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PERFORMANCE ANALYSIS OF NETWORKED CONTROL SYSTEMS WITH SNR CONSTRAINTS

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Abstract. *The performance analysis of single-input and single-output (SISO) networked control systems (NCSs) with signal-to-noise ratio (SNR) constraints is presented in this paper. The tracking performance is measured by the energy of the error variance response between the output of the plant and the reference signal. The objective is to obtain an optimal tracking performance, attainable by all possible stabilizing compensators. It is shown that the optimal tracking performance is constrained by nonminimum phase (NMP) zeros, unstable poles of a given plant, a given reference signal and SNR of a communication channel. The result obtained in this paper explicitly shows how the optimal tracking performance is limited by the SNR of a communication channel. A typical example is given to illustrate the theoretical results.*

Keywords: Networked control systems, Performance analysis, SNR constraints, Unstable poles, Nonminimum phase zeros

1. Introduction.

1.1. Antecedents and motivation.

In recent years, more and more attention has been drawn to the development of a general theory for networked control systems (NCSs) [1, 19], which considers control and communication issues simultaneously. In these studies, the main issues addressed include modeling of the networked control system and stabilization analysis with quantization effects [4, 6], time delays [22], packet losses [20], bandwidth constraint [10] and bit rate limitations [9]. Nowadays, network has been playing an important role in various fields, such as industrial control applications, manufacturing automation, automobiles, advanced aircraft, intelligent traffic, intelligent buildings. According to the application of networked control systems, we can know that the performance analysis of networked control systems is very important. However, there are some inherent shortcomings with NCSs, such as bandwidth constraints, signal-to-noise ratio, packet dropouts and quantization precision, time delays, which will degrade the performance of NCSs or even lead to system instability. When designing NCSs, the characteristics of the communication system should be explicitly taken into account of to ensure acceptable performance levels. This raises new challenges. A key question is, in NCSs, how to design the communication channels capacity to ensure achievable performance. It is very important to know how the communication parameters affect the performance of networked control systems, which provides a useful guidance in design of networked control systems.

The optimal tracking performance in control design has been an important area of research for many years [3, 16, 17, 21]. It is well known that the optimal performance is constrained by nonminimum phase zeros and unstable poles of a given plant for linear feedback control systems in paper [12]. In paper [3] studied the tracking performance of multi-input multi-output linear time-invariant systems. It was shown that plant nonminimum phase zeros, the zero directions, the unstable poles and pole direction had a negative effect on a feedback systems ability to reduce the tracking error. It had also pointed out that the two-parameter compensator could improve the performance of control systems. Recently, some efforts have been devoted to the same problem for networked control systems [7, 18], where the constraint factor from network under consideration is signal-to-noise ratio. The optimal design and tracking performance of scalar NCS's with signal-to-noise ratio constraints of a communication channel is studied in the paper [13], which has provided a description of the optimal tradeoff curve in the performance versus signal-to-noise ratio plane. The problem of elucidating the interplay between closed loop performance and SNR constraints is discussed in the paper [14], which has provided a characterization of the best achievable performance subjected to a given SNR constraints. A difficulty associated with the results in the paper [11] is that they do not address the question of what is the optimal performance for a given SNR constraints. Also, they do not address the problem of optimal design with SNR constraints. Partial solutions to the latter problem have been studied in the paper [7]. In that work, one degree-of-freedom control schemes for noisy discrete-time LTI plants have been studied with the assumption that an additive white Gaussian noise feedback channel with pre- and post-scaling factors. The adopted model can be found in many real systems. For example, in a robot surgery system with remote monitoring patient is the plant, the robot is the controller, the remote expert obtains information via the network transmission, and the instruction of the expert is then returned to the robot via the network transmission. If the communication channels capacity is limited, the instructions of the expert will partly lose and affect result of the operations. It is very important to understand how the channels capacity affects system performance.

1.2. **Contribution.** In the present work, we also use an additive noise channel model for the link between the sensor and the plant. Continuing with the work of [13], we study the optimal tracking performance issues pertaining to SISO networked control systems with SNR constraints of a communication channel by using one-parameter and two-parameter compensators, respectively. We give an explicit expression for the optimal tracking performance in terms of the SNR constraint of a communication channel and characteristics of a given plant. The results show that the optimal tracking performance is constrained by the nonminimum phase zeros, the unstable poles of a given plant and SNR constraints of a communication channel. The tracking performance is improved by two-parameter compensator scheme as shown in Figure 3. The results obtained in this work explicitly show how the optimal tracking performance is degraded by the SNR.

