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STATE ESTIMATION AND SLIDING MODE CONTROL OF UNCERTAIN SWITCHED HYBRID SYSTEMS

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Abstract. *This paper is concerned with the sliding mode control (SMC) design methodology for a class of uncertain switched hybrid systems with the unmeasurable states. A state observer is designed first, and then some conditions for the convergence of the estimated state error are obtained. By matrix transformation techniques, the solvability condition for the corresponding observer gain is established in terms of a set of linear matrix inequalities (LMIs). Based on the estimated states, a sliding surface is constructed, which guarantees the equivalent sliding motion restrict to the sliding surface under a designed switching law. A SMC law is then synthesized for the reaching motion such that the trajectories of the resulting closed-loop system can be driven onto a prescribed sliding surface and maintained there for all subsequent time. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approaches.*
Keywords: Uncertain switched systems, Observer, Sliding mode control (SMC), Linear matrix inequality (LMI)

1. **Introduction.** Switched systems consist of a family of subsystems, which described by continuous- or discrete-time dynamics, and a rule specifying the switching among them [1]. Switched systems have gained a great deal of attention in the past few years, since many real-world systems such as power systems and power electronics [2], transmission and stepper motors [3] can be modelled as switched systems. Many results have been reported for switched systems, such as stability and stabilization problems [3, 4, 5, 6], optimal performance analysis and control problems [7, 8], robust filtering problem [9, 10], and model reduction problem [11]. In practice, considering that the states of many actual switched systems are not all measurable, thus a state observer should be designed to estimate the system states, and then synthesize the system with observer-based controller. In [12, 13], an observer is designed based on common Lyapunov function for continuous-time and discrete-time switched systems, respectively. The observer-based switched state feedback stabilization for switched linear systems with time delay in the detection of switched signal is investigated in [14].

On the other hand, sliding mode control (SMC) belongs to the class of so-called variable structure controls, which consists of both a set of feedback control laws and a decision rule allowing us to select at any time the right law according to the current state of the system. More specifically, the principle of SMC is to constrain the time evolution of a system in the very neighborhood of a prescribed manifold of the state space called the sliding surface [15]. SMC has been proven to be an effective robust control strategy. It has been successfully applied to a wide variety of practical engineering systems such as robot manipulators, aircraft, underwater vehicles, spacecraft, flexible space structures, electrical motors, power systems, and automotive engines [16]. It also has been investigated for switched systems. To mention a few, Lian et al. proposed a strategy of model reference adaptive integral sliding mode variable structure control to solve the tracking problem for a class of uncertain switched systems with time-varying delay [17]; Saadaoui et al. designed an observer for switched mechanical systems based on the high order sliding mode technique [18]; Wu and Lam investigated the SMC of switched hybrid systems with time-varying delay [19]; Wu et al. studied the SMC of switched hybrid systems with stochastic perturbation [20]; Zhong et al. addressed the SMC problem for uncertain stochastic systems [21].

This paper is concerned with the design of SMC based on observer for a class of switched systems. This is a new problem in SMC and switched systems research areas. In this work, some conditions of the convergence of the estimated state error for the state observer are given, and the solution to the corresponding observer gain is formulated in the form of a set of linear matrix inequalities (LMIs). In addition, the sliding mode controller for each subsystem and a set of switching laws are designed by using restructured state. The designed SMC law is proved to guarantee that the state trajectories of the resultant SMC system can be driven onto the specified sliding surfaces for each mode in a finite time, and the overall switched closed-loop system is asymptotically stable. Finally, a numerical example is provided to illustrate the effectiveness of the proposed theory.

The rest of this paper is organized as follows. The SMC of uncertain switched hybrid systems is formulated in Section 2. Sections 3 and 4 present our main results on observer design and SMC design. A numerical example is given in Section 5 and we conclude this paper in Section 6.

Notations: The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent the identity matrix and a zero matrix, respectively; $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix, for a vector $a = (a_i) \in \mathbb{R}^n$. In symmetric block matrices or long matrix expressions, we use a star ($\star$) to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. **Problem Statement and Preliminaries.**

Consider a class of switched systems represented by the following state-space description:

$$\begin{aligned} \dot{x}(t) &= \left(A_{\sigma(t)} + \Delta A_{\sigma(t)}\right) x(t) + B\left(u_{\sigma(t)}(t) + f_{\sigma(t)}(x,t)\right), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u_{\sigma(t)}(t) \in \mathbb{R}^m$ is the control input vector; $y(t) \in \mathbb{R}^r$ is the measurable output vector; $f_{\sigma(t)}(x,t) \in \mathbb{R}^m$ represents the nonlinear uncertainties of the system. $A_{\sigma(t)}$, B and C are constant matrices of appropriate dimensions, and $\Delta A_{\sigma(t)}$ denotes the parameter uncertainty. Assume that B is of the full column rank of m, and C is of the full row rank of r. $\sigma(t) : [0,\infty) \to \varphi$ is the piecewise constant switching signal that may depend on either time t or state x. The value of $\sigma(t)$ is generated by restructured state $x(t)$ and other hybrid scheme. Systems (1) consists of a finite family of N continuous-time subsystems and a switching rule for switching between them. The rule defines a switching sequence that describes the temporal evolution of the discrete state. For each possible value $\sigma(t) = i$, $i \in \varphi$, we will denote the system matrices associated with mode i by

$$A_{\sigma(t)} = A_i, \quad \Delta A_{\sigma(t)} = \Delta A_i, \quad u_{\sigma(t)}(t) = u_i(t), \quad f_{\sigma(t)}(x,t) = f_i(x,t).$$

2.1. **Several definitions and theorems.** The following assumptions are introduced.

