi(t)-\alpha)M_x(t+N|t)H_1X(t) \\ &\quad -\alpha M_x(t+N|t)H\tilde{x}(t|t-1) - \xi(t)M_x(t+N|t)v(t) \end{aligned} \tag{48}$$

with the smoothing error $\tilde{x}(t+N|t) = x(t+N) - \hat{x}(t+N|t)$. Hence, the smoothing error covariance matrix is given by

$$\begin{aligned} P_x(t+N|t) &= \mathrm{E}[\tilde{x}(t+N|t)\tilde{x}^{\mathrm{T}}(t+N|t)] \\ &= P_x(t+N|t-1) + \alpha(1-\alpha)M_x(t+N|t)H_1q(t)H_1^{\mathrm{T}}M_x^{\mathrm{T}}(t+N|t) \\ &\quad +\alpha^2 M_x(t+N|t)HP_x(t|t-1)H^{\mathrm{T}}M_x^{\mathrm{T}}(t+N|t) \\ &\quad +\alpha M_x(t+N|t)Q_vM_x^{\mathrm{T}}(t+N|t) \\ &\quad -\alpha\mathrm{E}[\tilde{x}(t+N|t-1)\tilde{x}^{\mathrm{T}}(t|t-1)]H^{\mathrm{T}}M_x^{\mathrm{T}}(t+N|t) \\ &\quad -\alpha M_x(t+N|t)H\mathrm{E}[\tilde{x}(t|t-1)\tilde{x}^{\mathrm{T}}(t+N|t-1)] \end{aligned} \tag{49}$$

Let $\Delta_x^{(1)}(t+N,t|t-1) = \mathrm{E}[\tilde{x}(t+N|t-1)\tilde{x}^{\mathrm{T}}(t|t-1)]$, $\Delta_x^{(2)}(t+N,t|t-1) = \mathrm{E}[\tilde{x}(t+N|t-1)\tilde{\tilde{u}}^{\mathrm{T}}(t|t-1)]$, and note that $\tilde{x}(t+N|t-1) = x(t+N) - \hat{x}(t+N|t-1)$ where $\hat{x}(t+N|t-1)$ is uncorrelated with $\tilde{x}(t|t-1)$ and $\tilde{\tilde{u}}(t|t-1)$, then we have $\Delta_x^{(1)}(t+N,t|t-1) = \mathrm{E}[x(t+N)\tilde{x}^{\mathrm{T}}(t|t-1)]$ and $\Delta_x^{(2)}(t+N,t|t-1) = \mathrm{E}[x(t+N)\tilde{\tilde{u}}^{\mathrm{T}}(t|t-1)]$. Further, from (26), (29)-(31) we have

$$\begin{aligned} \Delta_x^{(1)}(t+N,t|t-1) &= \mathrm{E}[x(t+N)\tilde{x}^{\mathrm{T}}(t-1|t-1)]\Phi^{\mathrm{T}} + \mathrm{E}[x(t+N)\tilde{\tilde{u}}^{\mathrm{T}}(t-1|t-1)]B^{\mathrm{T}} \\ &= \mathrm{E}[x(t+N)\tilde{x}^{\mathrm{T}}(t-1|t-2)][\Phi - \alpha(\Phi K_x(t-1) + BK_{\tilde{u}}(t-1))H]^{\mathrm{T}} \\ &\quad +\mathrm{E}[x(t+N)\tilde{\tilde{u}}^{\mathrm{T}}(t-1|t-2)]B^{\mathrm{T}} \end{aligned} \tag{50}$$

$$\begin{aligned} \Delta_x^{(2)}(t+N,t|t-1) &= (1-\beta)\mathrm{E}[x(t+N)\tilde{\tilde{u}}^{\mathrm{T}}(t-1|t-1)] \\ &= (1-\beta)\mathrm{E}[x(t+N)\tilde{\tilde{u}}^{\mathrm{T}}(t-1|t-2)] \\ &\quad -\alpha(1-\beta)\mathrm{E}[x(t+N)\tilde{x}^{\mathrm{T}}(t-1|t-2)]H^{\mathrm{T}}K_{\tilde{u}}^{\mathrm{T}}(t-1) \end{aligned} \tag{51}$$

From (50), (51) and the definitions of $\Delta_x^{(1)}(t+N,t|t-1)$ and $\Delta_x^{(2)}(t+N,t|t-1)$, we have (43). Then, (49) can be rewritten as

$$\begin{aligned} P_x(t+N|t) &= \alpha[M_x(t+N|t) - \Delta_x^{(1)}(t+N,t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)]\Lambda(t)[M_x(t+N|t) \\ &\quad -\Delta_x^{(1)}(t+N,t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)]^{\mathrm{T}} + P_x(t+N|t-1) \\ &\quad -\alpha\Delta_x^{(1)}(t+N,t|t-1)H^{\mathrm{T}}\Lambda^{-1}(t)H\Delta_x^{(1)}(t+N,t|t-1)^{\mathrm{T}} \end{aligned} \tag{52}$$

From (52), the first term in the right hand side of (52) must be equal to zero to minimize the error covariance matrix $P_x(t+N|t)$. Then, we have (45) and the first equation of (42).

Remark 4.1. *Theorems 3.1, 4.1 and 4.2 give the linear optimal full-order estimators for system (1)-(4). When there is no control input, i.e., $B = 0$, the proposed full-order estimators are reduced to the results in [20] with only packet dropouts of single side from the sensor to the estimator. When there are no packet dropouts of both sides, i.e., $\alpha = 1$ and $\beta = 1$, they are reduced to the standard Kalman estimators [29].*

5. **Stability Analysis and Steady-State Estimators.** In the preceding sections, we have obtained the linear optimal full-order estimators including filter, predictor and smoother. In this section, we will investigate their stability and steady-state property for $0 < \alpha, \beta < 1$.