Communication channel capacity has been precisely characterized in the famous Shannon result for an additive white Gaussian noise (AWGN) channel with signal-to-noise ratio γ

$$C = B\log_2(1+\gamma),$$

where C and B represent the communication channel capacity and communication bandwidth, respectively.

It is noted that the obtained results focus on the issue of the optimal performance achievable by networked control systems, and in particular on how the optimal performance may be intrinsically limited by the properties of the plant and the SNR of communication channel. The proposed results provide a useful guideline in design of control systems including design of communication channels capacity.

1.3. **Paper structure.** The remainder of the paper is organized as follows. The problem formulation is discussed in Section 2. The optimal tracking performance by one-parameter and two-parameter compensators with SNR constraints of a communication channel is obtained in Section 3. A typical example is given to verify the proposed results in Section 4. The paper conclusions and future research directions are presented in Section 5.

2. Problem Statement and Preliminaries.

2.1. **Preliminaries.** We first describe the standard notation used throughout this paper. For any complex number z, we denote its complex conjugate by $\bar{z}$. $\hat{u}(s)$ denotes Laplace transform of any signal $u(t)$. Let the open-right and left-half plane be denoted by $\mathbb{C}_+ := \{s : \mathrm{Re}(s) > 0\}$ and $\mathbb{C}_- := \{s : \mathrm{Re}(s) < 0\}$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces containing functions that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm $\|\cdot\|_2$ and the norm in the space $\mathcal{L}_2$. The space $\mathcal{L}_2$ is the Hilbert space with inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(f^H(jw)g(jw)\right)dw,$$

which further induces the $\mathcal{L}_2$ norm $\|f\|_2^2 := \langle f, f\rangle$. For any f, $g \in \mathcal{L}_2$, they are orthogonal if $\langle f, g\rangle = 0$. It is well known that $\mathcal{L}_2$ can be decomposed into two orthogonal subspaces $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ given by [5]

$$\mathcal{H}_2 := \{f : f(s)\ \textit{analytic in}\ \mathbb{C}_+,$$
$$\|f\|_2^2 := \sup_{\sigma>0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\|f(\sigma + jw)\|^2 dw < 0\},$$

and

$$\mathcal{H}_2^\perp := \{f : f(s)\ \textit{analytic in}\ \mathbb{C}_-,$$
$$\|f\|_2^2 := \sup_{\sigma<0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\|f(\sigma + jw)\|^2 dw < 0\}.$$

Finally, $\mathbb{R}\mathcal{H}_\infty$ denotes the set of all stable, proper, rational transfer function.

2.2. **Problem statement.** Consider a SISO linear time-invariant networked control system based on SNR constraints as depicted in Figure 1, where the problem is to obtain the optimal tracking performance. Figure 1 represents the unity standard feedback control system with communication channel in feedback path. In this setup, G denotes the plant model and K denotes the single-degree-of-freedom compensator, whose transfer function are $G(s)$ and $K(s)$, respectively. The communication channel is characterized by an additive white noise n. The signal r, y, ω and ν represent the reference input, the system output, the communication channel input, the communication channel output, respectively. Throughout this article, symbols $\tilde{r}$, $\tilde{y}$, $\tilde{\omega}$, $\tilde{\nu}$, $\tilde{n}$ and $\tilde{e}$ are the Laplace transforms of signals $r(t)$, $y(t)$, $\omega(t)$, $\nu(t)$, $n(t)$ and $e(t)$, respectively.