Assumption 1. *Matrix pair (A_i, B) is controllable and (A_i, C) is detectable for all $i \in \varphi$.*

Assumption 2. *The uncertainty ΔA_i is norm-bounded, and it can be represented by*

$$\Delta A_i = E\Sigma_i(t)F, \quad i \in \varphi,$$

where E and F are known constant matrices of appropriate dimensions, and $\Sigma_i(t)$ are unknown time-varying uncertainties satisfying $\Sigma_i^T(t)\Sigma_i(t) \leq I$.

Assumption 3. *There exists a known nonnegative scalar-valued function $\eta_i(x,t)$ such that $\|f_i(x,t)\| \leq \eta_i(x,t)$ for all t and $i \in \varphi$.*

Since B is of full column rank by assumption, there exists a nonsingular matrix T such that

$$TB = \begin{bmatrix} 0_{(n-m)\times m} \\ B_1 \end{bmatrix},$$

where $B_1 \in \mathbb{R}^{m\times m}$ is nonsingular. We define the nonsingular matrix T as [25],

$$T = \begin{bmatrix} \tilde{B}^T \\ B^T \end{bmatrix},$$

where $\tilde{B}$ is an orthogonal complement of matrix B. By means of the state transformation $z(t) = Tx(t)$, system (1) becomes the following regular form:

$$\begin{aligned} \dot{z}(t) &= \left(\bar{A}_{\sigma(t)} + \Delta\bar{A}_{\sigma(t)}\right) z(t) + \bar{B}\left(u_{\sigma(t)}(t) + f_{\sigma(t)}(x,t)\right), \\ y(t) &= \bar{C}z(t), \end{aligned} \tag{2}$$

where $\bar{A}_\sigma = TA_\sigma T^{-1}$, $\Delta\bar{A}_\sigma = T\Delta A_\sigma T^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$ and

$$T^{-1} = \begin{bmatrix} \tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1} & B\left(B^TB\right)^{-1} \end{bmatrix}.$$

Let $z(t) = \begin{bmatrix} z_1^T(t) & z_2^T(t) \end{bmatrix}^T$ with $z_1(t) \in \mathbb{R}^{n-m}$, $z_2(t) \in \mathbb{R}^m$ and

$$\bar{A}_{\sigma(t)} + \Delta\bar{A}_{\sigma(t)} = \begin{bmatrix} \bar{A}_{\sigma 11} & \bar{A}_{\sigma 12} \\ \bar{A}_{\sigma 21} & \bar{A}_{\sigma 22} \end{bmatrix}$$

We know that system (2) can be expressed in the following regular form:

$$\begin{aligned} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} &= \begin{bmatrix} \bar{A}_{\sigma 11} & \bar{A}_{\sigma 12} \\ \bar{A}_{\sigma 21} & \bar{A}_{\sigma 22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times m} \\ B_1 \end{bmatrix} \left(u_\sigma(t) + f_\sigma(x,t)\right), \\ y(t) &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \end{aligned} \tag{3}$$

where $\bar{A}_{\sigma 11} \in \mathbb{R}^{r\times r}$, $\bar{A}_{\sigma 12} \in \mathbb{R}^{r\times(n-r)}$, $\bar{A}_{\sigma 21} \in \mathbb{R}^{(n-r)\times r}$, $\bar{A}_{\sigma 22} \in \mathbb{R}^{(n-r)\times(n-r)}$, $\bar{B}_{\sigma 1} \in \mathbb{R}^{r\times m}$, $\bar{A}_{\sigma 11} = \tilde{B}^TA_\sigma\tilde{B}(\tilde{B}^T\tilde{B})^{-1} + \tilde{B}^TE\Sigma_\sigma F\tilde{B}(\tilde{B}^T\tilde{B})^{-1}$, $\bar{A}_{\sigma 12} = \tilde{B}^TA_\sigma B(B^TB)^{-1} + \tilde{B}^TE\Sigma_\sigma FB$ $(B^TB)^{-1}$, $\bar{A}_{\sigma 21} = B^TA_\sigma\tilde{B}(\tilde{B}^T\tilde{B})^{-1} + B^TE\Sigma_\sigma F\tilde{B}(\tilde{B}^T\tilde{B})^{-1}$, $\bar{A}_{\sigma 22} = B^TA_\sigma B(B^TB)^{-1} + B^TE\Sigma_\sigma FB(B^TB)^{-1}$, $\bar{C}_1 = C\tilde{B}(\tilde{B}^T\tilde{B})^{-1}$ and $\bar{C}_2 = CB(B^TB)^{-1}$.

In order to develop the main design method, we need the following lemmas.

Lemma 2.1. [22] *Given real matrices R_1 and R_2 of appropriate dimensions and an unknown matrix $\Sigma(t)$ with $\Sigma^T(t)\Sigma(t) \leq I$, we have*

$$R_1\Sigma(t)R_2 + R_2^T\Sigma^T(t)R_1^T \leq \beta R_1R_1^T + \beta^{-1}R_2^TR_2, \tag{4}$$

where $\beta > 0$ is a constant.

Introduce a convex combination of system (1) without the matched uncertainties $f_i(x,t)$ as

$$\begin{aligned} \dot{x}(t) &= \left(\bar{A} + \Delta\bar{A}\right)x(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{5}$$

where $\bar{A} = \sum_{i=1}^k \alpha_iA_i$, $\Delta\bar{A} = \sum_{i=1}^k \alpha_i\Delta A_i$, and $\alpha_i > 0$ with $\sum_{i=1}^k \alpha_i = 1$.