We combine the state error $\tilde{x}(t|t)$ and input error $\tilde{\tilde{u}}(t|t)$ into a vector $[\tilde{x}^{\mathrm{T}}(t|t), \tilde{\tilde{u}}^{\mathrm{T}}(t|t)]^{\mathrm{T}}$. Now we check its covariance

$$P(t|t) = \mathrm{E}\{[\tilde{x}^{\mathrm{T}}(t|t), \tilde{\tilde{u}}^{\mathrm{T}}(t|t)]^{\mathrm{T}}[\tilde{x}^{\mathrm{T}}(t|t), \tilde{\tilde{u}}^{\mathrm{T}}(t|t)]\} = \begin{bmatrix} P_x(t|t) & P_{x\tilde{u}}(t|t) \\ P_{\tilde{u}x}(t|t) & P_{\tilde{u}}(t|t) \end{bmatrix}$$

by (26), (29)-(31), i.e., combining (21)-(25), and have the covariance matrix as

$$P(t|t) = \Psi(t)P(t-1|t-1)\Psi(t)^{\mathrm{T}} + Q(t) - K(t)S^{\mathrm{T}} - SK^{\mathrm{T}}(t) + K(t)R(t)K^{\mathrm{T}}(t) \tag{53}$$

where $\Psi(t)$, $Q(t)$, $K(t)$, S and $R(t)$ are defined by

$$\Psi(t)=\begin{bmatrix}(I_n-\alpha K_x(t)H)\Phi & (I_n-\alpha K_x(t)H)B\\ -\alpha K_{\tilde u}(t)H\Phi & (1-\beta)-\alpha K_{\tilde u}(t)HB\end{bmatrix},$$

$$K(t)=\begin{bmatrix}K_x(t)\\ K_{\tilde u}(t)\end{bmatrix},\quad S=\alpha\begin{bmatrix}\Gamma Q_w\Gamma^{\mathrm T}H^{\mathrm T}\\ 0\end{bmatrix},$$

$$Q(t)=\beta(1-\beta)\begin{bmatrix}0 & 0\\ I_r & -C\end{bmatrix}\begin{bmatrix}u(t)u^{\mathrm T}(t) & u(t)\bar X^{\mathrm T}(t)\\ \bar X(t)u^{\mathrm T}(t) & q(t)\end{bmatrix}\begin{bmatrix}0 & 0\\ I_r & -C\end{bmatrix}^{\mathrm T}+\begin{bmatrix}\Gamma Q_w\Gamma^{\mathrm T} & 0\\ 0 & 0\end{bmatrix},$$

$$R(t)=\alpha[(1-\alpha)H_1q(t)H_1^{\mathrm T}+\alpha H\Gamma Q_w\Gamma^{\mathrm T}H^{\mathrm T}+Q_v] \tag{54}$$

Note that (53) and (54) contain the terms relative to the input $u(t)$, the covariance matrix $P(t|t)$ is time-varying generally. The following theorem states the stability.

Theorem 5.1. *For system (1)-(4), if the matrix Φ is stable and the input $u(t)$ is bounded, the solution $P(t|t)$ to Equation (53) with an any initial condition $P(0|0)\ge 0$, i.e., $P_x(0|0)\ge 0$, $P_{\tilde u}(0|0)\ge 0$, $P_{x\tilde u}(0|0)$, $q(0)\ge 0$ and $\bar X(0)$, is bounded.*

Proof: From the stability of Φ and $0<\alpha,\beta<1$, it can be known that $\bar\Phi$ is stable. Then, $\bar X(t)$ computed by (11) is bounded from the bounded input $u(t)$. Further, let

$$\begin{aligned}A \;=\;& \Phi_0\otimes\Phi_0+\alpha\Phi_1\otimes\Phi_1+\beta\Phi_2\otimes\Phi_2+\alpha\Phi_0\otimes\Phi_1+\beta\Phi_0\otimes\Phi_2\\ &+\alpha\Phi_1\otimes\Phi_0+\alpha\beta\Phi_1\otimes\Phi_2+\beta\Phi_2\otimes\Phi_0+\alpha\beta\Phi_2\otimes\Phi_1\end{aligned} \tag{55}$$

We readily verify $\rho(A)<1$ where $\rho(A)$ is the spectrum radius of the matrix A and $\otimes$ is the Kronecker product. In this situation, we have that $q(t)$ computed by (10) is bounded from the bounded $u(t)$ and $\bar X(t)$. Then, we have that $Q(t)$, S and $R(t)$ are bounded.

From the stability of Φ and $0<\beta<1$, we know that $\begin{bmatrix}\Phi & B\\ 0 & (1-\beta)I_r\end{bmatrix}$ is stable. This means that the pair $\left(\begin{bmatrix}\Phi & B\\ 0 & (1-\beta)I_r\end{bmatrix},\begin{bmatrix}H & 0\end{bmatrix}\right)$ is detectable. Then there is a matrix $\begin{bmatrix}\bar K_x\\ \bar K_{\tilde u}\end{bmatrix}$ such that

$$\left(\begin{bmatrix}I_n & 0\\ 0 & I_r\end{bmatrix}-\alpha\begin{bmatrix}\bar K_x\\ \bar K_{\tilde u}\end{bmatrix}\begin{bmatrix}H & 0\end{bmatrix}\right)\begin{bmatrix}\Phi & B\\ 0 & (1-\beta)I_r\end{bmatrix}=\begin{bmatrix}(I_n-\alpha\bar K_xH)\Phi & (I_n-\alpha\bar K_xH)B\\ -\alpha\bar K_{\tilde u}H\Phi & (1-\beta)I_r-\alpha\bar K_{\tilde u}HB\end{bmatrix}$$

is stable. So, we define two stable suboptimal filters as follows

$$\begin{aligned}\hat x_{so}(t|t) \;=\;& (I_n-\alpha\bar K_xH)\Phi\hat x_{so}(t-1|t-1)+(I_n-\alpha\bar K_xH)B\hat{\tilde u}_{so}(t-1|t-1)\\ &+\bar K_xy(t)-(1-\alpha)\bar K_xy(t-1)\end{aligned} \tag{56}$$

$$\begin{aligned}\hat{\tilde u}_{so}(t|t) \;=\;& ((1-\beta)I_r-\alpha\bar K_{\tilde u}HB)\hat{\tilde u}_{so}(t-1|t-1)-\alpha\bar K_{\tilde u}H\Phi\hat x_{so}(t-1|t-1)\\ &+\beta u(t)+\bar K_{\tilde u}y(t)-(1-\alpha)\bar K_{\tilde u}y(t-1)\end{aligned} \tag{57}$$

