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# COMPUTATIONAL PROCEDURE OF OPTIMAL INVENTORY MODEL INVOLVING CONTROLLABLE BACKORDER DISCOUNTS AND VARIABLE LEAD TIME WITH DEFECTIVE UNITS

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Abstract. *This paper investigates the backorder rate inventory problem with defective units. In practice, the uncertainties of customer's demand and backorder rate are inherent; hence, backorder rate and customer's demand are not constant. In this study, we assume that the backorder rate is dependent on the amount of shortages and backorder price discounts, and the lead time demand follows a mixture of distributions. Here, we derive the mixture inventory system, and construct the algorithm procedure to optimize the order quantity, backorder price discount and lead time. Numerical examples are also included to illustrate the results.*

**Keywords:** Defective units, Order quantity, Backorder discount, Minimax distribution free procedure, Mixture of distributions

1. **Introduction.** In the deterministic or probabilistic inventory model, lead time is viewed as a prescribed constant or a stochastic variable, which therefore is not subject to control (e.g., [11,19]). In fact, lead time usually consists of the following components: order preparation, order transit, supplier lead time, delivery time, and setup time (see [20]). In many practical situations, lead time can be reduced by an added crashing cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock (SS), reduce the loss caused by stock-out, improve the service level to the customer, and increase the competitive ability in business.

Liao and Shyu [10] presented a probabilistic model in which the order quantity is predetermined and lead time is a unique decision variable. Ben-Daya and Raouf [1] extended the Liao and Shyu's [10] model by considering both the lead time and the order quantity as decision variables. These studies [1,10] assumed that the probability of allowable stock-out during the lead time is very small, and hence the shortages are neglected. In [13,16,25], Ben-Daya and Raouf's [1] model is considered with the stock-out component. Ouyang and Wu [15] considered both lead time and the order quantity as the decision variables of a mixture of backorders and lost sales inventory model in which the shortages are allowed and consider the demand of the lead time with normal distribution. Besides, they assumed that an arrival order may contain some defective units, and the number of defective units in an arrival order of size $Q$ which is a random variable with binomial probability distribution. Ouyang and Chuang [13] (also see Lee [9]) took the stock shortages into consideration and assumed that the backorder rate is dependent on the length of lead time through the amount of shortages. Similarly, from Pan and Hsiao [17] the backorder rate was assumed in proportion to the price discount. In this paper, we develop a new general form of backorder rate with the idea of Ouyang and Chuang [13] (also see Lee [9]) and Pan and Hsiao [17] considerations. Thus, the backorder rate is dependent on the amount of shortages and backorder price discounts. Furthermore, this article proposes an inventory model involving backorder discounts with defective units.

In real environments, there exist many uncertain factors which can affect the value of the lead time demand. The demand of the different customers is not identical in the lead time, so we cannot only use a single distribution (such as Ouyang and Wu [15], Ouyang et al. [12]) to describe the lead time demand. It is more reasonable that mixture distribution is applied to describe the lead time demand than single distribution is used. In fact, mixture of normal distributions has been successfully applied in many fields including economics, marketing, and finance (Clark [2], Zangari [26], Venkataraman [21], Duffie and Pan [3], Hull and White [7], Wang [22]). Many studies (Wilson [23,24], Zangari [26], Venkataraman [21], Duffie and Pan [3], Hull and White [7]) show that the distributions of daily changes, such as returns in equity, foreign exchanges, and commodity markets, are frequently asymmetric with fat tails. The assumption of normality is far from perfect and often inappropriate. Mixture of normal distributions is a more general and flexible distribution for fitting the market data of daily changes. Recently the mixture of normal distributions has become a popular model for the distribution of daily changes in market variables with fat tails. In this study, we assume that the lead time demand follows the mixture of distributions (see Everitt and Hand [4]). Moreover, it is considered that an inventory model involves controllable backorder discounts and variable lead time with defective units. We first assume that the lead time demand follows a mixture of normal distributions and develop an algorithm procedure to find the optimal order quantity, the optimal lead time and the optimal backorder price discount. And then we consider that any mixture of distribution functions (d.f.s), say $F_* = pF_1 + (1-p)F_2$, of the lead time demand has only known finite first and second moments (and hence, mean and variance are also known and finite) but we make no assumption on the distribution form of $F_*$. That is, $F_1$ and $F_2$ of $F_*$ belong to the class $\Omega$ of all single d.f.s with finite mean and variance. Our goal is to solve a mixture inventory model with defective units by using the minimax distribution free approach. The minimax distribution free approach for our inventory model is to find the most unfavorable d.f.s $F_1$ and $F_2$ of $F_*$ for each decision variable and then to minimize over the decision variables. Furthermore, the purpose of this paper is to develop an algorithm procedure for the mixture inventory model with defective units to find the optimal order quantity, optimal lead time and the optimal backorder price discount when the distribution of the lead time demand is mixture of normal distributions or mixture of free distributions. Finally, first two numerical examples with known (or given) parameters of the mixture of distributions are also given to illustrate that when $p = 0$ or 1, the model considers only one kind of customers' demand; when $0 < p < 1$, the model considers two kinds of customers' demand for the fixed fraction of backorder parameters $\varepsilon, \delta$ and defective rate $\theta$. It implies that the minimum expected total annual cost of two kinds of customers' demand is larger than the minimum expected total annual cost of one kind of customers' demand. Thus, the minimum expected total annual cost increases as the distance between $p$ and 0 (or 1) increases for the fixed $\varepsilon, \delta$ and $\theta$. Next, in third example, if the parameters of the mixture of distributions for the lead time demand are unknown, the maximum likelihood (ML) method is the most popular technique for deriving estimators of these parameters. That is, we directly take derivatives with respect to these parameters of this likelihood functions based on the ML method and use them to find the maximum. It turns out that these equations can become quite difficult since these equations are nonlinear and no analytic solutions can be found. Hence, we use an easier algorithm that is guaranteed to converge to the ML estimators. The algorithm is called expectation maximization (EM) algorithm which was introduced by Hastie et al. [6] and used to obtain the estimates of unknown parameters of the mixture of distributions for the lead time demand. Hence, if the true distribution of the lead time demand is mixture of normal distributions or mixture of free distributions, we use a single distribution (such as Ouyang and Wu [15], Ouyang et al. [12]) instead of the true distribution of the lead time demand; then the minimum expected total annual cost will be overestimated.

## 2. Notations and Assumptions.

To establish the mathematical model, the notations and assumptions of the model are as follows:

### 2.1. Notations.

- $A$ Fixed ordering cost per order;
- $D$ Average demand per year;
- $h$ Inventory holding cost per non-defective unit per year;
- $h^c$ Inventory holding cost per defective unit per year;
- $L$ Length of lead time, decision variable;
- $Q$ Order quantity, decision variable;
- $r$ Reorder point;
- $X$ Lead time demand with the mixtures of distribution;
- $\beta$ Fraction of the demand backordered during the stock out period, $\beta \in [0, 1]$;
- $\pi_0$ Gross marginal profit per unit;
- $\pi_x$ Backorder price discount offered by the supplier unit, $0 \leq \pi_x \leq \pi_0$;
- $\delta, \varepsilon$ Backorder parameters, $0 \leq \delta \leq 1,\ 0 \leq \varepsilon < \infty$;
- $p$ The weight of the component normal distributions, $0 \leq p \leq 1$;
- $B(r)$ The expected shortage quantity at the end of cycle;
- $q$ The allowable stock out probability;
- $k$ The safety factor which satisfies $P(X > r) = q$;
- $v$ Inspecting cost per unit;
- $Y$ Number of non-defective items in a lot, a random variable;
- $\theta$ The probability of defect, $0 < \theta < 1$;
- $x^+$ Maximum value of $x$ and 0, i.e., $x^+ = \max\{x, 0\}$;
- $x^-$ Maximum value of $-x$ and 0, i.e., $x^- = \max\{-x, 0\}$;

$$I_{(0<X<r)} = \begin{cases} 1, & 0 < x < r, \\ 0, & \text{o.w.} \end{cases}$$

### 2.2. Assumptions.

A1: Inventory is continuously reviewed. Replenishments are made whenever the inventory level (based on the number of non-defective items) falls to the reorder point $r$.

A2: The reorder point $r$ = expected demand during lead time + safety stock (SS), and SS = $k \times$ (standard deviation of lead time demand), that is $r = \mu_* L + k\sigma_* \sqrt{L}$, where

$\mu_* = p\mu_1 + (1-p)\mu_2$, $\sigma_* = \sqrt{1+p(1-p)\eta^2}\sigma$, $\mu_1 = \mu_* + (1-p)\eta\sigma\Big/\sqrt{L}$, $\mu_2 = \mu_* - p\eta\sigma\Big/\sqrt{L}$ (i.e., $\mu_1 - \mu_2 = \eta\sigma\Big/\sqrt{L}$, $\eta \in R$ is given), and $k$ is the safety factor which satisfies $P(X > r) = q$.

A3: An arrival order may contain some defective units. We assume that the number of non-defective units $y$ in an arriving order of size $Q$ be a binomial random variable with parameters $Q$ and $1-\theta$, where $\theta$ represents the probability of defect. Upon arrival of order, the entire items are inspected and defective units in each lot will be returned to the vendor at the time of delivery of the next lot.

A4: The lead time $L$ consists of $n$ mutually independent components. The $i$th component has a minimum duration $a_i$, a normal duration $b_i$ and a crashing cost $c_i$ per unit time. Furthermore, these $c_i$ are assumed to be arranged such that $c_1 \le c_2 \le \cdots \le c_n$.

A5: The components of the lead time are crashed one at a time, starting with the component of least $c_i$ and so on.

A6: Let $L_0 \equiv \sum_{j=1}^{n} b_j$ and $L_i$ be the length of lead time with components $1, 2, \cdots, i$ crashed to their minimum duration, then $L_i$ can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$. Thus, the lead time crashing cost $R(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $R(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j)$.