Lemma 2.2. *If there exists matrix $P > 0$, state feedback gain K, constant $\lambda > 0$, and scalars $\alpha_i > 0$ with $\sum_{i=1}^k \alpha_i = 1$ satisfying*

$$\left(\bar{A} - BK\right)^T P + P\left(\bar{A} - BK\right) + \lambda^2PEE^TP + \frac{1}{\lambda^2}F^TF + C^TC < 0, \tag{6}$$

then system (5) is robustly stabilizable.

Proof: Choose a Lyapunov function of the form:

$$V(x) = x^TPx + \int_0^t x^TC^TCxds$$

Substituting $u = -Kx$ into (5), we have

$$\begin{aligned} \dot{V}(x) &= x^T\left[\left(\bar{A} + \Delta\bar{A} - BK\right)^T P + P\left(\bar{A} + \Delta\bar{A} - BK\right)\right]x + x^TC^TCx \\ &= x^T\left[\left(\bar{A} - BK\right)^T P + P\left(\bar{A} - BK\right) + \Delta\bar{A}^TP + P\Delta\bar{A} + C^TC\right]x. \end{aligned} \tag{7}$$

By Lemma 2.1, it follows that

$$\begin{aligned} \Delta\bar{A}^TP + P\Delta\bar{A} &= \left(\sum_{i=1}^n \alpha_i\Delta A_i\right)^T P + P\left(\sum_{i=1}^n \alpha_i\Delta A_i\right) \\ &= \left[E\left(\sum_{i=1}^n \alpha_i\Sigma_i(t)\right)F\right]^T P + P\left[E\left(\sum_{i=1}^n \alpha_i\Sigma_i(t)\right)F\right] \\ &\leq \lambda^2PEE^TP + \frac{1}{\lambda^2}F^TF. \end{aligned}$$

Hence, we have

$$\dot{V}(x) \leq x^T\left[\left(\bar{A} - BK\right)^T P + P\left(\bar{A} - BK\right) + \lambda^2PEE^TP + \frac{1}{\lambda^2}F^TF + C^TC\right]x < 0.$$

This completes the proof.

3. Observer Design.

From Assumption 1, we can construct the observer for system (1) as follows:

$$\dot{\tilde{x}}(t) = (A_i + \Delta A_i)\tilde{x}(t) + B\left(u_i(t) + f_i(\tilde{x},t)\right) + L_i\left(y(t) - C\tilde{x}(t)\right), \tag{8}$$

where $\tilde{x}(t)$ is the restructured state. Let $e(t) = x(t) - \tilde{x}(t)$, from (1) and (8), the observer error system is expressed as

$$\dot{e}(t) = (A_i + \Delta A_i - L_iC)e(t) + B\left(f_i(x,t) - f_i(\tilde{x},t)\right). \tag{9}$$

We suppose that $f_i(x,t)$ satisfies the Lipschitz condition:

$$\|f_i(x,t) - f_i(\tilde{x},t)\| \le \gamma_i \|x - \tilde{x}\|,$$

where $\gamma_i > 0$ is a constant.

Lemma 3.1. *For system (1) with $u_i = 0$, there exist Lyapunov function $V(x(t))$ satisfies*

$$\begin{aligned} \dot{V}(x(t)) &< 0, \quad \forall t \in (t_k, t_{k+1}), \\ V(x(t_{k+1})) &< V(x(t_k)) \end{aligned}$$

where $V(x(t_k)) = \lim_{t\to t_k} V(x(t))$, then the system (1) is asymptotic stability [23].

Theorem 3.1. *For system (1), suppose that there exist matrix X_i such that for $i \in \varphi$, the following liner matrix inequality hold:*

$$\begin{bmatrix} \alpha I - P & 0 & 0 & 0 & 0 \\ \star & P - \beta I & 0 & 0 & 0 \\ \star & \star & -\frac{1}{\gamma_i^2} I & 0 & P \\ \star & \star & \star & -\frac{1}{\lambda^2} I & E^T P \\ \star & \star & \star & \star & M_1 \end{bmatrix} < 0, \tag{10}$$

where $M_1 = A_i^T P + PA_i - C^T X_i - X_i C + \frac{1}{\lambda^2} F^T F - \delta P + I$, $\alpha > 0$, $\beta > 0$, $P > 0$ satisfies (6), and δ satisfies

$$\delta > \begin{cases} \dfrac{\ln(\beta/\alpha)}{T_a}, & \alpha < \beta \\ 0, & \alpha \ge \beta \end{cases}$$

Moreover, the observer matrix for system (1) with the average dwell time T_a defined as in [19] *can be designed by $L_i = P^{-1} X_i$.*

Proof: Substituting $L_i = P^{-1} X_i$ into (10), it follows that

$$\alpha I < P < \beta I, \tag{11}$$

$$(A_i + \Delta A_i - L_i C)^T P + P(A_i + \Delta A_i - L_i C) + \gamma_i^2 P^T P + I < \delta P, \tag{12}$$

when the ith subsystem is activated, the observer error system is expressed as

$$\dot{e}(t) = (A_i + \Delta A_i - L_i C) e(t) + B (f_i(x,t) - f_i(\tilde{x},t)), \tag{13}$$