Combining (56) and (57), let the covariance matrix of the suboptimal filter $[\hat x_{so}^{\mathrm T}(t|t),\ \hat{\tilde u}_{so}^{\mathrm T}(t|t)]^{\mathrm T}$ be $P_{so}(t|t)$ under the same initial values with (53). From the suboptimality, we have $P(t|t)\le P_{so}(t|t)$. From the stability of (56) and (57), we have the bounded solution $P_{so}(t|t)$ for any initial value. So, $P(t|t)$ is bounded.

Since the covariance matrix $P(t|t)$ depends on the input $u(t)$, the covariance matrix $P(t|t)$ has not the steady-state value generally, particularly for the time-varying $u(t)$. To investigate the steady-state filter, we first check Equations (10) and (11).

Theorem 5.2. *For system (1)-(4), if matrix Φ is stable and the input is constant, i.e., $u(t) = u$, the solutions $q(t)$ and $\bar{X}(t)$ to Equations (10) and (11) with arbitrary initial conditions $q(0) \geq 0$ and $\bar{X}(0)$ converge exponentially to the unique solutions $q \geq 0$ and $\bar{X}$ to the following algebraic Lyapunov equation and simple difference equation*

$$\begin{aligned} q \;=\; & \Phi_0 q \Phi_0^{\mathrm{T}} + \alpha \Phi_1 q \Phi_1^{\mathrm{T}} + \beta \Phi_2 q \Phi_2^{\mathrm{T}} + \alpha \Phi_0 q \Phi_1^{\mathrm{T}} + \beta \Phi_0 q \Phi_2^{\mathrm{T}} + \alpha \Phi_1 q \Phi_0^{\mathrm{T}} \\ & + \alpha\beta \Phi_1 q \Phi_2^{\mathrm{T}} + \beta \Phi_2 q \Phi_0^{\mathrm{T}} + \alpha\beta \Phi_2 q \Phi_1^{\mathrm{T}} + \beta B_1 u u^{\mathrm{T}} B_1^{\mathrm{T}} \\ & + \beta[\Phi_0 + \alpha\Phi_1 + \Phi_2]\bar{X} u^{\mathrm{T}} B_1^{\mathrm{T}} + \beta B_1 u \bar{X}^{\mathrm{T}} [\Phi_0 + \alpha\Phi_1 + \Phi_2]^{\mathrm{T}} + Q_0 \end{aligned} \tag{58}$$

$$\bar{X} = \bar{\Phi}\bar{X} + \bar{B}u \tag{59}$$

i.e., $q = \lim_{t\to\infty} q(t)$ and $\bar{X} = \lim_{t\to\infty} \bar{X}(t)$. Moreover, we have $\lim_{t\to\infty} Q(t) = Q$ and $\lim_{t\to\infty} R(t) = R$. Q and R are defined by (54) where $q(t)$ and $\bar{X}(t)$ are replaced by q and $\bar{X}$, respectively.

Proof: From the proof of Theorem 5.1, we know that the stability of Φ and $0 < \alpha, \beta < 1$ mean the stability of $\bar{\Phi}$ and $\rho(A) < 1$. Note that the input is a constant u, then $\bar{X}(t)$ computed by (11) with an any initial value $\bar{X}(0)$ will converge exponentially to $\bar{X}$ computed by (59), i.e., $\bar{X} = \lim_{t\to\infty} \bar{X}(t)$, and $q(t)$ computed by (10) with any initial value $q(0) \geq 0$ will converge exponentially to the solution q to the algebraic Lyapunov Equation (58), i.e., $q = \lim_{t\to\infty} q(t)$. Further, we have $\lim_{t\to\infty} Q(t) = Q$ and $\lim_{t\to\infty} R(t) = R$.

The following theorem gives a sufficient condition for the existence of the steady-state filters for systems with packet dropouts of both sides.

Theorem 5.3. *For system (1)-(4) with the constant input u, if matrix Φ is stable and the pair $\left(\begin{bmatrix} \Phi & B \\ 0 & (1-\beta) I_r \end{bmatrix} - SR^{-1} \begin{bmatrix} H & 0 \end{bmatrix}, \bar{Q} \right)$ is stabilizable where $\bar{Q}$ satisfies $\bar{Q}\bar{Q}^{\mathrm{T}} = Q - SR^{-1}S^{\mathrm{T}}$, the solution $P(t|t)$ to Equation (53) with an arbitrary initial condition $P(0|0)$, i.e., $P_x(0|0) \geq 0$, $P_{\tilde{u}}(0|0) \geq 0$, $P_{x\tilde{u}}(0|0)$, $q(0) \geq 0$ and $\bar{X}(0)$, converges exponentially to the unique positive semi-definite solution Σ to the following algebraic Riccati equation*

$$\Sigma = \Psi \Sigma \Psi^{\mathrm{T}} + Q - KS^{\mathrm{T}} - SK^{\mathrm{T}} + KRK^{\mathrm{T}} \tag{60}$$