## 3. Model Formulation.

### 3.1. The mixture of normal distributions model.

The lead time demand $X$ is assumed to be a mixture of normal distributions $F_* = pF_1 + (1-p)F_2$, where $F_i$ has a normal distribution with finite mean $\mu_i L$ and standard deviation $\sigma\sqrt{L}$, $i = 1, 2$ and $\mu_1 - \mu_2 = \eta\sigma\Big/\sqrt{L}$, $\eta \in R$ is given. Therefore, the lead time demand $X$ has mixtures of probability density function (p.d.f.) which is given by

$$f(x) = p\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}} e^{-\frac{1}{2}\left[(x-\mu_1 L)\big/\sigma\sqrt{L}\right]^2} + (1-p)\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}} e^{-\frac{1}{2}\left[(x-\mu_2 L)\big/\sigma\sqrt{L}\right]^2},$$

where $\mu_1 - \mu_2 = \eta\sigma\Big/\sqrt{L}$, $\eta \in R$, $x \in R$, $0 \le p \le 1$, $\sigma > 0$. Moreover, the mixture of normal distributions is unimodal for all $p$ if $(\mu_1 - \mu_2)^2 < 27\sigma^2/(8L)$ $\left(\text{or } \eta < \sqrt{\frac{27}{8}}\right)$ (see Everitt and Hand [4]).

As mentioned earlier, we assume that shortages are allowed and the reorder point is $r = \mu_* L + k\sigma_*\sqrt{L}$. Then the safety factor, $k$, satisfies $P(X > r) = 1 - p\Phi(r_1) - (1-p)\Phi(r_2) = q$, where $\Phi$ represents the cumulative distribution function (c.d.f.) of the standard normal random variable, $q$ represents the allowable stock-out probability during the lead time, $r_1 = (r - \mu_1 L)/(\sigma\sqrt{L}) = k\sqrt{1+p(1-p)\eta^2} - (1-p)\eta$ and $r_2 = (r - \mu_2 L)/(\sigma\sqrt{L}) = k\sqrt{1+p(1-p)\eta^2} + p\eta$.

The stock-out occurs as $x > r$ and the shortage is $x - r$; then the expected demand shortage at the end of the cycle is given by

$$B(r) = E[X - r]^+ = \int_r^{\infty} (x - r) dF_*(x) = \sigma\sqrt{L}\Psi(r_1, r_2, p), \tag{1}$$

where $\Psi(r_1, r_2, p) = p[\phi(r_1) - r_1(1 - \Phi(r_1))] + (1-p)[\phi(r_2) - r_2(1 - \Phi(r_2))]$, $r_1 = (r - \mu_1 L)\ /(\sigma\sqrt{L})$, $r_2 = (r - \mu_2 L)\Big/(\sigma\sqrt{L})$ and $\phi$, $\Phi$ be the standard normal p.d.f. and c.d.f., respectively (also see Ravindran et al. [18], Wu and Tsai [25]). For backorder rate $\beta$, the expected number of backorders per cycle is $\beta B(r)$, the expected lost sales per cycle is $(1-\beta)B(r)$, and the stock-out cost per cycle is $[\pi_x\beta + \pi_0(1-\beta)]B(r)$.

The expected net inventory level just before the order arrives is $E[(X-r)^- I_{(0<X<r)}] - \beta B(r)$ and the expected net inventory level at the beginning of the cycle, given that there are $y$ non-defective items in an arriving order of size $Q$, is $y + E[(X-r)^- I_{(0<X<r)}] - \beta B(r)$. Therefore, the expected holding cost per cycle is $h\frac{y}{D}\left\{\frac{y}{2} + E[(X-r)^- I_{(0<X<r)}] - \beta B(r)\right\}$.

Hence, the cost per cycle given that there are $y$ non-defective items in an arriving order size $Q$ is

$$\begin{aligned} C(y) &= \text{ordering cost} + \text{non-defective holding cost} + \text{stock-out cost} \\ &\quad + \text{defective holding cost} + \text{inspecting cost} + \text{lead time crashing cost} \\ &= A + h\frac{y}{D}\left\{\frac{y}{2} + E[(X-r)^- I_{(0<X<r)}] - \beta B(r)\right\} \\ &\quad + [\pi_x\beta + \pi_0(1-\beta)]B(r) + h^c\frac{y}{D}(Q-y) + vQ + R(L), \end{aligned} \tag{2}$$

where

$$\begin{aligned} & E[(X-r)^- I_{(0<X<r)}] - \beta B(r) \\ &= \int_0^r (r-x)dF_*(x) \\ &= \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right]\right. \\ &\quad \left. + (1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + (1-\beta)B(r). \end{aligned} \tag{3}$$

Because the expected length of the cycle time is $E(T|Q) = \frac{E(Y|Q)}{D}$ and the cycle cost under the lot of size $Q$ is

$$\begin{aligned} E(C|Q) &= A + \frac{h}{2D}E(Y^2|Q) + \frac{h}{D}E(Y|Q)\left\{E[(X-r)^- I_{(0<X<r)}] - \beta B(r)\right\} \\ &\quad + [\pi_x\beta + \pi_0(1-\beta)]B(r) + \frac{h^cQ}{D}E(Y|Q) - \frac{h^c}{D}E(Y^2|Q) + vQ + R(L), \end{aligned} \tag{4}$$

the expected total annual cost is

$$\begin{aligned} EAC^N(Q,L) &= \frac{E(C|Q)}{E(T|Q)} \\ &= \frac{AD}{E(Y|Q)} + \frac{h}{2}\frac{E(Y^2|Q)}{E(Y|Q)} + h\left\{E[(X-r)^- I_{(0<X<r)}] - \beta B(r)\right\} \\ &\quad + [\pi_x\beta + \pi_0(1-\beta)]\frac{DB(r)}{E(Y|Q)} + h^cQ - h^c\frac{E(Y^2|Q)}{E(Y|Q)} \\ &\quad + \frac{DvQ}{E(Y|Q)} + \frac{D}{E(Y|Q)}R(L). \end{aligned} \tag{5}$$

For a given lot of size $Q$, we assume that the number of non-defective units is a random variable ($Y$), which has a binomial distribution with parameter $Q$ and $1-\theta$. That is, $Y$ has the binomial p.d.f. as

$$P(Y = y) = C_y^Q(1-\theta)^y\theta^{Q-y}, \quad y = 0, 1, 2, \cdots, Q \text{ and } 0 < \theta < 1. \tag{6}$$

In this case, we know that

$$E(Y|Q) = Q(1-\theta) \tag{7}$$

and

$$E(Y^2|Q) = Q(1-\theta)[\theta + Q(1-\theta)]. \tag{8}$$

Therefore, substituting the results of (1), (3), (7) and (8) into (5), we get

$$\begin{aligned} EAC^N(Q,L) \;=\; & \frac{AD}{Q(1-\theta)} + \frac{h}{2}[\theta + Q(1-\theta)] + h\sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right.\right. \\ & \left. -\phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right] + (1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right. \\ & \left.\left. -\phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + h(1-\beta)B(r) + [\pi_x\beta + \pi_0(1-\beta)]\frac{DB(r)}{Q(1-\theta)} \\ & + h^c\theta(Q-1) + \frac{Dv}{1-\theta} + \frac{D}{Q(1-\theta)}R(L). \end{aligned} \tag{9}$$

In the real market, as unsatisfied demands occur, the longer the length of lead time is, the larger the amount of shortages is, the smaller the proportion of customers can wait, and hence the smaller the backorder rate would be. In addition, the larger backorder price discount is, and hence the larger the backorder rate would be. Therefore, we consider the backorder rate that is proposed by combining Ouyang and Chuang [13] with Pan and Hsiao [17] at the same time. Thus, we define the backorder rate to be $\beta = \frac{\pi_x}{\pi_0}\cdot\frac{\delta}{1+\varepsilon B(r)}$, where the backorder parameters $\delta$ and $\varepsilon$ with $0 \le \delta \le 1$ and $0 \le \varepsilon < \infty$. Hence, the total expected annual cost (9) is express as

$$\begin{aligned} & EAC^N(Q,\pi_x,L) \\ = & \frac{AD}{Q(1-\theta)} + \frac{h}{2}[Q(1-\theta)+\theta] + h\sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right]\right. \\ & \left. +(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + h\left(1 - \frac{\pi_x}{\pi_0}\frac{\delta}{1+\varepsilon B(r)}\right)B(r) \\ & + \frac{D}{Q(1-\theta)}\pi_0\left[1 - \frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\frac{\delta}{1+\varepsilon B(r)}\right]B(r) + h^c\theta(Q-1) + \frac{Dv}{1-\theta} + \frac{D}{Q(1-\theta)}R(L) \\ = & \frac{AD}{Q(1-\theta)} + \frac{h}{2}[Q(1-\theta)+\theta] + h\sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right]\right. \\ & \left. +(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + h\left(1 - \frac{\alpha_1}{1+\varepsilon B(r)}\right)B(r) \\ & + \frac{D}{Q(1-\theta)}\pi_0\left[1 - \frac{\alpha_2}{1+\varepsilon B(r)}\right]B(r) + h^c\theta(Q-1) + \frac{Dv}{1-\theta} + \frac{D}{Q(1-\theta)}R(L), \end{aligned} \tag{10}$$

where $\alpha_1 = \frac{\pi_x}{\pi_0}\delta$, $\alpha_2 = \frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\delta$, and $B(r) = \sigma\sqrt{L}\,\Psi(r_1,r_2,p)$.