The Lyapunov function is chosen as

$$V(e) = e^T(t) P e(t).$$

then we have

$$\begin{aligned} \dot{V}(e) &= e^T(t) \left[((A_i + \Delta A_i) - L_i C)^T P + P((A_i + \Delta A_i) - L_i C) \right] e(t) \\ &\quad + (f_i(x,t) - f_i(\tilde{x},t))^T B^T P e(t) + e^T(t) PB (f_i(x,t) - f_i(\tilde{x},t)) \\ &\le e^T(t) \left[(A_i - L_i C)^T P + P(A_i - L_i C) \right] e(t) \\ &\quad + e^T(t) \left(\Delta A_i P + P \Delta A_i^T \right) e(t) + 2\gamma_i \|PBe(t)\| \|e(t)\| \\ &= e^T(t) \left[(A_i - L_i C)^T P + P(A_i - L_i(t) C) \right] e(t) \end{aligned}$$

$$
\begin{aligned}
&+e^T(t)\left[\left(\sum_{l=1}^{n}\alpha_l\Delta A_j\right)^T P + P\left(\sum_{l=1}^{n}\alpha_l\Delta A_i\right)\right] e(t) + 2\gamma_i \|PBe(t)\| \|e(t)\| \\
=\ & e^T(t)\left[(A_i - L_iC)^T P + P\left(A_i - L_i(t)C\right)\right] e(t) \\
&+e^T(t)\left\{\left[E\left(\sum_{l=1}^{n}\alpha_l\Sigma_i(t)\right)F\right]^T P + P\left[E\left(\sum_{l=1}^{n}\alpha_l\Sigma_i(t)\right)F\right]\right\} e(t) \\
&+2\gamma_i \|PBe(t)\| \|e(t)\| \\
\le\ & e^T(t)\left[(A_j - L_iC)^T P + P\left(A_i - L_iC\right)\right] e(t) \\
&+e^T(t)\left[\lambda^2 PEE^TP + \frac{1}{\lambda^2}F^TF\right] e(t) + 2\gamma_i \|PBe(t)\| \|e(t)\| \\
\le\ & e^T(t)\left[(A_i - L_iC)^T P + P\left(A_i - L_iC\right)\right. \\
&\left.+\lambda^2 PEE^TP + \frac{1}{\lambda^2}F^TF + \gamma_i^2 P^TP + I\right] e(t). \qquad (14)
\end{aligned}
$$

Considering (12), it follows that

$$\dot{V}(e) < -\delta e^T(t)Pe(t) = -\delta V(e),$$

and by (11) we have

$$\alpha \|e\|^2 < V(e) < \beta \|e\|^2 .$$

Therefore, by Lyapunov stability theory, we can conclude that the observer error system in (14) is asymptotic stability. This completes the proof.

4. **Sliding Mode Control.**

4.1. **Sliding mode dynamics analysis.**

In this section, we consider the sliding mode control problem for system (1).

According to the sliding mode control theory in [24], we know that the first subsystem in (3) represents the sliding mode dynamics. We design the following linear sliding surface function:

$$s(t) = Hz_1(t) + z_2(t), \qquad (15)$$

where $H \in \mathbb{R}^{m\times(n-m)}$ is the parameter to be design later.

Remark 4.1. *Please note that the sliding surface function defined in (15) does not switch with the switching signal, that is, there is a unique nonswitched sliding surface function. The reason why we use this kind of sliding surface function, not a switching one like $s(t) = H_iz_1(t) + z_2(t)$, is to avoid repetitive jumps of the trajectories of the state components of the closed-loop system between sliding surfaces and hence the possible instability.*

When the system trajectories reach onto the sliding surface $s(t) = 0$, that is, $z_2(t) = -Hz_1(t)$, the sliding mode dynamics is attained. By substituting $z_2(t) = -Hz_1(t)$ into the first subsystem in (15) yields the sliding mode dynamics:

$$
\begin{aligned}
\dot{z}_1(t) &= \left(\bar{A}_{\sigma 11} - \bar{A}_{\sigma 12}H\right) z_1(t), \\
y(t) &= \left(\bar{C}_1 - \bar{C}_2H\right) z_1(t). \qquad (16)
\end{aligned}
$$

Now, we will analyze the stability of the sliding mode dynamics in (16) and give the following theorem. Defining the following matrices [25]:

$$\bar{P} = T^{-T} P T^{-1} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}, \tag{17}$$

where

$$\begin{aligned} \bar{P}_{11} &= (\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T P\tilde{B}(\tilde{B}^T\tilde{B})^{-1}, \\ \bar{P}_{12} &= (\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T PB(B^TB)^{-1}, \\ \bar{P}_{22} &= (B^TB)^{-1}B^TPB(B^TB)^{-1}. \end{aligned}$$

Let

$$\Re = T\left(\bar{A} - BK\right)T^{-1} = \begin{bmatrix} \Re_{11} & \Re_{12} \\ \Re_{21} & \Re_{22} \end{bmatrix}, \tag{18}$$

with $\Re_{11} = \tilde{B}^T\bar{A}\tilde{B}(\tilde{B}^T\tilde{B})^{-1}$ and $\Re_{12} = \tilde{B}^T\bar{A}B(B^TB)^{-1}$.