with $\Psi = \begin{bmatrix} (I_n - \alpha K_x H)\Phi & (I_n - \alpha K_x H)B \\ -\alpha K_{\tilde{u}} H \Phi & (1-\beta) - \alpha K_{\tilde{u}} H B \end{bmatrix}$. Furthermore, we have that $\Sigma = \lim_{t\to\infty} P(t|t)$, $K_x = \lim_{t\to\infty} K_x(t)$, $K_{\tilde{u}} = \lim_{t\to\infty} K_{\tilde{u}}(t)$ and $K = \begin{bmatrix} K_x \\ K_{\tilde{u}} \end{bmatrix}$, and the steady-state filters

$$\begin{aligned} \hat{x}(t|t) \;=\; & (I_n - \alpha K_x H)\Phi \hat{x}(t-1|t-1) + (I_n - \alpha K_x H) B \hat{\tilde{u}}(t-1|t-1) \\ & + K_x y(t) - (1-\alpha) K_x y(t-1) \end{aligned} \tag{61}$$

$$\begin{aligned} \hat{\tilde{u}}(t|t) \;=\; & ((1-\beta) I_r - \alpha K_{\tilde{u}} H B)\hat{\tilde{u}}(t-1|t-1) - \alpha K_{\tilde{u}} H \Phi \hat{x}(t-1|t-1) \\ & + \beta u + K_{\tilde{u}} y(t) - (1-\alpha) K_{\tilde{u}} y(t-1) \end{aligned} \tag{62}$$

are asymptotically stable.

Proof: For system (1)-(4) with the constant input, we introduce a new Riccati equation

$$\breve{P}(t|t) = \breve{\Psi}(t)\breve{P}(t-1|t-1)\breve{\Psi}^{\mathrm{T}}(t) + Q - \breve{K}(t)S^{\mathrm{T}} - S\breve{K}^{\mathrm{T}}(t) + \breve{K}(t)R\breve{K}^{\mathrm{T}}(t) \tag{63}$$

with the initial value $\breve{P}(0|0) = P(0|0)$. $\breve{\Psi}(t)$ and $\breve{K}(t)$ have the same definitions as $\Psi(t)$ and $K(t)$ except that $q(t)$ and $\bar{X}(t)$ are replaced by q and $\bar{X}$. Then, from $q = \lim_{t\to\infty} q(t)$ and $\bar{X} = \lim_{t\to\infty} \bar{X}(t)$, we have $\lim_{t\to\infty} \left\| \Psi(t) - \breve{\Psi}(t) \right\| = 0$, $\lim_{t\to\infty} \left\| K(t) - \breve{K}(t) \right\| = 0$ and $\lim_{t\to\infty} \left\| P(t) - \breve{P}(t) \right\| = 0$.

On the other hand, from the proof of Theorem 5.1, we have that the pair

$$\left(\begin{bmatrix} \Phi & B \\ 0 & (1-\beta)I_r \end{bmatrix}, \begin{bmatrix} H & 0 \end{bmatrix}\right)$$

is detectable. Also, the pair

$$\left(\begin{bmatrix} \Phi & B \\ 0 & (1-\beta)I_r \end{bmatrix} - SR^{-1}\begin{bmatrix} H & 0 \end{bmatrix}, \bar{Q}\right)$$

is stabilizable where $\bar{Q}$ satisfies $\bar{Q}\bar{Q}^{\mathrm{T}} = Q - SR^{-1}S^{\mathrm{T}}$, then the solution $\breve{P}(t|t)$ to Equation (63) will converge exponentially to the unique positive semi-definite solution Σ to the algebraic Riccati Equation (60), i.e., $\lim_{t\to\infty} \breve{P}(t|t) = \Sigma$, as well as $\lim_{t\to\infty} \breve{\Psi}(t) = \Psi$ and $\lim_{t\to\infty} \breve{K}(t) = K$. Also, the matrix Ψ is stable, which means the stability of the filters (61) and (62) [29].

Based on the preceding analysis, we have

$$\begin{aligned} 0 &\le \lim_{t\to\infty} \|P(t|t) - \Sigma\| \\ &= \lim_{t\to\infty} \left\|P(t|t) - \breve{P}(t|t) + \breve{P}(t|t) - \Sigma\right\| \\ &\le \lim_{t\to\infty} \left\|P(t|t) - \breve{P}(t|t)\right\| + \lim_{t\to\infty} \left\|\breve{P}(t|t) - \Sigma\right\| = 0 \end{aligned} \tag{64}$$

which means $\lim_{t\to\infty} P(t|t) = \Sigma$. Furthermore, we also have $\lim_{t\to\infty} \Psi(t) = \Psi$ and $\lim_{t\to\infty} K(t) = K$.

From Theorems 3.1, 4.1 and 4.2, it can be known that the existence of the steady-state filter implies that of the steady-state predictor and smoother.

Remark 5.1. *The proof of the existence of steady-state filters is different from [24] where the packet dropout rate from the controller to the actuator is removed by approximately treating. Here, we introduce an additional Riccati equation (63) which plays a bridge between (53) and (60). Furthermore, we do not remove the packet dropout rate from the controller to the actuator since the packet dropout rate almost exists in networks and it is independent of the transmitted signals. Certainly, they have the same steady-state value when the time t approaches infinite except for the different transient process.*

Remark 5.2. *In Theorems 5.1-5.3, the stability and the steady-state property require matrix Φ to be stable. This condition is, in fact, necessary as in robust filtering [30] because the presence stochastic or deterministic parameter uncertainty implies that the filtering error dynamics cannot be decoupled from the system state. It is clear from (26) if the system is unstable, its unbounded state will drive the filtering error to infinite.*

6. **Simulation Examples.**

Example 6.1. *Consider an example (1)-(4) similar to [22-24]*

$$\Phi = \begin{bmatrix} 1.7240 & -0.7788 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \Gamma = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \; H = \begin{bmatrix} 0.0286 & 0.0264 \end{bmatrix}, \tag{65}$$

In simulation, we set the variance $Q_w = 1$ and $Q_v = 1$. We take 100 sampling data. The initial values $x(0) = [2, -2]^{\mathrm{T}}$ and $P_0 = 0.1I_2$, where I_2 is the 2×2 identity matrix. Our aim is to find the linear optimal full-order filters $\hat{x}(t|t)$ and $\hat{\hat{u}}(t|t)$, predictors $\hat{x}(t|t-1)$ and $\hat{\hat{u}}(t|t-1)$, and smoothers $\hat{x}(t|t+1)$ and $\hat{\hat{u}}(t|t+1)$.