In order to find the minimum total expected annual cost, we can take the first partial derivatives of $EAC^N(Q,\pi_x,L)$ with respect to $Q$, $\pi_x$ and $L$ respectively, and we can obtain

$$\begin{aligned} \frac{\partial EAC^N(Q,\pi_x,L)}{\partial Q} \;=\; & -\frac{AD}{Q^2(1-\theta)} + \frac{h}{2}(1-\theta) - \frac{D}{Q^2(1-\theta)}\pi_0\left[1 - \frac{\alpha_2}{1+\varepsilon B(r)}\right]B(r) \\ & + h^c\theta - \frac{D}{Q^2(1-\theta)}R(L), \end{aligned} \tag{11}$$

$$\frac{\partial EAC^N(Q,\pi_x,L)}{\partial \pi_x} = -\frac{h}{\pi_0}\frac{\delta}{1+\varepsilon B(r)}B(r) - \frac{D}{Q(1-\theta)}\left(1 - \frac{2\pi_x}{\pi_0}\right)\frac{\delta}{1+\varepsilon B(r)}B(r), \tag{12}$$

$$\begin{aligned} \frac{\partial EAC^N(Q,\pi_x,L)}{\partial L} \;=\; & \frac{h\sigma}{2\sqrt{L}}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right]\right. \\ & \left. +(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} \\ & + \frac{h\mu_*}{2}\left\{p\left(r_1 + \frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right. \\ & \left. +(1-p)\left(r_2 + \frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right\} \\ & + \frac{hB(r)}{2L}\left(1 - \frac{\alpha_1}{[1+\varepsilon B(r)]^2}\right) + \frac{D\pi_0}{Q(1-\theta)}\frac{B(r)}{2L}\left(1 - \frac{\alpha_2}{[1+\varepsilon B(r)]^2}\right) - \frac{D}{Q(1-\theta)}c_i, \end{aligned} \tag{13}$$

where $\alpha_1 = \frac{\pi_x}{\pi_0}\delta$, $\alpha_2 = \frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\delta$, and $B(r) = \sigma\sqrt{L}\,\Psi(r_1,r_2,p)$.

It is clear that for any given $r_1$, $r_2$ and $p$, we have $\Psi(r_1, r_2, p) > 0$. Hence, for fixed $L \in [L_i, L_{i-1}]$, $EAC^N(Q, \pi_x, L)$ is convex in $Q$ and $\pi_x$, since

$$\begin{aligned}\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial Q^2} &= \frac{2AD}{Q^3(1-\theta)} + \frac{2D}{Q^3(1-\theta)}\pi_0\left(1-\frac{\alpha_2}{1+\varepsilon B(r)}\right)B(r) \\ &\quad + \frac{2D}{Q^3(1-\theta)}R(L) > 0,\end{aligned}$$

$$\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial \pi_x^2} = \frac{2D}{Q(1-\theta)\pi_0}\frac{\delta}{1+\varepsilon B(r)}B(r) > 0,$$

and

$$\begin{aligned}&\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial Q^2}\cdot\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial \pi_x^2} - \left[\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial Q\partial \pi_x}\right]^2 \\ &= \frac{4D^2}{Q^4(1-\theta)^2\pi_0}\frac{\delta}{1+\varepsilon B(r)}B(r)[A+R(L)] + \frac{D^2}{Q^4(1-\theta)^2\pi_0}\frac{\delta}{1+\varepsilon B(r)}[B(r)]^2\left(4-\frac{\delta}{1+\varepsilon B(r)}\right)^2 > 0.\end{aligned}$$

However, for fixed $Q$ and $\pi_x$, $EAC^N(Q, L)$ is concave function of $L \in [L_i, L_{i-1}]$, since

$$\begin{aligned}\frac{\partial^2 EAC^N(Q,\pi_x,L)}{\partial L^2} &= -\frac{h\sigma}{4L^{3/2}}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right]\right. \\ &\quad \left. +(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right]\right\} \\ &\quad -\frac{h\mu_* p}{4L}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\left\{\frac{\mu_*\sqrt{L}}{\sigma}\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right. \\ &\quad \left.\times\left(r_1+\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)-\left(r_1+\frac{2\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right\} \\ &\quad -\frac{h\mu_*(1-p)}{4L}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\left\{\frac{\mu_*\sqrt{L}}{\sigma}\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right. \\ &\quad \left.\times\left(r_2+\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)-\left(r_2+\frac{2\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right\} \\ &\quad -\frac{h}{4L^2}\frac{B(r)\left\{(1-\alpha_1)[1+3\varepsilon B(r)]+3[\varepsilon B(r)]^2+[\varepsilon B(r)]^3\right\}}{[1+\varepsilon B(r)]^3} \\ &\quad -\frac{D\pi_0}{Q(1-\theta)}\frac{B(r)\left\{(1-\alpha_2)[1+3\varepsilon B(r)]+3[\varepsilon B(r)]^2+[\varepsilon B(r)]^3\right\}}{4L^2[1+\varepsilon B(r)]^3} \\ &< 0, \text{ if } \min\left\{\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\,,\ \frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right\} > \sqrt{2}.\end{aligned}$$

Therefore, for fixed $Q$ and $\pi_x$, the minimum expected total annual cost will occur at the end points of the interval $[L_i, L_{i-1}]$. Solving for $Q$ and $\pi_x$ by setting Equations (11) and (12) to zero, we have

$$Q=\left\{\frac{2D}{(1-\theta)[h(1-\theta)+2h^c\theta]}\left\{A+R(L)+\pi_0\left[1-\frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\frac{\delta}{1+\varepsilon B(r)}\right]B(r)\right\}\right\}^{\frac{1}{2}} \tag{14}$$

and

$$\pi_x = \frac{1}{2}\left[h\frac{Q(1-\theta)}{D}+\pi_0\right]. \tag{15}$$

Substituting the result of (15) into (14), we can rewrite (14) as

$$Q=\left\{\frac{2D\left\{A+R(L)+\pi_0\left[1-\frac{1}{4}\left(\frac{\delta}{1+\varepsilon B(r)}\right)\right]B(r)\right\}}{h(1-\theta)^2\left[1-\frac{h}{2D\pi_0}\frac{\delta}{1+\varepsilon B(r)}B(r)\right]+2h^c\theta(1-\theta)}\right\}^{\frac{1}{2}}. \tag{16}$$

Thus, we can establish the following iterative algorithm to find the optimal order quantity, backorder price discount and lead time.

**Algorithm 3.1.**

Step 1. For given $A$, $D$, $h$, $h^c$, $v$, $\sigma$, $\pi_0$, $\eta$, $p$, $q$, $\varepsilon$, $\delta$, $\theta$, $a_i$, $b_i$, $c_i$, and $i = 1, 2, \cdots, n$.

Step 2. Given $\eta$, $p$ and $q$, using the computer software Intel Visual Fortran V9.0 [8] and the subroutine ZREAL from IMSL to solve $k$ via the equation $1 - p\Phi(r_1) - (1-p)\Phi(r_2) = q$ where $r_1 = k\cdot\sqrt{1+p(1-p)\eta^2} - \eta(1-p)$ and $r_2 = k\cdot\sqrt{1+p(1-p)\eta^2} + \eta p$. Further, we obtain $r_1$ and $r_2$.

Step 3. Given $a_i$, $b_i$ and $c_i$, $i = 1, 2, \cdots, n$, use assumption A6 and Equation (1) to compute $L_i$ and $B(r)$, respectively.
Step 4. For each $L_i$, $i = 0, 1, 2, \cdots, n$, use Equation (16) to compute $Q_i$. Then, $\pi_{x_i}$can be computed by using Equation (15).
Step 5. Compare $\pi_{x_i}$ and $\pi_0$. If $\pi_{x_i} < \pi_0$, then $\pi_{x_i}$ in Step 4 is optimal; if $\pi_{x_i} \geq \pi_0$, then take $\pi_{x_i} = \pi_0$ into Step 6.
Step 6. For each pair $(Q_i, \pi_{x_i}, L_i)$, compute the corresponding expected total annual cost $EAC^N(Q_i, \pi_{x_i}, L_i)$, $i = 1, 2, \cdots, n$.
Step 7. Find $\min\limits_{i=0,1,\cdots,n} EAC^N(Q_i, \pi_{x_i}, L_i)$.
If $EAC^N(Q_s, \pi_{x_s}, L_s) = \min\limits_{i=0,1,\cdots,n} EAC^N(Q_i, \pi_{x_i}, L_i)$, then $(Q_s, \pi_{x_s}, L_s)$ is the optimal solution.

3.2. **The mixture of distributions free model.** In this subsection, we relax the restriction about the form of the mixture of d.f.s of lead time demand, i.e., we assume here that the lead time demand $X$ has a mixture of d.f.s $F_* = pF_1 + (1-p)F_2$, where $F_i$ has finite mean $\mu_i L$ and standard deviation $\sigma\sqrt{L}$, $i = 1, 2$, $\mu_1 - \mu_2 = \eta\sigma\Big/\sqrt{L}$, $\eta \in R$, but we make no assumption on the mixtures of d.f.'s form of $F_*$. Now, we try to use a minimax distribution free procedure to solve this problem. Let $\Omega$ denote the class of all single c.d.f. (include $F_1$ and $F_2$) with finite mean and standard deviation, then the minimax distribution free approach for our problem is to find the most unfavorable c.d.f.s $F_1$ and $F_2$ in $\Omega$ for each decision variable and then to minimize over the decision variables. Thus, our problem is to solve

$$\min_{Q>0,\ \pi_x>0,\ L>0}\ \max_{F_1,\ F_2\in\Omega} EAC^F(Q, \pi_x, L), \tag{17}$$

where

$$\begin{aligned} EAC^F(Q, \pi_x, L) \;=\; & \tfrac{AD}{Q(1-\theta)} + \tfrac{h}{2}[Q(1-\theta)+\theta] + h\left[k\sigma_*\sqrt{L} + (1-\beta)B(r)\right] \\ & + \tfrac{DB(r)}{Q(1-\theta)}[\pi_x\beta + \pi_0(1-\beta)] + h^c\theta(Q-1) + \tfrac{Dv}{1-\theta} + \tfrac{DR(L)}{Q(1-\theta)}, \end{aligned}$$

$\beta = \frac{\pi_x}{\pi_0}\cdot\frac{\delta}{1+\varepsilon B(r)}$ and $B(r) = E(X-r)^+$.

In addition, we need the following Proposition 3.1 to solve the above problem of the model (17).