Theorem 4.1. *For a given constant $\lambda > 0$, suppose that (6) has solution $P > 0$ such that for $i \in \varphi$, then the sliding mode dynamics in (16) is the asymptotically stable. Moreover, if the conditions above are feasible, the matrix H in (15) is given by*

$$H = \left[\left(B^TB\right)^{-1}B^TPB\left(B^TB\right)^{-1}\right]^{-1}\left(B^TB\right)^{-1}B^TP\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1}$$

that is, the sliding surface can be designed as

$$\begin{aligned} s(t) = S\tilde{x}(t) \ = \ &\left\{\left[\left(B^TB\right)^{-1}B^TPB\left(B^TB\right)^{-1}\right]^{-1}\right. \\ &\left.\times\left(B^TB\right)^{-1}B^TP\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1}\tilde{B}^T + B^T\right\}\tilde{x}(t) = 0. \end{aligned} \tag{19}$$

Proof: According to Lemma 2.2, we know that the sliding motion (15) can be rewritten equivalently as

$$\begin{aligned} \dot{z}_1(t) &= \left(\hat{A}_{\sigma 11} - \hat{A}_{\sigma 12}H + \hat{E}\Sigma_\sigma(t)\hat{F}\right)z_1(t), \\ y(t) &= \hat{C}z_1(t), \end{aligned}$$

where $\hat{A}_{\sigma 11} = \tilde{B}^TA_\sigma\tilde{B}(\tilde{B}^T\tilde{B})^{-1}$, $\hat{A}_{\sigma 12} = \tilde{B}^TA_\sigma B(B^TB)^{-1}$, $\tilde{E} = \tilde{B}^TE$, $\tilde{F} = F\tilde{B}(\tilde{B}^T\tilde{B})^{-1} - FB(B^TB)^{-1}H$ and $\tilde{C} = C\tilde{B}(\tilde{B}^T\tilde{B})^{-1} - CB(B^TB)^{-1}H$.

From (17) and (18), we have

$$\begin{bmatrix} \Pi & \bar{P}TE & FT^{-1} & CT^{-1} \\ \star & -\lambda^2 I & 0 & 0 \\ \star & \star & -\frac{1}{\lambda^2}I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0, \tag{20}$$

where $\Pi = \Re^T\bar{P} + \bar{P}\Re$.

Pre- and post-multiplying (20) by $\begin{bmatrix} I_{n-m} & -\bar{P}_{12}\bar{P}_{22}^{-1} \end{bmatrix}$ and $\begin{bmatrix} I_{n-m} & -\bar{P}_{12}\bar{P}_{22}^{-1} \end{bmatrix}^T$, respectively, we have

$$\begin{aligned} &\left(\Re_{11} - \Re_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^T\right)^T\Xi + \Xi\left(\Re_{11} - \Re_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^T\right) + \lambda^2\Xi\tilde{B}^TEE^T\tilde{B}\Xi \\ &\quad + \left[\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1} - B\left(B^TB\right)^{-1}\bar{P}_{22}^{-1}\bar{P}_{12}^T\right]^T\left(\frac{1}{\lambda^2}F^TF + C^TC\right) \\ &\quad\quad \times\left[\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1} - B\left(B^TB\right)^{-1}\bar{P}_{22}^{-1}\bar{P}_{12}^T\right] < 0, \end{aligned} \tag{21}$$

where $\Xi = \bar{P}_{11} - \bar{P}_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^T$, since $\bar{P} > 0$, it follows that $\Xi > 0$. Setting

$$H = \bar{P}_{22}^{-1}\bar{P}_{12}^T = \left[\left(B^TB\right)^{-1}B^TPB\left(B^TB\right)^{-1}\right]^{-1}\left(B^TB\right)^{-1}B^TP\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1},$$

thus inequality (21) becomes

$$(\Re_{11} - \Re_{12}H)^T\Xi + \Xi(\Re_{11} - \Re_{12}H) + \lambda^2\Xi\hat{E}\hat{E}^T\Xi + \frac{1}{\lambda^2}\hat{F}^T\hat{F} + \hat{C}^T\hat{C} < 0, \tag{22}$$

that is,

$$\begin{bmatrix} \Pi & \Xi\hat{E} & \hat{F} & \hat{C} \\ \star & -\lambda^2 I & 0 & 0 \\ \star & \star & -\frac{1}{\lambda^2}I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0,$$

where $\Pi = (\Re_{11} - \Re_{12}H)^T\Xi + \Xi(\Re_{11} - \Re_{12}H)$.

Substituting $\bar{A} = \sum_{i=1}^k \alpha_i A_i$ into the inequality (22) and denoting

$$Q_i = \left(\bar{A}_{i11} - \bar{A}_{i12}H\right)^T\Xi + \Xi\left(\bar{A}_{i11} - \bar{A}_{i12}H\right) + \lambda^2\Xi\hat{E}\hat{E}^T\Xi + \frac{1}{\lambda^2}\hat{F}^T\hat{F} + \hat{C}^T\hat{C}, \quad i \in \varphi,$$

we have

$$\alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_k Q_k < 0.$$

Define the regions

$$\Omega_i = \left\{z_1 | z_1^T Q_i z_1 < 0\right\}, \quad i \in \varphi.$$

Obviously, we have $\bigcup_{i\in\varphi}\Omega_i = \mathbb{R}^{(n-m)}/\{0\}$.

Design the switching law as

$$\sigma(0) = \min\arg\left\{\Omega_i | z_1(0) \in \Omega_i\right\}, \tag{23}$$

$$\sigma(t) = \begin{cases} i, & \text{if } z_1(t) \in \Omega_i \text{ and } \sigma(t^-) = i \\ \min\arg\left\{\Omega_i | z_1(0) \in \Omega_i\right\}, & \text{others} \end{cases}. \tag{24}$$

Chose the Lyapunov function as

$$V(t) = z_1^T(t)\Xi z_1(t). \tag{25}$$

Considering the previous definition, it is obvious that $V(t)$ can never be negative. Thus, provided that its time derivative is kept negative, its current value necessarily decreases with time to converge towards zero. The time derivative of (25) can be expressed as:

$$\dot{V}(t) = 2z_1^T(t)\Xi\dot{z}_1(t).$$

Then, by (22) the time derivative of (25) along the trajectory of system (16) satisfies $\dot{V}(t) < 0$ and the sliding motion is robustly stabilizable under the switching law (24).