Under $\alpha = 0.2$, $\beta = 0.8$ and $u(t) = 2\sin(0:0.1:0.1*100)$, the linear optimal filter for the first state component is shown in Figure 1. Figure 2 shows the estimation error

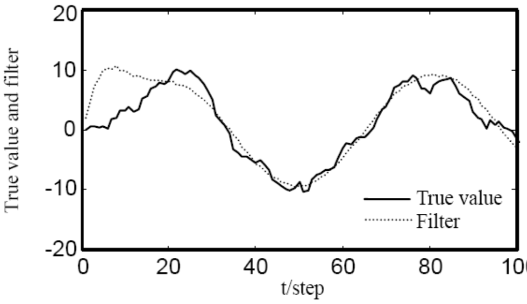


FIGURE 1. Linear optional filter for the first state component under $\alpha = 0.2$ and $\beta = 0.8$

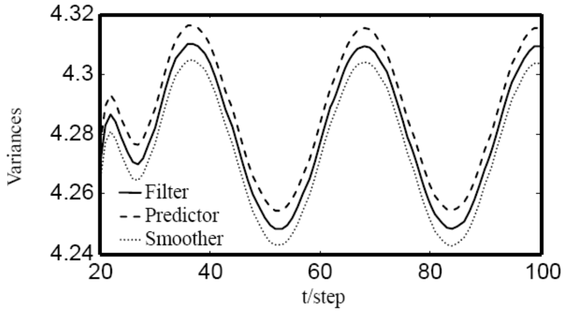
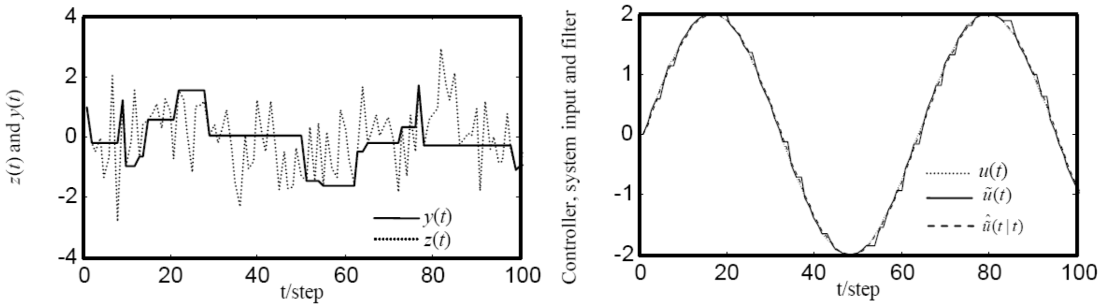


FIGURE 2. Estimation error variances for the first state component of the optimal estimators under $\alpha = 0.2$ and $\beta = 0.8$



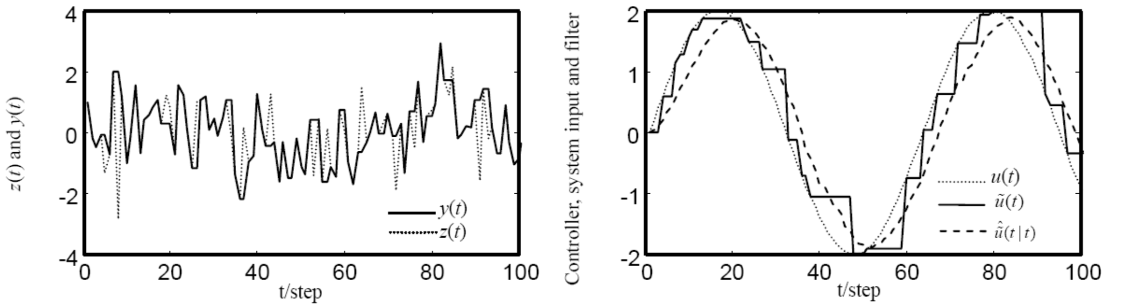
(a) Measurements at sensor and estimator

(b) Controller $u(t)$, system input $\tilde{u}(t)$ and filter $\hat{\tilde{u}}(t|t)$

FIGURE 3. Packet dropouts of measurement and input under $\alpha = 0.2$ and $\beta = 0.8$ and input filter

variances of the state filter, predictor and smoother within $20 \leq t \leq 100$. From Figure 2, it can be seen that the smoother has the best accuracy while the predictor has the worst, and the variances are bounded, which means the stability of the estimators. Figure 3 with $\alpha = 0.2$, $\beta = 0.8$ and Figure 4 with $\alpha = 0.8$, $\beta = 0.2$ show the effects of packet dropouts on the measurements and control inputs, respectively, where (a)s of Figure 3 and Figure 4 compare the difference between the measurements $z(t)$ and $y(t)$, and (b)s of Figure 3 and Figure 4 compare the difference between the controllers $u(t)$ and $\tilde{u}(t)$. Moreover, the input filters $\hat{\tilde{u}}(t|t)$ are also shown in (b)s of Figure 3 and Figure 4. We see that the input filters have effective estimation accuracy. Figure 5 shows the comparison of square roots of mean square errors (SRMSE) of our prior filter and those in [22-24] by 100-time Monte-Carlo test. We see that our filter has better accuracy than [22,24]. Moreover, our full-order filter has the reduced online computational cost than the augmented filter in [24] though they have the same accuracy.