**Proposition 3.1. (Gallego and Moon [5]):** *For $F_1$ and $F_2$ $\in \Omega$,*

$$E(X_i - r)^+ \leq \frac{1}{2}\left\{\sqrt{\sigma^2 L + (r-\mu_i L)^2} - (r - \mu_i L)\right\}, \quad i = 1, 2, \tag{18}$$

*where random variables $X_i$ has a single distribution function $F_i$, $i = 1, 2$. Moreover, the upper bound of (18) is tight.*

So, by using $F_* = pF_1 + (1-p)F_2$ and inequality (18), we obtain

$$\begin{aligned} B(r) \;=\; & \textstyle\int_r^\infty (x-r)dF_* = p\int_r^\infty (x-r)dF_1 + (1-p)\int_r^\infty (x-r)dF_2 \\ \leq\; & p\cdot\tfrac{1}{2}\left\{\sqrt{\sigma^2 L + (r-\mu_1 L)^2} - (r-\mu_1 L)\right\} \\ & +(1-p)\cdot\tfrac{1}{2}\left\{\sqrt{\sigma^2 L + (r-\mu_2 L)^2} - (r-\mu_2 L)\right\} \\ =\; & \tfrac{1}{2}(\mu_* L - r) + \tfrac{p}{2}\left[\sqrt{\sigma^2 L + (r-\mu_1 L)^2}\right] + \tfrac{(1-p)}{2}\left[\sqrt{\sigma^2 L + (r-\mu_2 L)^2}\right], \end{aligned}$$

where

$$r - \mu_1 L = r - \mu_* L - (1-p)\eta\sigma\sqrt{L} = k\sigma_* L - (1-p)\eta\sigma\sqrt{L}$$

and

$$r-\mu_2 L = r-\mu_* L + p\eta\sigma\sqrt{L} = k\sigma_* L + p\eta\sigma\sqrt{L} \ \left(\because \mu_1-\mu_2=\frac{\eta\sigma}{\sqrt{L}}\right).$$

Then

$$\begin{aligned} B(r) \le\ & \tfrac{1}{2}\left[-k\sqrt{1+p(1-p)\eta^2}\sigma\sqrt{L}\right] + \tfrac{p}{2}\sigma\sqrt{L}\left[\sqrt{1+[k\sqrt{1+p(1-p)\eta^2}-(1-p)\eta]^2}\right] \\ & + \tfrac{(1-p)}{2}\sigma\sqrt{L}\left[\sqrt{1+[k\sqrt{1+p(1-p)\eta^2}+p\eta]^2}\right]. \end{aligned} \tag{19}$$

From the definition of the backorder rate $\beta$ and inequality (19), we have

$$\begin{aligned} \beta =\ & \left\{\tfrac{\pi_x}{\pi_0}\tfrac{\delta}{1+\varepsilon B(r)}\right\} \\ \ge\ & \tfrac{\pi_x}{\pi_0}\delta\cdot\left\{1+\varepsilon\left\{\tfrac{1}{2}\left(-k\sqrt{1+p(1-p)\eta^2}\sigma\sqrt{L}\right)\right.\right. \\ & +\tfrac{p}{2}\sigma\sqrt{L}\left[\sqrt{1+\left[k\sqrt{1+p(1-p)\eta^2}-(1-p)\eta\right]^2}\right] \\ & \left.\left.+\tfrac{1-p}{2}\sigma\sqrt{L}\left[\sqrt{1+\left[k\sqrt{1+p(1-p)\eta^2}+p\eta\right]^2}\right]\right\}\right\}^{-1}, \end{aligned} \tag{20}$$

where $0 \le \delta \le 1$, $0 \le \varepsilon \le \infty$, $0 \le p \le 1$, and $\sigma > 0$.

By using the inequality (19), the model (17) can be reduced to

$$\min_{Q>0,\ \pi_x>0,\ L>0} EAC^U(Q,\pi_x,L)\ , \tag{21}$$

where

$$\begin{aligned} EAC^U(Q,\pi_x,L) =\ & \tfrac{AD}{Q(1-\theta)} + h\left[\tfrac{Q(1-\theta)+\theta}{2} + k\sqrt{1+p(1-p)\eta^2}\sigma\sqrt{L}\right. \\ & \left.+\left(1-\tfrac{\pi_x}{\pi_0}\tfrac{\delta}{1+\Delta^U(L)}\right)\left(\tfrac{\Delta^U(L)}{\varepsilon}\right)\right] \\ & +\tfrac{D}{Q(1-\theta)}\pi_0\left[1-\tfrac{\pi_x}{\pi_0}\left(1-\tfrac{\pi_x}{\pi_0}\right)\tfrac{\delta}{1+\Delta^U(L)}\right]\left(\tfrac{\Delta^U(L)}{\varepsilon}\right) \\ & +h^c\theta(Q-1)+\tfrac{Dv}{1-\theta}+\tfrac{D}{Q(1-\theta)}R(L), \end{aligned}$$

$$\begin{aligned} \Delta^U(L) =\ & \varepsilon\cdot\left\{\tfrac{1}{2}\left[-k\sqrt{1+p(1-p)\eta^2}\sigma\sqrt{L}\right]\right. \\ & +\tfrac{p}{2}\sigma\sqrt{L}\left[\sqrt{1+[k\sqrt{1+p(1-p)\eta^2}-(1-p)\eta]^2}\right] \\ & \left.+\tfrac{(1-p)}{2}\sigma\sqrt{L}\left[\sqrt{1+[k\sqrt{1+p(1-p)\eta^2}+p\eta]^2}\right]\right\} \end{aligned} \tag{22}$$

and $EAC^U(Q,L)$ and $\Delta^U(L)/\varepsilon$ are the upper bound for $EAC^F(Q,L)$and $B(r)$. Moreover, the upper bounds are tight, since the upper bound of (18) is tight.

Taking the partial derivatives of $EAC^U(Q,\pi_x,L)$ with respect to $Q$, $\pi_x$ and $L$, we obtain

$$\begin{aligned} \tfrac{\partial EAC^U(Q,\pi_x,L)}{\partial Q} =\ & -\tfrac{AD}{Q^2(1-\theta)}+\tfrac{h}{2}(1-\theta)-\tfrac{D}{Q^2(1-\theta)}\pi_0\left(1-\tfrac{\alpha_2}{1+\Delta^U(L)}\right)\left(\tfrac{\Delta^U(L)}{\varepsilon}\right) \\ & +h^c\theta-\tfrac{D}{Q^2(1-\theta)}R(L), \end{aligned} \tag{23}$$

$$\begin{aligned} \tfrac{\partial EAC^U(Q,\pi_x,L)}{\partial \pi_x} =\ & -\tfrac{h}{\pi_0}\tfrac{\delta}{1+\Delta^U(L)}\left(\tfrac{\Delta^U(L)}{\varepsilon}\right) \\ & -\tfrac{D}{Q(1-\theta)}\left(1-2\tfrac{\pi_x}{\pi_0}\right)\left(\tfrac{\delta}{1+\Delta^U(L)}\tfrac{\Delta^U(L)}{\varepsilon}\right), \end{aligned} \tag{24}$$

and

$$\begin{aligned}\frac{\partial EAC^U(Q,\pi_x,L)}{\partial L} &= \frac{hk}{2\sqrt{L}}\left[\sqrt{1+p(1-p)\eta^2\sigma}\right]+h\left(\frac{\Delta^U(L)}{2\varepsilon L}\right)\left(1-\frac{\alpha_1}{[1+\Delta^U(L)]^2}\right)\\ &+\frac{D\pi_0}{Q(1-\theta)}\left(\frac{\Delta^U(L)}{2\varepsilon L}\right)\left(1-\frac{\alpha_2}{[1+\Delta^U(L)]^2}\right)-\frac{D}{Q(1-\theta)}c_i.\end{aligned}$$

Since $B(r)$ is the expected shortage quantity at the end of cycle, we know that $B(r)>0$ if shortages occur; $B(r)=0$, otherwise. It is clear that $B(r)$ is positive. By examining the second-order sufficient conditions (SOSCs), it can be easily verified that $EAC^U(Q,\pi_x,L)$ is a convex in $Q$ and $\pi_x$, since

$$\begin{aligned}\frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial Q^2} &= \frac{2AD}{Q^3(1-\theta)}+\frac{2D}{Q^3(1-\theta)}\pi_0\left(1-\frac{\alpha_2}{1+\Delta^U(L)}\right)\left(\frac{\Delta^U(L)}{\varepsilon}\right)\\ &+\frac{2D}{Q^3(1-\theta)}R(L)>0,\\ \frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial \pi_x^2} &= \frac{2D}{Q(1-\theta)\pi_0}\frac{\delta}{1+\Delta^U(L)}\left(\frac{\Delta^U(L)}{\varepsilon}\right)>0,\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial Q^2}\cdot\frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial \pi_x^2}-\left[\frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial Q\partial \pi_x}\right]^2\\ &=\frac{4D^2}{Q^4(1-\theta)^2\pi_0}\frac{\delta}{1+\Delta^U(L)}\left(\frac{\Delta^U(L)}{\varepsilon}\right)[A+R(L)]\\ &+\frac{D^2}{Q^4(1-\theta)^2}\frac{\delta}{1+\Delta^U(L)}\left(\frac{\Delta^U(L)}{\varepsilon}\right)^2\left[4-\frac{\delta}{1+\Delta(L)}\right]>0.\end{aligned}$$

However, for fixed $Q$ and $\pi_x$, $EAC^U(Q,\pi_x,L)$ is concave in $L\in[L_i,L_{i-1}]$ because

$$\begin{aligned}\frac{\partial^2 EAC^U(Q,\pi_x,L)}{\partial L^2} &= -\frac{hk}{4L^{3/2}}[\sqrt{1+p(1-p)\eta^2\sigma}]\\ &-h\frac{\Delta^U(L)}{4\varepsilon L^2}\left\{\frac{(1-\alpha_1)\left[1+3\Delta^U(L)\right]+3\left[\Delta^U(L)\right]^2+\left[\Delta^U(L)\right]^3}{[1+\Delta^U(L)]^3}\right\}\\ &-\frac{D\pi_0}{Q(1-\theta)}\frac{\Delta^U(L)}{4\varepsilon L^2}\left\{\frac{(1-\alpha_2)\left[1+3\Delta^U(L)\right]+3\left[\Delta^U(L)\right]^2+\left[\Delta^U(L)\right]^3}{[1+\Delta^U(L)]^3}\right\}<0.\end{aligned}$$

Therefore, for fixed $Q$ and $\pi_x$, the minimum expected total annual cost will occur at the end points of the interval $L\in[L_i,L_{i-1}]$. Solving for $Q$ and $\pi_x$ by setting Equations (23) and (24) to zero, we have

$$\begin{aligned}Q &= \left\{\frac{2D}{(1-\theta)[h(1-\theta)+2h^c\theta]}\left(A+R(L)\right.\right.\\ &\left.\left.+\pi_0\left[1-\frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\frac{\delta}{1+\Delta^U(L)}\right]\left(\frac{\Delta^U(L)}{\varepsilon}\right)\right)\right\}^{\frac{1}{2}},\end{aligned}\tag{25}$$

and

$$\pi_x=\frac{1}{2}\left(h\frac{Q(1-\theta)}{D}+\pi_0\right),\tag{26}$$

where $\Delta^U(L)$ is expressed as Equation (22).