4.2. **Sliding mode controller synthesis.** In this section, we will design a sliding mode controller to drive the system's trajectories onto the predefined sliding surface $s(t) = 0$.

Theorem 4.2. *Suppose that the conditions in Theorem 3.1 are satisfied, and the linear sliding surface is given by (15). Then the trajectory of the closed-loop system (3) can be driven onto the sliding surface $s(t) = 0$ in a finite time with the control*

$$\begin{aligned} u_i(t) = & -(SB)^{-1}SA_i\tilde{x} - (SB)^{-1}SL(y - C\tilde{x}) \\ & -(SB)^{-1}\left(\|SE\|\,\|F\tilde{x}\| + \|SB\|\,\eta_i(x,t) + \mu\right)\operatorname{sign}(s), \quad i \in \varphi, \end{aligned} \tag{26}$$

where μ is a positive scalar.

Proof: Consider the switching function as

$$s(t) = M\tilde{z}_1(t) + \tilde{z}_2(t) = S\tilde{x},$$

and choose the Lyapunov function as

$$V(t) = \frac{1}{2}s^T(t)s(t).$$

Then, the derivative of the sliding function $s(t) = S\tilde{x}(t)$ along the trajectory of system (1) is

$$\dot{s}(t) = S\left[(A_i + \Delta A_i)\,\tilde{x}(t) + B\left(u_i(t) + f_i(\tilde{x}, t)\right) + L\left(y - C\tilde{x}\right)\right], \quad i \in \varphi,$$

we have

$$\begin{aligned} \dot{V}(t) &= s^T(t)\dot{s}(t) \\ &= s^T(t)\left[S\left(A_i + \Delta A_i\right)\tilde{x}(t) + SBu_i(t) + SBf_i(\tilde{x}, t) + SL\left(y - C\tilde{x}\right)\right]. \end{aligned} \tag{27}$$

Substituting the controllers (26) into the above equation, we have

$$\begin{aligned} \dot{V}(t) \;=\;& s^T(t)\left\{S\left(A_i + \Delta A_i\right)\tilde{x}(t) - SB\left[(SB)^{-1}MA_i\tilde{x} - (SB)^{-1}SL\left(y - C\tilde{x}\right)\right.\right. \\ & \left. -(SB)^{-1}\left(\|SE\|\,\|F\tilde{x}\| + \|SB\|\,\eta_i(\tilde{x}, t) + \mu\right)\mathrm{sign}(s)\right] \\ & \left. +SBf_i(\tilde{x}, t) + SL(t)\left(y - C\tilde{x}\right)\right\}, \\ \dot{V}(t) \;=\;& s^T(t)\left\{S\Delta A_i\tilde{x}(t) - SB\left[(SB)^{-1}\left(\|SE\|\,\|F\tilde{x}\| + \|SB\|\,\eta_i(\tilde{x}, t) + \mu\right)\mathrm{sign}(s)\right]\right. \\ & \left. +MBf_i(\tilde{x}, t)\right\}. \end{aligned} \tag{28}$$

From Assumption 2, we have

$$SBf_i(\tilde{x}, t) - \|SB\|\,\eta_i(\tilde{x}, t) < 0, \tag{29}$$

and by Assumption 3 we have

$$S\Delta A_i\tilde{x}(t) - \|SE\|\,\|F\tilde{x}\| < 0. \tag{30}$$

From (29) and (30), it follows that

$$\dot{V}(t) \le -\mu\,\|s(t)\| = \sqrt{2}\mu V^{\frac{1}{2}}(t) < 0. \tag{31}$$

It is shown from (31) that there exists an instant $t^* = \sqrt{2V(0)}/\mu$ such that $V(t) = 0$ (equivalently, $s(t) = 0$) when $t \ge t^*$. Thus, the system trajectories can be driven onto the predefined sliding surface in a finite time. This completes the proof.

5. **Numerical Example.** In this section, we provide a numerical examples to illustrate the developed theories. Consider the switched hybrid systems composed of two three-order subsystems, and the parameters are given as follows:

$$A_1 = \begin{bmatrix} -20 & -1 & 1 \\ -1 & -10 & -1 \\ 0 & -1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 1 & -1 \\ 0 & -15 & -8 \\ -4 & 0 & -2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix},$$

and the uncertainty $\Delta A_i = E\Sigma_i(t)F$ with

$$E = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad \Sigma_1 = -0.8 \in [-1, 1],$$