Figures 1-5 have shown the performance of the state and input estimators under the bounded time-varying input. Next, we check the steady-state property under the time-invariant input. We set $u(t) = 10$, $\alpha = 0.5$ and $\beta = 0.1$. Figure 6 shows the performance of the filter, predictor and smoother for the input. Figure 7 shows that estimation error variances for the input, where (a) shows the variances via time t, and (b) shows the variances via α under $\beta = 0.1$ or β under $\alpha = 0.5$ at time $t = 15$. It can be seen that the estimation accuracy becomes better as α or β increases. Moreover, the accuracy improves



(a) Measurements at sensor and estimator (b) Controller $u(t)$, system input $\tilde{u}(t)$ and filter $\hat{\tilde{u}}(t|t)$

FIGURE 4. Packet dropouts of measurement and input under $\alpha = 0.8$ and $\beta = 0.2$ and input filter

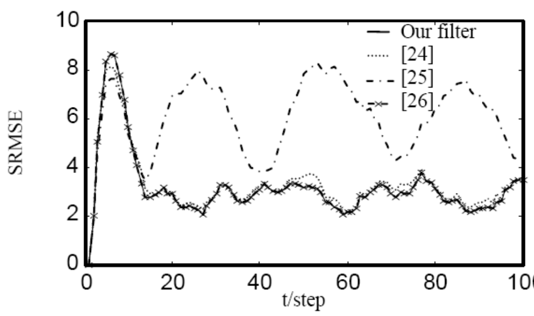


FIGURE 5. Comparison of SRMSEs for the first state component of our filter and filters in [22-24] under $\alpha = 0.8$ and $\beta = 0.2$

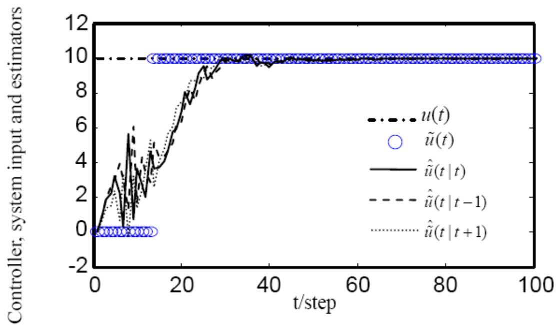
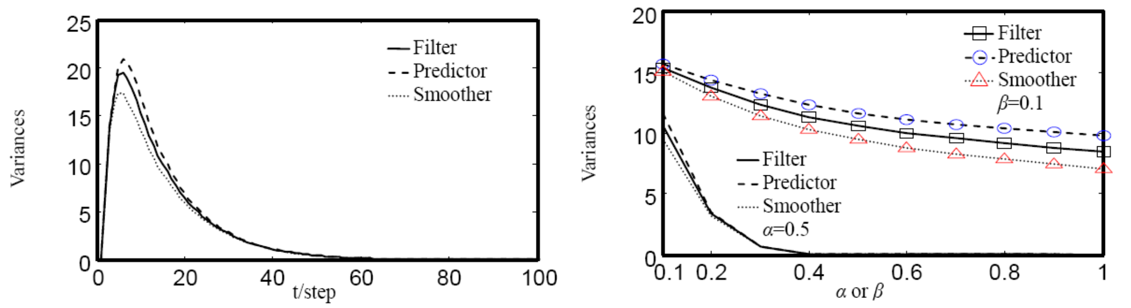


FIGURE 6. Controller $u(t)$, system input $\tilde{u}(t)$ and estimators $\hat{\tilde{u}}(t|t)$, $\hat{\tilde{u}}(t|t-1)$ and $\hat{\tilde{u}}(t|t+1)$ under $\alpha = 0.5$ and $\beta = 0.1$



(a) Estimation error variances for the system input $\tilde{u}(t)$ under $\alpha = 0.5$ and $\beta = 0.1$ (b) Estimation error variances for system input $\tilde{u}(t)$ under $\alpha = 0.5$, $0.1 \leq \beta \leq 1$ and $\beta = 0.1$, $0.1 \leq \alpha \leq 1$, at $t = 15$

FIGURE 7. Estimation error variances for the linear optimal filter, predictor and smoother of the system input

faster as β increases, which is reasonable since the input is time-invariant and it will almost be used later once it is received. Figure 8 shows the estimation error variances of the filter, predictor and smoother for the first component of the state. Figure 9 shows the

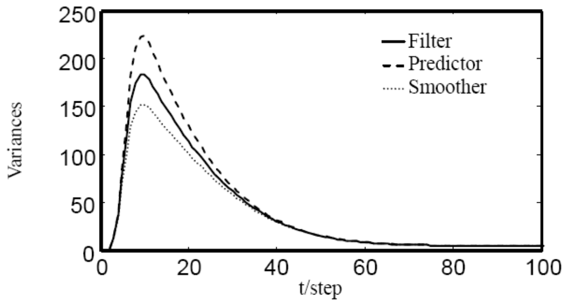


FIGURE 8. Estimation error variances for the first state component under $\alpha = 0.5$ and $\beta = 0.1$

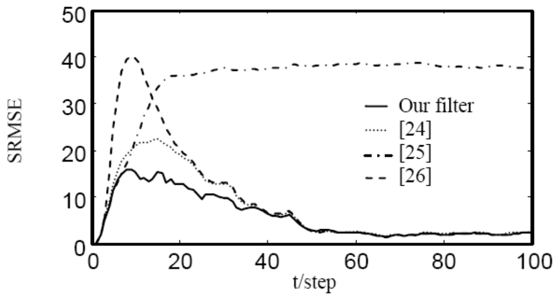


FIGURE 9. SRMSE for the first state component of our filter and filters in [22-24] under $\alpha = 0.5$ and $\beta = 0.1$

comparison of square roots of mean square errors (SRMSEs) for our filter and those in [22-24] by 100-time Monte-Carlo test. We see that our filter has better accuracy than [22-24] before the filters enter the steady state. Furthermore, our filter has the same steady-state performance as [24] and better accuracy than [22,23] as the time t approaches infinity. Though our filter and the augmented filter in [24] have the same steady-state performance, our filter has the reduced online computational cost (see Remark 3.1). At the same time, the H_∞ filter in [23] has the bad accuracy in mean square error.