Substituting (26) into (25), $Q$ can be rewritten as

$$Q=\left\{\frac{2D\left[A+R(L)+\pi_0\left(1-\frac{1}{4}\left(\frac{\delta}{1+\Delta^U(L)}\right)\right)\left(\frac{\Delta^U(L)}{\varepsilon}\right)\right]}{h(1-\theta)^2\left(1-\frac{h}{2\pi_0 D}\frac{\delta}{1+\Delta^U(L)}\frac{\Delta^U(L)}{\varepsilon}\right)+2h^c\theta(1-\theta)}\right\}^{\frac{1}{2}}.\tag{27}$$

In practice, since the d.f. $F_*$ of the lead time demand $X$ is unknown, even if the value of $q$ is given, we cannot get the exact value of safety factor $k$. Thus, in order to find the value of $k$, we need the following proposition.

**Proposition 3.2. (Ouyang and Wu [14]):** *Let $Y$ be a random variable which has a p.d.f. $f_Y(y)$ with finite mean $\mu L$ and standard deviation $\sigma\sqrt{L}$ $(>0)$, the for any real number $d>\mu L$,*

$$P(Y>d)\le\frac{\sigma^2 L}{\sigma^2 L+(d-\mu L)^2}.\tag{28}$$

So, by using $F_* = pF_1 + (1-p)F_2$, the recorder point $r = \mu_* L + k\sigma_*\sqrt{L}$ and the Proposition 3.2, we get

$$\begin{aligned} P(X > r) &\le p\frac{\sigma^2 L}{\sigma^2 L + (r-\mu_1 L)^2} + (1-p)\frac{\sigma^2 L}{\sigma^2 L + (r-\mu_2 L)^2} \\ &= \frac{p}{1+\left[k\sqrt{1+p(1-p)\eta^2}-(1-p)\eta\right]^2} + \frac{1-p}{1+\left[k\sqrt{1+p(1-p)\eta^2}+p\eta\right]^2} \end{aligned} \tag{29}$$

Further, it is assumed that the allowable stock-out probability $q$ during lead time is given, that is, $q = P(X > r)$, then from Equation (29), we get $0 \le k \le \sqrt{\frac{1}{q}-1} + |\eta|$. It is easy to verify that $EAC^U(Q, \pi_x, L)$ has a smooth curve for $0 \le k \le \sqrt{\frac{1}{q}-1} + |\eta|$. Hence, we can establish the following algorithm to obtain the suitable $k$ and hence the optimal $Q$, $\pi_x$ and $L$ can be obtained.

**Algorithm 3.2.**

Step 1. For given $A$, $D$, $h$, $h^c$, $v$, $\sigma$, $\pi_0$, $\eta$, $p$, $q$, $\delta$, $\varepsilon$, $\theta$, $a_i$, $b_i$, $c_i$, and $i = 1, 2, \cdots, n$.

Step 2. Given $\eta$ and $q$, we divide the interval $\left[0, \sqrt{\frac{1}{q}-1} + |\eta|\right]$ into $m$ equal subintervals, where $m$ is large enough. Let $k_0 = 0$, $k_m = \sqrt{\frac{1}{q}-1} + |\eta|$, then $k_j = k_{j-1} + \frac{k_m - k_0}{m}$, $j = 1, 2, \cdots, m-1$ can be obtained.

Step 3. Given $a_i$, $b_i$ and $c_i$, use assumption A6 to compute $L_i$, $i = 1, 2, \cdots, n$.

Step 4. For each $L_i$, $i = 1, 2, \cdots, n$, compute $Q_{i,k_j}$ by using Equation (27) for given $k_j$, $j = 0, 1, 2, \cdots, m$. Then, $\pi_{x_{i,k_j}}$ can be computed by using Equation (26).

Step 5. Compare $\pi_{x_{i,k_j}}$ and $\pi_0$. If $\pi_{x_{i,k_j}} < \pi_0$, then $\pi_{x_{i,k_j}}$ in Step 4 is optimal; if $\pi_{x_{i,k_j}} \ge \pi_0$, then take $\pi_{x_{i,k_j}} = \pi_0$ into Step 6.

Step 6. For each $(Q_{i,k_j}, \pi_{x_{i,k_j}}, L_i)$ and $k_j$, the corresponding expected total annual cost $EAC^U(Q_{i,k_j}, \pi_{x_{i,k_j}}, L_i)$, $i = 1, 2, \cdots,$ $n$ and $j = 0, 1, \cdots, m$ can be computed.

Step 7. Find $\min\limits_{k_j \in \{k_0, k_1, \cdots, k_m\}} EAC^U(Q_{i,k_j}, \pi_{x_{i,k_j}}, L_i)$ and let

$$EAC^U(Q_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i) = \min_{k_j \in \{k_0, k_1, \cdots, k_m\}} EAC^U(Q_{i,k_j}, \pi_{x_{i,k_j}}, L_i),$$

then find $\min\limits_{i=0,1,\cdots,n} EAC^U(Q_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i)$. If

$$EAC^U(Q_f, \pi_{x_f}, L_f) = \min_{i=0,1,\cdots,n} EAC^U(Q_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i),$$

then $(Q_f, \pi_{x_f}, L_f)$ is the optimal solution.

4. **Numerical Examples.** In order to illustrate the above solution procedure, we consider an inventory system with the following data: $D = 600$ units/year, $A = \$200$ per order, $h = \$20$, $h^c = \$12$, $v = \$1.6$, $\pi = \$50$, $\pi_0 = \$150$, $\mu_* = 11$ units/week, $\sigma = 7$ units/week, $\theta = 0.00, 0.15, 0.30, 0.45$, $\varepsilon = 0, 2, 100, \infty$ (backorder case), $q = 0.2$ and the lead time has three components with shown in Table 1.

Table 1. Lead time data

| Lead time component $i$ | Normal duration $b_i$ (days) | Minimum duration $a_i$ (days) | Unit crashing cost $c_i$ (\$/days) |
|---|---|---|---|
| 1 | 20 | 6 | 0.4 |
| 2 | 20 | 6 | 1.2 |
| 3 | 16 | 9 | 5.0 |

**Example 4.1.** *We assume here that the lead time demand follows a mixture of normal distributions and want to solve the case when* $\eta = 0.7$, $p = 0, 0.4, 0.8, 1$, $\varepsilon = 0, 2, 100, \infty$, $\theta = 0.00, 0.15, 0.30, 0.45$ *and* $\delta = 1.0, 0.6$. *A summary of these optimal results is presented in* $(Q_s, \pi_{x_s}, L_s)$ *and* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *of Tables 2 and 3. From Tables 2 and 3, for fixed p, defective rate* $\theta$ *and* $\delta = 0.6, 1.0$, *we can get the order quantity* $Q_s$, *the backorder price discount* $\pi_{x_s}$ *and the minimum expected total annual cost* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *decrease as* $\varepsilon$ *decreases (i.e., the backorder rate* $\beta$ *increases ); the order quantity* $Q_s$, *the backorder price discount* $\pi_{x_s}$ *and the minimum expected total annual cost* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *decrease as* $\delta$ *increases for fixed p, defective rate* $\theta$ *and* $\varepsilon$. *Moreover, for fixed p,* $\varepsilon$ *and* $\delta = 0.6, 1.0$, *the backorder price discount* $\pi_{x_s}$ *decreases, the order quantity* $Q_s$ *and the minimum expected total annual cost* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *increase as* $\theta$ *increases. For fixed* $\varepsilon$, *defective rate* $\theta$ *and* $\delta = 0.6, 1.0$, *when p = 0 or 1, the model considers only one kind of customers' demand; when* $0 < p < 1$, *the model considers two kinds of customers' demand. It implies that* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *of two kinds of customers' demand are larger than* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *of one kind of customers' demand. Thus, the minimum expected total annual cost* $EAC^N(Q_s, \pi_{x_s}, L_s)$ *increases as the distance between p and 0 (or 1) increases for fixed* $\varepsilon$ *and defective rate* $\theta$. *Hence, if the true distribution of the lead time demand is mixture of normal distribution, the minimum expected total annual cost will be underestimated for treating a single normal distribution as the true distribution of the lead time demand. In addition, no matter what values of p and* $\theta$, *the optimal lead time* $L_s$ *is approached to 3 weeks for* $\delta$= *0.6, 1.0 and* $\varepsilon = 2, 100, \infty$; *the optimal lead time* $L_s$ *is approached to 4 weeks for* $\delta = 1.0$ *and* $\varepsilon = 0$.