$$\Sigma_2 = -0.2 \in [-1, 1], \quad f_1 = f_2 = 0.$$

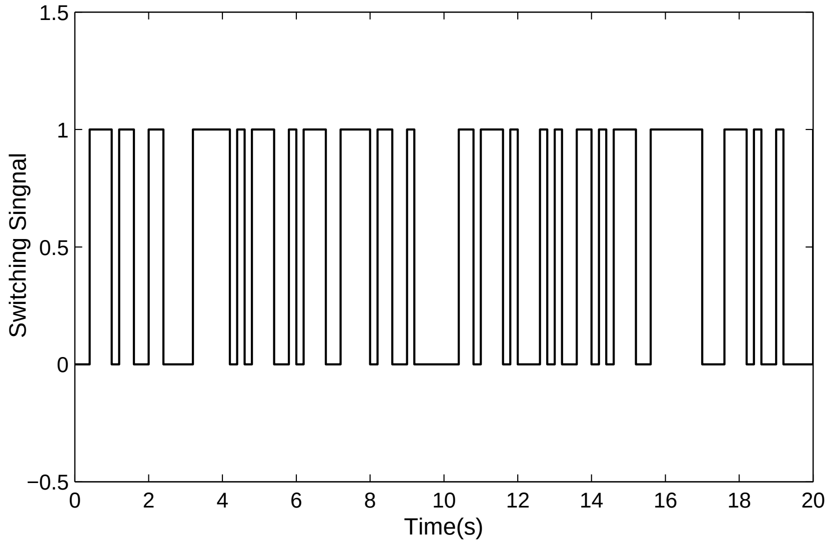

Figure 1. Switching signal

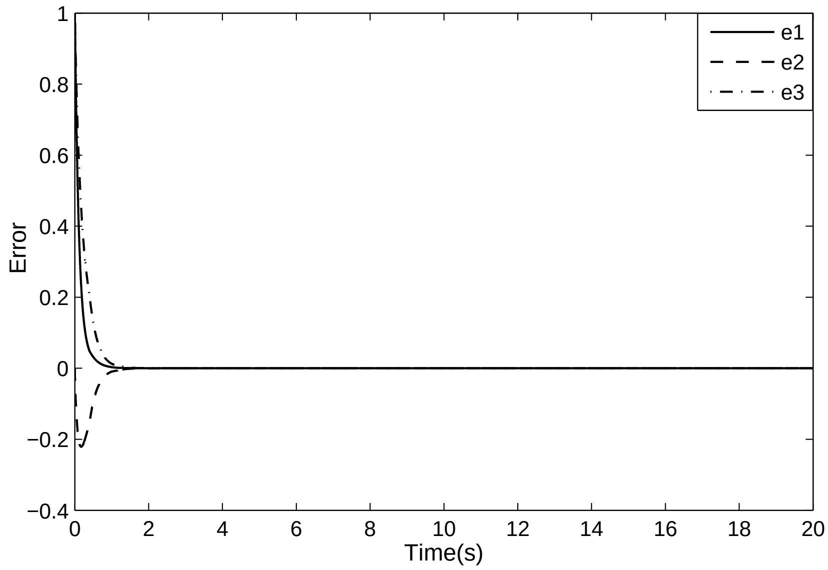

Figure 2. States of the observer error system

We choose the constant $\lambda = 0.6$, $\mu = 1$, and the convex combination coefficients $\alpha_1 = \alpha_2 = 0.5$. Let $K = -B^T P$, solving (6) leads to the solution

$$P = \begin{bmatrix} 0.2654 & 0.0720 & -0.1406 \\ 0.0720 & 0.2348 & -0.0138 \\ -0.1406 & -0.0138 & 0.4420 \end{bmatrix}.$$

Choose the constants as $\alpha = 0.1$, $\beta = 2$, $\gamma = 0.1$, $\mu = 1$, $\delta = 2$. By substituting P into (10) and then solving (10), we have

$$X_1 = \begin{bmatrix} 21.3936 \\ -24.6660 \\ -22.7819 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 20.3314 \\ -17.0977 \\ -19.4633 \end{bmatrix},$$

thus,

$$L_1 = P^{-1}X_1 = \begin{bmatrix} 106.7534 \\ -139.1011 \\ -21.9412 \end{bmatrix}, \quad L_2 = P^{-1}X_2 = \begin{bmatrix} 95.7108 \\ -103.1795 \\ -16.8189 \end{bmatrix}.$$

According to (18), we have

$$H = \left[\left(B^TB\right)^{-1} B^TPB\left(B^TB\right)^{-1}\right]^{-1}\left(B^TB\right)^{-1} B^TP\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1} = \begin{bmatrix} -0.0108 & 0.4744 \end{bmatrix},$$

$$\begin{aligned} S &= \left[\left(B^TB\right)^{T} B^TPB\left(B^TB\right)^{-1}\right]^{-1}\left(B^TB\right)^{-1} B^TP\tilde{B}\left(\tilde{B}^T\tilde{B}\right)^{-1}\tilde{B}^T + B^T \\ &= \begin{bmatrix} -0.4292 & -0.3189 & 1.0905 \end{bmatrix}. \end{aligned}$$

The sliding function is given as

$$s(t) = S\tilde{x}(t) = \begin{bmatrix} -0.4292 & -0.3189 & 1.0905 \end{bmatrix}\tilde{x}(t).$$

According to (26), the controllers for subsystems given as

$$\begin{aligned} u_1(t) &= \begin{bmatrix} -7.1220 & -2.0223 & 4.4504 \end{bmatrix}\tilde{x}(t) - 576.6452\left(y(t) - C\tilde{x}(t)\right) \\ &\quad - \left(0.0882\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2} + 0.8\right)\operatorname{sign}(s(t)), \\ u_2(t) &= \begin{bmatrix} 2.1164 & -3.4837 & -0.6396 \end{bmatrix}\tilde{x}(t) + 572.7613\left(y(t) - C\tilde{x}(t)\right) \\ &\quad - \left(0.0882\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2} + 0.8\right)\operatorname{sign}(s(t)). \end{aligned}$$

We checked that the above observer error system is stable for a switching signal given in Figure 1, the states of the observer error system are shown in Figure 2 with the initial condition given by $x(0) = \begin{bmatrix} -1 & 0.5 & 1 \end{bmatrix}$. Here, to prevent the control signals from chattering, we replace $\operatorname{sign}(s(t))$ with $s(t)/\left(0.01 + \|s(t)\|\right)$. Figure 3 shows the state response of the closed-loop switched system. The sliding function and the control input are given in Figures 4 and 5, respectively.