Example 6.2. *Consider the following mass-spring system shown in Figure 10.*

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{\mu}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & -\frac{\mu}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} \tilde{u}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} w(t) \quad (66)$$

$$z(t) = Hx(t) + v(t) \quad (67)$$

where $x(t) = [\ x_1(t) \quad x_2(t) \quad \dot{x}_1(t) \quad \dot{x}_2(t)\ ]^{\mathrm{T}}$, x_1, x_2 *and* m_1, m_2 *are the positions and masses, respectively,* k_1 *and* k_2 *are the spring constants,* μ *is the viscous friction coefficient between the masses and the horizontal surface. The process noise and measurement noise are uncorrelated. Our aim is to find the linear optimal full-order filters* $\hat{x}(t|t)$ *and* $\hat{\tilde{u}}(t|t)$.

In the simulation, we take $Q_w = 1$, $Q_v = I_2$, $\alpha = 0.8$, $\beta = 0.5$, $u(t) = \sin(4\pi t/100)$, $m_1 = 1$, $m_2 = 0.5$, $k_1 = 1$, $k_2 = 1$, $\mu = 0.5$, and the sampling period $T = 1$s, then we have the parameters of the corresponding discretized system of system (66)-(67) as follows:

$$\Phi = \begin{bmatrix} 0.3273 & 0.3089 & 0.5610 & 0.0951 \\ 0.5227 & 0.4224 & 0.1902 & 0.4541 \\ -0.9318 & 0.3708 & 0.0468 & 0.2138 \\ 0.5278 & -0.7180 & 0.4276 & -0.0317 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0549 \\ 0.6325 \\ 0.1902 \\ 0.9082 \end{bmatrix},$$
$$\Gamma = \begin{bmatrix} 1.2567 \\ 1.3592 \\ 0.2924 \\ 0.5895 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (68)$$

Optimal linear filters for the mass positions are shown in Figure 11. Figure 12 gives the filter of the input $\tilde{u}(t)$. It can be seen that our filters designed are effective.

7. **Conclusions.** For NCSs with multiple packet dropouts in data transmissions of both sides from the sensor to the estimator and from the controller to the actuator, we have

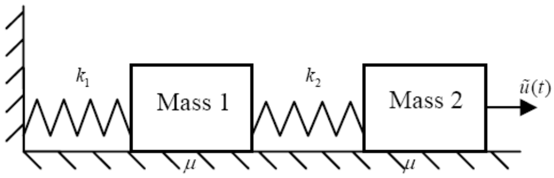
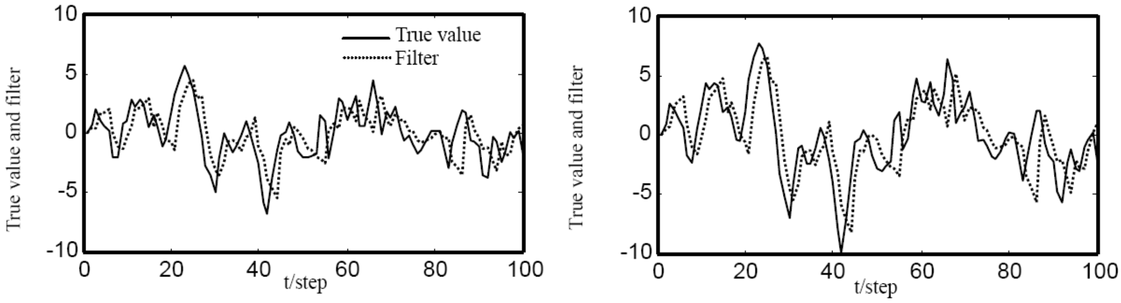


FIGURE 10. Mass-spring system



(a) Filter for the position of mass 1

(b) Filter for the position of mass 2

FIGURE 11. Linear optimal filters for the mass positions under $\alpha = 0.8$ and $\beta = 0.5$

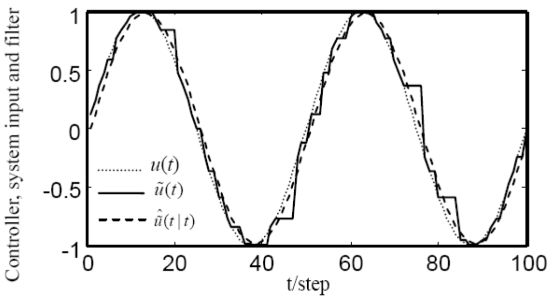


FIGURE 12. Controller $u(t)$, system input $\tilde{u}(t)$ and filter $\hat{\tilde{u}}(t|t)$ under $\alpha = 0.8$ and $\beta = 0.5$

derived the linear optimal full-order filter, predictor and smoother for the state and input in the least mean square sense via completing square approach. Our solutions are given in terms of a Riccati recursion, a Lyapunov recursion and a simple difference recursion. They have the reduced online computational cost. The sufficient conditions for the stability of the estimators and the existence of the steady-state estimators have been given.

Acknowledgment. This work was supported by Natural Science Foundation of China under Grant NSFC-60874062, NSFC-61174139, Program for New Century Excellent Talents in University under Grant NCET-10-0133, Program for High-qualified Talents under Grant Hdtd2010-03, and Electronic Engineering Province Key Laboratory. The author is indebted to the editor and referees for valuable comments and suggestions.

REFERENCES

[1] V. Gupta, D. Spanos, B. Hassibi and R. M. Murray, Optimal LQG control across a packet-dropping links, *Proc. of American Control Conference*, pp.360-365, 2005.

[2] L. Schenato, Optimal estimation in networked control systems subject to random delay and packet loss, *Proc. of the 45th IEEE Conference on Decision and Control*, San Diego, CA, pp.5615-5620, 2006.