**Example 4.2.** *We consider the inventory problem with the same parameters as those in Example 4.1. We assume here that the probability distribution of the lead time demand is mixture of free distributions. By using the proposed Algorithm 3.2 and setting* $m = 500$, *the optimal results are summarized in* $(Q_f, \pi_{x_f}, L_f)$ *and* $EAC^U(Q_f, \pi_{x_f}, L_f)$ *of Tables 2 and 3. From Tables 2 and 3, when* $\eta = 0.7$, *for fixed p, defective rate* $\theta$ *and* $\delta = 0.6, 1.0$, *we can get the order quantity* $Q_f$, *the backorder price discount* $\pi_{x_f}$ *and the minimum expected total annual cost* $EAC^U(Q_f, \pi_{x_f}, L_f)$ *decrease as* $\varepsilon$ *decreases (i.e., the backorder rate* $\beta$ *increases ); the order quantity* $Q_f$, *the backorder price discount* $\pi_{x_f}$ *and the minimum expected total annual cost* $EAC^U(Q_f, \pi_{x_f}, L_f)$ *decrease as* $\delta$ *increases for fixed p, defective rate* $\theta$ *and* $\varepsilon$. *Moreover, for fixed p,* $\varepsilon$ *and* $\delta = 0.6, 1.0$, *the backorder price discount* $\pi_{x_f}$ *decreases, the order quantity* $Q_f$ *and the minimum expected total annual cost* $EAC^U(Q_f, \pi_{x_f}, L_f)$ *increase as* $\theta$ *increases; the minimum expected total annual cost* $EAC^U(Q_f, \pi_{x_f}, L_f)$ *increases and then decreases as p increases for fixed* $\varepsilon$, $\theta$ *and* $\delta = 0.6, 1.0$. *In addition, no matter what values of p,* $\varepsilon$ *and* $\theta$, *the optimal lead time* $L_f$ *is approached to 3 weeks for* $\delta = 0.6, 1.0$. *Finally, the expected total annual cost* $EAC^N(Q_f, \pi_{x_f}, L_f)$ *is obtained by substituting* $Q_f$, $\pi_{x_f}$ *and* $L_f$ *into Equation (10) which is the mixture of normal distributions model. The expected value of additional information (EVAI) is the largest amount that one is willing to pay for the knowledge of* $F_1$ *and* $F_2$ *and is equal to* $EAC^N(Q_f, \pi_{x_f}, L_f) - EAC^N(Q_s, \pi_{x_s}, L_s)$. *From Tables 2 and 3, we observe that for fixed p and* $\varepsilon$, *EVAI increases as* $\theta$ *increases. Moreover, we also observe that for fixed* $\theta$ *and* $\varepsilon = 2, 100, \infty$ *and, EVAI increases and then decreases as p increases when* $\delta = 1.0$ *and* $0.6$; *for fixed* $\theta$ *and* $\varepsilon = 0$, *EVAI increases and then decreases as p increases when* $\delta = 0.6$; *for fixed* $\theta$ *and* $\varepsilon = 0$, *EVAI decreases and then increases as p increases when* $\delta = 1.0$.

**Example 4.3.** *From Figure 1, we have the density plot of the fictitious demand data set from Hastie et al. [6, p.237], which contains 20 past data* = $\{-0.39, 0.12, 0.94, 1.67, 1.76,$

TABLE 2. Summary of the optimal solution procedure ($L_i$ in weeks and $\eta = 0.7, \delta = 1.0$)

| $\varepsilon$ | $\theta$ | $(Q_f, \pi_{x_f}, L_f)$ | $EAC^U(f)$ | $EAC^N(f)$ | $(Q_s, \pi_{x_s}, L_s)$ | $EAC^N(s)$ | EVAI | CP |
|---|---|---|---|---|---|---|---|---|
| $p = 0.0$ (or 1) | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.674,3) | 4811.220 | 4515.343 | (166.,77.770,3) | 4513.274 | 2.069 | 1.0005 |
| | 0.15 | (170.,77.405,3) | 5302.093 | 5021.259 | (178.,77.517,3) | 5017.471 | 3.788 | 1.0008 |
| | 0.30 | (184.,77.151,3) | 5955.979 | 5694.775 | (193.,77.251,3) | 5690.541 | 4.234 | 1.0007 |
| | 0.45 | (205.,76.880,3) | 6892.579 | 6658.185 | (215.,76.968,3) | 6653.341 | 4.844 | 1.0007 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.673,3) | 4809.729 | 4513.900 | (166.,77.769,3) | 4511.829 | 2.071 | 1.0005 |
| | 0.15 | (170.,77.404,3) | 5300.447 | 5019.684 | (178.,77.515,3) | 5015.891 | 3.792 | 1.0008 |
| | 0.30 | (184.,77.150,3) | 5954.150 | 5693.026 | (193.,77.250,3) | 5688.787 | 4.240 | 1.0007 |
| | 0.45 | (205.,76.879,3) | 6890.501 | 6656.198 | (215.,76.967,3) | 6651.349 | 4.850 | 1.0007 |
| 2 | | | | | | | | |
| | 0.00 | (158.,77.636,3) | 4758.488 | 4461.697 | (164.,77.729,3) | 4459.756 | 1.941 | 1.0004 |
| | 0.15 | (167.,77.370,3) | 5244.783 | 4962.560 | (175.,77.479,3) | 4958.940 | 3.620 | 1.0007 |
| | 0.30 | (182.,77.118,3) | 5892.346 | 5629.802 | (190.,77.217,3) | 5625.554 | 4.248 | 1.0008 |
| | 0.45 | (202.,76.851,3) | 6820.289 | 6584.395 | (211.,76.938,3) | 6579.536 | 4.859 | 1.0007 |
| 0 | | | | | | | | |
| | 0.00 | (156.,77.601,3) | 4624.543 | 4310.958 | (155.,77.576,4) | 4301.933 | 9.025 | 1.0021 |
| | 0.15 | (165.,77.340,3) | 5106.696 | 4796.495 | (165.,77.341,4) | 4782.701 | 13.794 | 1.0029 |
| | 0.30 | (177.,77.071,3) | 5747.255 | 5445.984 | (179.,77.094,4) | 5425.603 | 20.381 | 1.0038 |
| | 0.45 | (195.,76.787,3) | 6659.351 | 6377.313 | (200.,76.830,4) | 6347.197 | 30.116 | 1.0047 |
| $p = 0.4$ | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.670,3) | 4821.562 | 4574.607 | (168.,77.807,3) | 4570.345 | 4.262 | 1.0009 |
| | 0.15 | (169.,77.401,3) | 5311.963 | 5085.774 | (180.,77.550,3) | 5079.031 | 6.744 | 1.0013 |
| | 0.30 | (183.,77.135,3) | 5963.355 | 5766.992 | (196.,77.281,3) | 5757.877 | 9.115 | 1.0016 |
| | 0.45 | (204.,76.866,3) | 6895.749 | 6738.994 | (218.,76.994,3) | 6728.566 | 10.427 | 1.0015 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.668,3) | 4820.069 | 4573.184 | (168.,77.806,3) | 4568.917 | 4.267 | 1.0009 |
| | 0.15 | (169.,77.400,3) | 5310.314 | 5084.221 | (180.,77.549,3) | 5077.469 | 6.752 | 1.0013 |
| | 0.30 | (183.,77.134,3) | 5961.515 | 5765.271 | (195.,77.280,3) | 5756.144 | 9.126 | 1.0016 |
| | 0.45 | (203.,76.865,3) | 6893.658 | 6737.039 | (217.,76.993,3) | 6726.598 | 10.440 | 1.0016 |
| 2 | | | | | | | | |
| | 0.00 | (158.,77.632,3) | 4768.878 | 4520.731 | (166.,77.766,3) | 4516.654 | 4.077 | 1.0009 |
| | 0.15 | (167.,77.367,3) | 5254.708 | 5026.833 | (177.,77.512,3) | 5020.315 | 6.518 | 1.0013 |
| | 0.30 | (180.,77.102,3) | 5900.065 | 5701.841 | (193.,77.248,3) | 5692.691 | 9.149 | 1.0016 |
| | 0.45 | (200.,76.837,3) | 6823.844 | 6665.009 | (214.,76.965,3) | 6654.542 | 10.467 | 1.0016 |
| 0 | | | | | | | | |
| | 0.00 | (156.,77.596,3) | 4636.205 | 4358.567 | (157.,77.611,4) | 4357.754 | 0.813 | 1.0002 |
| | 0.15 | (165.,77.336,3) | 5117.845 | 4848.156 | (167.,77.372,4) | 4842.702 | 5.454 | 1.0011 |
| | 0.30 | (177.,77.067,3) | 5757.828 | 5502.889 | (182.,77.122,4) | 5490.986 | 11.902 | 1.0022 |
| | 0.45 | (195.,76.786,3) | 6669.254 | 6441.323 | (202.,76.855,4) | 6419.929 | 21.394 | 1.0033 |
| $p = 0.8$ | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.673,3) | 4817.024 | 4542.023 | (167.,77.785,3) | 4539.179 | 2.844 | 1.0006 |
| | 0.15 | (170.,77.402,3) | 5307.667 | 5050.163 | (179.,77.530,3) | 5045.212 | 4.951 | 1.0010 |
| | 0.30 | (183.,77.140,3) | 5959.683 | 5727.169 | (194.,77.263,3) | 5720.646 | 6.523 | 1.0011 |
| | 0.45 | (204.,76.870,3) | 6893.276 | 6694.135 | (216.,76.979,3) | 6686.672 | 7.463 | 1.0011 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.671,3) | 4815.532 | 4540.646 | (167.,77.784,3) | 4537.741 | 2.905 | 1.0006 |
| | 0.15 | (170.,77.401,3) | 5306.020 | 5048.597 | (179.,77.529,3) | 5043.640 | 4.957 | 1.0010 |
| | 0.30 | (183.,77.139,3) | 5957.846 | 5725.431 | (194.,77.263,3) | 5718.899 | 6.532 | 1.0011 |
| | 0.45 | (204.,76.869,3) | 6891.189 | 6692.162 | (216.,76.978,3) | 6684.689 | 7.473 | 1.0011 |
| 2 | | | | | | | | |
| | 0.00 | (158.,77.634,3) | 4764.315 | 4488.330 | (165.,77.744,3) | 4485.583 | 2.747 | 1.0006 |
| | 0.15 | (167.,77.368,3) | 5250.383 | 4991.352 | (176.,77.493,3) | 4986.597 | 4.755 | 1.0010 |
| | 0.30 | (181.,77.107,3) | 5896.291 | 5662.111 | (191.,77.230,3) | 5655.567 | 6.544 | 1.0012 |
| | 0.45 | (201.,76.841,3) | 6821.257 | 6620.253 | (213.,76.949,3) | 6612.767 | 7.486 | 1.0011 |
| 0 | | | | | | | | |
| | 0.00 | (156.,77.599,3) | 4630.977 | 4332.924 | (155.,77.591,4) | 4328.219 | 4.705 | 1.0011 |
| | 0.15 | (165.,77.338,3) | 5112.887 | 4820.124 | (166.,77.353,4) | 4810.696 | 9.428 | 1.0020 |
| | 0.30 | (177.,77.069,3) | 5753.170 | 5471.729 | (180.,77.105,4) | 5455.800 | 15.929 | 1.0029 |
| | 0.45 | (195.,76.787,3) | 6664.941 | 6405.974 | (201.,76.840,4) | 6380.400 | 25.573 | 1.0040 |

Note: $EAC^U(Q_f, \pi_{x_f}, L_f)$, $EAC^N(Q_f, \pi_{x_f}, L_f)$ and $EAC^N(Q_s, \pi_{x_s}, L_s)$ will be denoted by the symbol $EAC^U(f)$, $EAC^N(f)$ and $EAC^N(s)$, respectively.