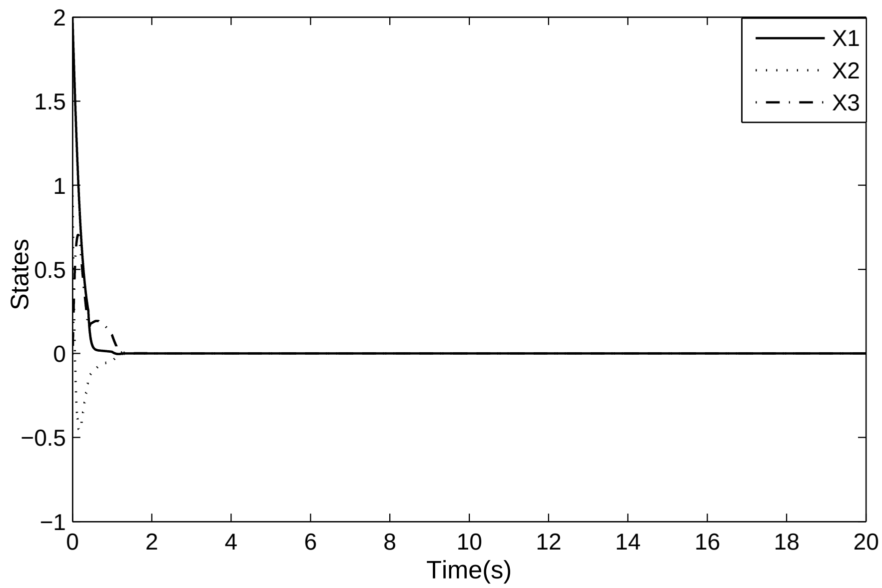

FIGURE 3. States of the closed-loop switched system

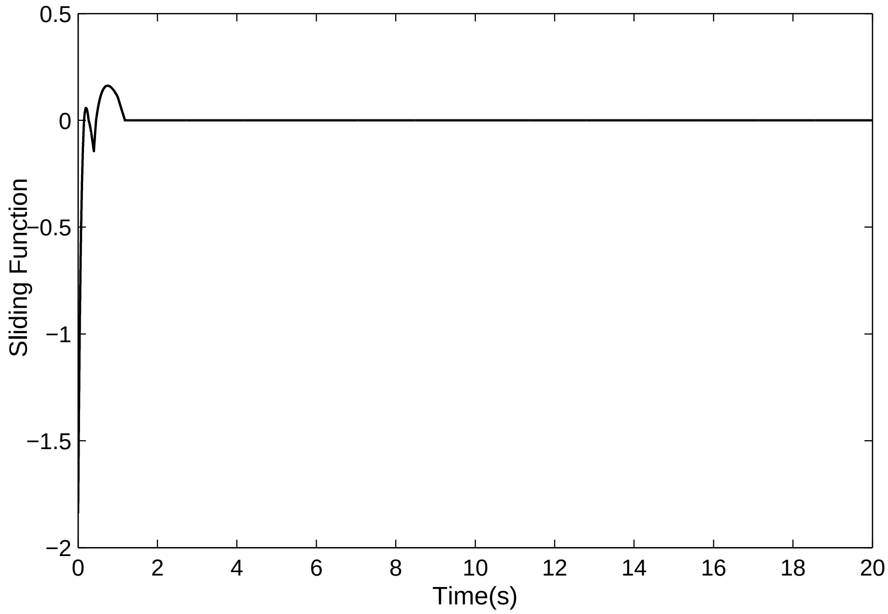

Figure 4. Sliding function

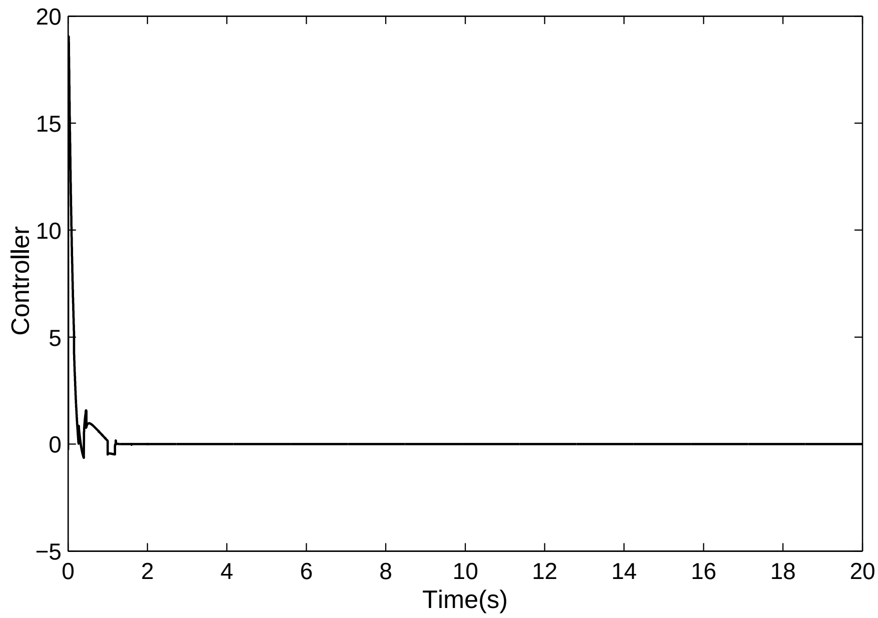

Figure 5. Control input

6. **Conclusions.** In this paper, the state estimation and observer-based sliding mode control strategy of switched linear systems have been presented. Some sufficient conditions have been proposed to guarantee the stability of the sliding mode dynamics, and the switching law which guarantees asymptotic stability of the overall switched closed-loop system has also been designed. Simulation results have demonstrated the effectiveness of the proposed theory.

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