[3] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan and S. Sastry, Kalman filtering with intermittent observations, *IEEE Trans. on Automatic Control*, vol.49, no.9, pp.1453-1464, 2004.

[4] H. Zhang and L. Xie, *Control and Estimation of Systems with Input/Output Delays*, Springer, Berlin, 2007.

[5] A. S. Matveev and A. V. Savkin, Optimal computer control via communication channels with irregular transmission times, *International Journal of Control*, vol.76, no.2, pp.165-177, 2003.

[6] F. Yang, Z. Wang, D. W. C. Ho and M. Gani, Robust H-infinity control with missing measurements and time-delays, *IEEE Trans. on Automatic Control*, vol.52, no.9, pp.1666-1672, 2007.

[7] A. Ray, L. W. Liou and J. H. Shen, State estimation using randomly delayed measurements, *Journal Dynamic System, Measurement, and Control*, vol.115, no.1, pp.19-26, 1993.

[8] E. Yaz and A. Ray, Linear unbiased state estimation under randomly varying bounded sensor delay, *Applied Mathematics Letters*, vol.11, no.1, pp.27-32, 1998.

[9] S. Nakamori, R. Caballero-Aguila, A. Hermoso-Carazo and J. Linares-Perez, Recursive estimators of signals from measurements with stochastic delays using covariance information, *Applied Mathematics and Computation*, vol.162, no.1, pp.65-79, 2005.

[10] Z. Wang, D. W. C. Ho and X. Liu, Robust filtering under randomly varying sensor delay with variance constraints, *IEEE Trans. on Circuits and Systems-II: Express Briefs*, vol.51, no.6, pp.320-326, 2004.

[11] Y. I. Yaz, E. E. Yaz and M. J. Mohseni, LMI-based state estimation for some discrete-time stochastic models, *Proc. of IEEE International Conference on Control Applications*, pp.456-460, 1998.

[12] Z. Wang, F. Yang, D. W. C. Ho and X. Liu, Robust H-infinity filtering for stochastic time-delay systems with missing measurements, *IEEE Trans. on Signal Processing*, vol.54, no.7, pp.2579-2587, 2006.

[13] F. Carravetta, A. Germani and M. Raimondi, Polynomial filtering for discrete-time stochastic linear systems with multiplicative state noise, *IEEE Trans. on Automatic Control*, vol.42, no.8, pp.1106-1126, 1997.

[14] A. De Santis, A. Germani and M. Raimondi, Optimal quadratic filtering of linear discrete-time non-Gaussian systems, *IEEE Trans. on Automatic Control*, vol.40, no.7, pp.1274-1278, 1995.

[15] N. Xiao, L. H. Xie and M. Y. Fu, Kalman filtering over unreliable communication networks with bounded Markovian packet dropouts, *International Journal of Robust and Nonlinear Control*, vol.19, no.16, pp.1770-1786, 2009.

[16] B. Azimi-Sadjadi, Stability of networked control systems in the presence of packet losses, *Proc. of the 43rd IEEE Conference on Decision and Control*, Atlantis, Bahamas, pp.676-681, 2004.

[17] V. B. Kolmanovskii, T. L. Maizenberg and J. P. Richard, Mean square stability of difference equations with a stochastic delay, *Nonlinear Analysis*, vol.52, no.3, pp.795-804, 2003.

[18] M. Sahebsara, T. Chen and S. L. Shah, Optimal H_2 filtering with random sensor delay, multiple packet dropout and uncertain observations, *International Journal of Control*, vol.80, no.2, pp.292-301, 2007.

[19] S. L. Sun, L. H. Xie, W. D. Xiao and Y. C. Soh, Optimal linear estimation for systems with multiple packet dropouts, *Automatica*, vol.44, no.5, pp.1333-1342, 2008.

[20] S. L. Sun, L. H. Xie and W. D. Xiao, Optimal full-order and reduced-order estimators for discrete-time systems with multiple packet dropouts, *IEEE Trans. on Signal Processing*, vol.56, no.8, pp.4031-4038, 2008.

[21] S. L. Sun, L. H. Xie, W. D. Xiao and N. Xiao, Optimal filtering for systems with multiple packet dropouts, *IEEE Trans. on Circuits and Systems II: Express Briefs*, vol.55, no.7, pp.695-699, 2008.

[22] M. Sahebsara, T. Chen and S. L. Shah, Optimal H_2 filtering in networked control systems with multiple packet dropout, *IEEE Trans. on Automatic Control*, vol.52, no.8, pp.1508-1513, 2007.

[23] M. Sahebsara, T. Chen and S. L. Shah, Optimal H-infinity filtering in networked control systems with multiple packet dropouts, *Systems & Control Letters*, vol.57, no.9, pp.696-702, 2008.

[24] Y. Liang, T. Chen and Q. Pan, Optimal linear state estimator with multiple packet dropouts, *IEEE Trans. on Automatic Control*, vol.55, no.6, pp.1428-1433, 2010.

[25] J. Ma and S. L. Sun, Information fusion estimators for systems with multiple sensors of different packet dropout rates, *Information Fusion*, vol.12, no.3, pp.213-222, 2011.

[26] Y. Q. Xia, J. Z. Shang, J. Chen and G. P. Liu, Networked data fusion with packet losses and variable delays, *IEEE Trans. on Systems, Man, and Cybernetics, Part B*, vol.39, no.5, pp.1107-1120, 2009.

[27] S. Gillijns and B. D. Moor, Unbiased minimum-variance input and state estimation for linear discrete-time systems, *Automatica*, vol.43, no.1, pp.111-116, 2007.
[28] C. S. Hsieh, Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs, *Automatica*, vol.45, no.9, pp.2149-2153, 2009.
[29] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall, Englewood Cliffs, New Jersey, 1979.
[30] F. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation*, CRC Press, Boca Raton, FL, 2007.