Table 3. Summary of the optimal solution procedure ($L_i$ in weeks and $\eta = 0.7$, $\delta = 0.6$)

| $\varepsilon$ | $\theta$ | $(Q_f, \pi_{x_f}, L_f)$ | $EAC^U(f)$ | $EAC^N(f)$ | $(Q_s, \pi_{x_s}, L_s)$ | $EAC^N(s)$ | $EVAI$ | CP |
|---|---|---|---|---|---|---|---|---|
| $p = 0.0$ (or 1) | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.674,3) | 4811.220 | 4515.343 | (166.,77.770,3) | 4513.274 | 2.069 | 1.0005 |
| | 0.15 | (170.,77.405,3) | 5302.093 | 5021.259 | (178.,77.517,3) | 5017.472 | 3.788 | 1.0008 |
| | 0.30 | (184.,77.151,3) | 5955.979 | 5694.775 | (193.,77.251,3) | 5690.542 | 4.234 | 1.0007 |
| | 0.45 | (205.,76.880,3) | 6892.579 | 6658.185 | (215.,76.968,3) | 6653.341 | 4.844 | 1.0007 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.673,3) | 4810.326 | 4514.478 | (166.,77.770,3) | 4512.407 | 2.070 | 1.0005 |
| | 0.15 | (170.,77.404,3) | 5301.105 | 5020.314 | (178.,77.516,3) | 5016.523 | 3.791 | 1.0008 |
| | 0.30 | (184.,77.150,3) | 5954.881 | 5693.726 | (193.,77.251,3) | 5689.489 | 4.237 | 1.0007 |
| | 0.45 | (205.,76.880,3) | 6891.333 | 6656.993 | (215.,76.967,3) | 6652.146 | 4.848 | 1.0007 |
| 2 | | | | | | | | |
| | 0.00 | (159.,77.652,3) | 4779.676 | 4483.236 | (165.,77.745,3) | 4481.253 | 1.983 | 1.0004 |
| | 0.15 | (168.,77.383,3) | 5267.812 | 4986.186 | (176.,77.494,3) | 4982.452 | 3.734 | 1.0007 |
| | 0.30 | (183.,77.131,3) | 5917.911 | 5655.902 | (191.,77.231,3) | 5651.660 | 4.242 | 1.0008 |
| | 0.45 | (203.,76.863,3) | 6849.333 | 6614.038 | (213.,76.950,3) | 6609.185 | 4.853 | 1.0007 |
| 0 | | | | | | | | |
| | 0.00 | (158.,77.631,3) | 4703.019 | 4393.667 | (161.,77.677,3) | 4393.190 | 0.477 | 1.0001 |
| | 0.15 | (167.,77.366,3) | 5188.804 | 4887.446 | (172.,77.432,3) | 4886.129 | 1.316 | 1.0003 |
| | 0.30 | (179.,77.094,3) | 5833.854 | 5547.595 | (185.,77.154,4) | 5542.318 | 5.277 | 1.0010 |
| | 0.45 | (199.,76.825,3) | 6754.009 | 6491.549 | (205.,76.883,4) | 6479.796 | 11.752 | 1.0018 |
| $p = 0.4$ | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.670,3) | 4821.562 | 4574.607 | (168.,77.807,3) | 4570.345 | 4.262 | 1.0009 |
| | 0.15 | (169.,77.401,3) | 5311.963 | 5085.774 | (180.,77.550,3) | 5079.031 | 6.744 | 1.0013 |
| | 0.30 | (183.,77.135,3) | 5963.355 | 5766.992 | (196.,77.281,3) | 5757.877 | 9.115 | 1.0016 |
| | 0.45 | (204.,76.866,3) | 6895.749 | 6738.994 | (218.,76.994,3) | 6728.566 | 10.428 | 1.0015 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.669,3) | 4820.667 | 4573.753 | (168.,77.807,3) | 4569.489 | 4.265 | 1.0009 |
| | 0.15 | (169.,77.400,3) | 5310.974 | 5084.843 | (180.,77.550,3) | 5078.094 | 6.749 | 1.0013 |
| | 0.30 | (183.,77.134,3) | 5962.251 | 5765.959 | (195.,77.281,3) | 5756.837 | 9.122 | 1.0016 |
| | 0.45 | (204.,76.866,3) | 6894.494 | 6737.821 | (217.,76.994,3) | 6727.386 | 10.435 | 1.0016 |
| 2 | | | | | | | | |
| | 0.00 | (159.,77.647,3) | 4790.047 | 4542.356 | (167.,77.782,3) | 4538.220 | 4.136 | 1.0009 |
| | 0.15 | (168.,77.381,3) | 5277.715 | 5050.490 | (178.,77.528,3) | 5043.899 | 6.591 | 1.0013 |
| | 0.30 | (181.,77.115,3) | 5925.493 | 5728.011 | (194.,77.261,3) | 5718.876 | 9.135 | 1.0016 |
| | 0.45 | (202.,76.849,3) | 6852.734 | 6694.729 | (216.,76.976,3) | 6684.278 | 10.451 | 1.0016 |
| 0 | | | | | | | | |
| | 0.00 | (158.,77.627,3) | 4714.091 | 4445.963 | (163.,77.710,3) | 4444.416 | 1.547 | 1.0003 |
| | 0.15 | (167.,77.363,3) | 5199.381 | 4944.326 | (174.,77.462,3) | 4941.306 | 3.020 | 1.0006 |
| | 0.30 | (179.,77.091,3) | 5843.875 | 5610.358 | (189.,77.202,3) | 5604.966 | 5.392 | 1.0010 |
| | 0.45 | (198.,76.812,3) | 6761.593 | 6563.188 | (210.,76.925,3) | 6554.910 | 8.278 | 1.0013 |
| $p = 0.8$ | | | | | | | | |
| $\infty$ | | | | | | | | |
| | 0.00 | (160.,77.673,3) | 4817.024 | 4542.023 | (167.,77.785,3) | 4539.179 | 2.844 | 1.0006 |
| | 0.15 | (170.,77.402,3) | 5307.667 | 5050.163 | (179.,77.530,3) | 5045.212 | 4.951 | 1.0010 |
| | 0.30 | (183.,77.140,3) | 5959.683 | 5727.169 | (194.,77.263,3) | 5720.646 | 6.523 | 1.0011 |
| | 0.45 | (204.,76.870,3) | 6893.276 | 6694.135 | (216.,76.979,3) | 6686.672 | 7.463 | 1.0011 |
| 100 | | | | | | | | |
| | 0.00 | (160.,77.672,3) | 4816.129 | 4541.163 | (167.,77.785,3) | 4538.316 | 2.847 | 1.0006 |
| | 0.15 | (170.,77.402,3) | 5306.679 | 5049.224 | (179.,77.530,3) | 5044.269 | 4.955 | 1.0010 |
| | 0.30 | (183.,77.139,3) | 5958.581 | 5726.126 | (194.,77.263,3) | 5719.598 | 6.528 | 1.0011 |
| | 0.45 | (204.,76.870,3) | 6892.024 | 6692.952 | (216.,76.978,3) | 6685.483 | 7.469 | 1.0011 |
| 2 | | | | | | | | |
| | 0.00 | (159.,77.649,3) | 4785.494 | 4509.908 | (166.,77.761,3) | 4507.111 | 2.796 | 1.0006 |
| | 0.15 | (168.,77.382,3) | 5273.403 | 5014.960 | (177.,77.508,3) | 5010.142 | 4.818 | 1.0010 |
| | 0.30 | (182.,77.120,3) | 5921.759 | 5688.245 | (192.,77.243,3) | 5681.709 | 6.536 | 1.0012 |
| | 0.45 | (202.,76.853,3) | 6850.192 | 6649.932 | (214.,76.961,3) | 6642.456 | 7.477 | 1.0011 |
| 0 | | | | | | | | |
| | 0.00 | (158.,77.630,3) | 4709.173 | 4417.542 | (161.,77.690,3) | 4416.699 | 0.843 | 1.0002 |
| | 0.15 | (167.,77.365,3) | 5194.721 | 4913.179 | (173.,77.444,3) | 4911.254 | 1.925 | 1.0004 |
| | 0.30 | (179.,77.092,3) | 5839.502 | 5575.743 | (187.,77.186,3) | 5571.908 | 3.835 | 1.0007 |
| | 0.45 | (198.,76.816,3) | 6757.859 | 6523.670 | (207.,76.894,4) | 6515.573 | 8.097 | 1.0012 |

Note: $EAC^U(Q_f, \pi_{x_f}, L_f)$, $EAC^N(Q_f, \pi_{x_f}, L_f)$ and $EAC^N(Q_s, \pi_{x_s}, L_s)$ will be denoted by the symbol $EAC^U(f)$, $EAC^N(f)$ and $EAC^N(s)$, respectively.

$2.44, 3.72, 4.28, 4.92, 5.53, 0.06, 0.48, 1.01, 1.68, 1.80, 3.25, 4.12, 4.60, 5.28, 6.22\}$. *We assume that the past data set (except the first data) is demand data with unit lead time (i.e., $L = 1$ week). Due to the apparent bi-modality, a single normal distribution would not be appropriate. So, we use a model of mixture of two normal distributions to fit the 19 demand data and assume that the parameters of the mixture of two normal distributions for the lead time demand are unknown. The maximum likelihood (ML) method is the most popular technique for deriving estimators of these parameters. That is, we directly take derivatives with respect to these parameters of this likelihood functions based on the ML method and use them to find the maximum. It turns out that these equations can become quite difficult since these equations are nonlinear and no analytic solutions can be found. Hence, we use an easier algorithm that is guaranteed to converge to the ML estimators. The algorithm is called expectation maximization (EM) algorithm which was introduced by Hastie et al. [6] and used to obtain the estimates of unknown parameters* $\mu_1$, $\mu_2$, $\sigma$ *and* $p$ *of the mixture of normal distributions for the lead time demand. Then, we obtain* $\hat{\mu}_1 = 1.21$, $\hat{\mu}_2 = 4.64$, $\hat{\sigma} = 1.18$ *and* $\hat{p} = 0.53$. *Second, we also obtain that* $\mu_* = 2.8221$, $\eta \approx -3.0$ *and* $\sigma_* \approx 2.12$ *via the equations* $\mu_* = p\mu_1 + (1-p)\mu_2$, $\mu_1 - \mu_2 = \eta\sigma\Big/\sqrt{L}$ *and* $\sigma_* = \sqrt{1 + p(1-p)\eta^2}\sigma$, *respectively. Next, we consider the case when* $\mu_* = 2.8221$, $\sigma = 1.18$, $\eta = -3.0$, $p = 0.53$ *and the other data is the same as Example 4.1. Besides, we also consider the situation that the data is modelled as a single normal distribution. In this situation, we take* $\mu_* = 2.8221$, $\sigma_* = 2.12$, $p = 0$ *and the other data is the same as Example 4.1. From Tables 4 and 5, we find that (i) the optimal solution is similar between the two cases; (ii) the minimum expected total annual cost will be overestimated by modelling the lead time demand data as a single normal distribution, when the lead time demand data follows a mixture of two normal distributions.*

## 5. **Concluding Remarks.**

This article proposed two cases for the inventory problem involving controllable backorder rate with defective units. In the real market as shortages occur as unsatisfied demands occur, the longer the length of lead time is, the smaller the proportion of backorder would be; the higher the backorder price discount is, the

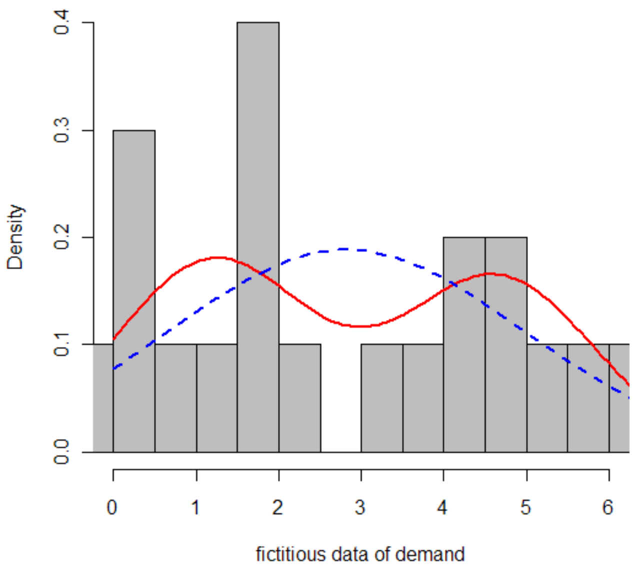


FIGURE 1. Density plot for demand data set. Solid curve denotes maximum likelihood fit of mixture of normal densities and dash curve presents fit of a single normal density.

TABLE 4. Summary of the optimal solution procedure ($L_i$ in weeks and $\delta = 0.6$)

| | | Mixture of two normal distributions | | Single normal distribution | |
|---|---|---|---|---|---|
| $\varepsilon$ | $\theta$ | $(Q_*, \pi_{x_*}, L_*)$ | $EAC(Q_*, \pi_{x_*}, L_*)$ | $(Q', \pi'_x, L')$ | $EAC(Q', \pi'_x, L')$ |
| | | $p = 0.53$ | | $p = 0$ | |
| $\infty$ | 0.00 | (127, 77.118, 6) | 3612.492 | (133, 77.211, 4) | 3694.444 |
| | 0.15 | (136, 76.924, 6) | 4037.756 | (142, 77.009, 4) | 4131.045 |
| | 0.30 | (148, 76.721, 6) | 4609.215 | (154, 76.797, 4) | 4717.098 |
| | 0.45 | (164, 76.504, 6) | 5433.250 | (171, 76.571, 4) | 5561.063 |
| 100 | 0.00 | (127, 77.117, 6) | 3611.395 | (133, 77.211, 4) | 3693.388 |
| | 0.15 | (136, 76.923, 6) | 4036.554 | (142, 77.008, 4) | 4129.889 |
| | 0.30 | (147, 76.720, 6) | 4607.878 | (154, 76.796, 4) | 4715.812 |
| | 0.45 | (164, 76.504, 6) | 5431.730 | (171, 76.570, 4) | 5559.601 |
| 2 | 0.00 | (126, 77.097, 6) | 3586.631 | (131, 77.191, 4) | 3668.107 |
| | 0.15 | (134, 76.905, 6) | 4009.431 | (140, 76.990, 4) | 4102.204 |
| | 0.30 | (146, 76.704, 6) | 4577.717 | (153, 76.780, 4) | 4685.034 |
| | 0.45 | (163, 76.490, 6) | 5397.419 | (170, 76.556, 4) | 5524.596 |
| 0 | 0.00 | (125, 77.080, 6) | 3564.563 | (130, 77.171, 4) | 3642.956 |
| | 0.15 | (133, 76.889, 6) | 3985.258 | (139, 76.972, 4) | 4074.661 |
| | 0.30 | (145, 76.690, 6) | 4550.835 | (150, 76.754, 6) | 4653.534 |
| | 0.45 | (161, 76.477, 6) | 5366.837 | (167, 76.533, 6) | 5486.241 |

TABLE 5. Summary of the optimal solution procedure ($L_i$ in weeks and $\delta = 1.0$)

| | | Mixture of two normal distributions | | Single normal distribution | |
|---|---|---|---|---|---|
| $\varepsilon$ | $\theta$ | $(Q_*, \pi_{x_*}, L_*)$ | $EAC(Q_*, \pi_{x_*}, L_*)$ | $(Q', \pi'_x, L')$ | $EAC(Q', \pi'_x, L')$ |
| | | $p = 0.53$ | | $p = 0$ | |
| $\infty$ | 0.00 | (127, 77.118, 6) | 3612.492 | (133, 77.211, 4) | 3694.444 |
| | 0.15 | (136, 76.924, 6) | 4037.756 | (142, 77.009, 4) | 4131.045 |
| | 0.30 | (148, 76.721, 6) | 4609.215 | (154, 76.797, 4) | 4717.097 |
| | 0.45 | (164, 76.504, 6) | 5433.250 | (171, 76.571, 4) | 5561.063 |
| 100 | 0.00 | (127, 77.116, 6) | 3610.663 | (133, 77.210, 4) | 3692.684 |
| | 0.15 | (136, 76.922, 6) | 4035.752 | (142, 77.008, 4) | 4129.117 |
| | 0.30 | (147, 76.720, 6) | 4606.987 | (154, 76.796, 4) | 4714.954 |
| | 0.45 | (164, 76.503, 6) | 5430.716 | (171, 76.570, 4) | 5558.625 |
| 2 | 0.00 | (125, 77.084, 6) | 3569.258 | (131, 77.177, 4) | 3650.418 |
| | 0.15 | (134, 76.893, 6) | 3990.401 | (140, 76.977, 4) | 4082.833 |
| | 0.30 | (145, 76.693, 6) | 4556.555 | (152, 76.769, 4) | 4663.496 |
| | 0.45 | (161, 76.480, 6) | 5373.344 | (169, 76.546, 4) | 5500.100 |
| 0 | 0.00 | (123, 77.054, 6) | 3532.148 | (127, 77.125, 6) | 3602.567 |
| | 0.15 | (132, 76.866, 6) | 3949.750 | (136, 76.930, 6) | 4028.675 |
| | 0.30 | (143, 76.669, 6) | 4511.346 | (148, 76.727, 6) | 4601.219 |
| | 0.45 | (159, 76.459, 6) | 5321.910 | (165, 76.509, 6) | 5426.733 |

larger the backorder rate would be. So, we consider that the backorder rate is dependent on the amount of the shortages and the backorder price discount based on the idea of Ouyang and Chuang [13] (also see Lee [9]) and Pan and Hsiao [17]. In the first case, we assume that the lead time demand possesses a mixture of normal distributions and the number of non-defective units is a random variable. In the second case, we relax the assumption about probability distributional form of the lead time demand and apply the minimax distribution-free procedure to solving the problem. In addition, we also develop an algorithmic procedure to find the optimal solution. Numerical examples are performed to investigate the results of our proposed models.

In practice, the uncertainties of customer's demand and backorder rate are inherent. It is not appropriate to describe the lead time demand by using a single distribution. Besides, backorder rate can be controlled by the amount of the shortages and the backorder price discount. In real environments, it is often that an arrival order lot may contain some defective units. From these perspectives, it is reasonable using the mixture inventory model we proposed. Most importantly, we make every endeavour to develop an inventory model which can fit in with the real situation.

In future research on this problem, it would be interesting to deal with a service level constraint or treat the reorder point as a decision variable. In addition, we assume that the crashing cost function in the lead time is a piecewise linear function in the thesis. It would be of interest to examine a non-linear relationship that exists between the crashing cost and the lead time in the future.

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