According to Figure 1, then

$$\tilde{\nu} = \tilde{\omega} + \tilde{n},$$

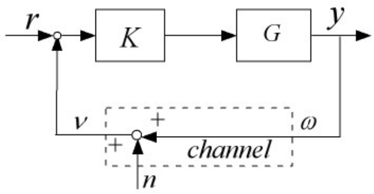

FIGURE 1. The unity feedback system with communication channel

where $\tilde{n}$ is an additive noise of a communication channel. The additive noise $\tilde{n}$ is assumed to be a zero mean white noise sequence, and is uncorrelated with the reference signal $\tilde{r}$, having variance σ_n^2 and power spectral density

$$\Phi_n(jw) = \sigma_n^2, \ \ \forall w \in \left(-\infty \quad \infty \right).$$

The power spectral density of a communication channel input is

$$\Phi_\omega(jw) = \sigma_\omega^2, \ \ \forall w \in \left(-\infty \quad \infty \right).$$

The signal-to-noise ratio of communication channel is denoted by

$$\gamma \triangleq \frac{\sigma_\omega^2}{\sigma_n^2}. \tag{1}$$

For the given reference signal r, the tracking error of the system is defined as $e = r - y$. It can be easily seen that

$$\tilde{e} = \tilde{r} - \tilde{y} = S(s)\tilde{r} + T(s)\tilde{n},$$

where

$$S(s) \triangleq \frac{1}{(1 + G(s)K(s))}, \quad T(s) \triangleq 1 - S(s). \tag{2}$$

In this paper, we shall derive an explicit expression for the optimal performance of tracking a random variable. The reference input r is supposed to be a zero-mean i.i.d. and variance (denote by σ^2) of a wide-sense stationary random process [8].

We are interested in NCS performance, the tracking performance of the system is defined as the variance of e

$$J = \sigma_e^2 = \|S(s)\|_2^2 \sigma^2 + \sigma_n^2 \|T(s)\|_2^2. \tag{3}$$

The optimal tracking performance is measured by the possible minimal tracking error achievable by all possible linear stabilizing controllers (denoted by $\mathcal{K}$), determined as

$$J^* = \inf_{K \in \mathcal{K}} J. \tag{4}$$

From Figure 1, we can obtain

$$\sigma_\omega^2 = \|T(s)\|_2^2 \sigma^2 + \sigma_n^2 \|T(s)\|_2^2.$$

From (1), we can obtain

$$\gamma = \frac{\sigma_\omega^2}{\sigma_n^2} = \frac{\sigma^2}{\sigma_n^2} \|T(s)\|_2^2 + \|T(s)\|_2^2. \tag{5}$$

For the rational transfer function G, we consider a coprime factorization of G as

$$G = \frac{N}{M}, \tag{6}$$

where N, $M \in \mathbb{R}\mathcal{H}_\infty$, and satisfy the Bezout identity

$$MX - NY = 1, \tag{7}$$

for some X, $Y \in \mathbb{RH}_\infty$. It is well known [15] that a stabilizing compensator K can be characterized by the Youla parameterization

$$\mathcal{K} := \left\{ K : K = -\frac{(Y - MQ)}{X - NQ}, \ \ Q \in \mathbb{RH}_\infty \right\}. \tag{8}$$

It is well known [23] that a nonminimum phase transfer function can be factorized as a minimum phase part and an all pass factor. Then one has

$$N = L_z N_m, \quad M = B_p M_m, \tag{9}$$

where L_z and B_p are the all pass factors; N_m and M_m are the minimum phase parts. L_z includes all the right half plane zeros of the plant $z_i \in \mathbb{C}_+$, $i = 1, \cdots, n$, and B_p includes all the right half plane poles of the plant $p_j \in \mathbb{C}_+$, $j = 1, \cdots, m$. We consider a coprime factorization of L_z and B_p as

$$L_z(s) = \prod_{i=1}^{n} \frac{s - z_i}{s + \bar{z}_i}, \quad B_p(s) = \prod_{j=1}^{m} \frac{s - p_j}{s + \bar{p}_j}. \tag{10}$$

3. Main Results.

3.1. Optimal performance of one-parameter with SNR.

Consider the system setup shown in Figure 1. According to (3) and (5), we can obtain J

$$J = \|S(s)\|_2^2 \sigma^2 + \frac{\|T(s)\|_2^2 \|T(s)\|_2^2 \sigma^2}{\gamma - \|T(s)\|_2^2}. \tag{11}$$

From (4) and (11), we can rewrite J^*

$$J^* \geq \inf_{Q \in \mathbb{RH}_\infty} \|S(s)\|_2^2 \sigma^2 + \inf_{Q \in \mathbb{RH}_\infty} \frac{\|T(s)\|_2^2 \|T(s)\|_2^2 \sigma^2}{\gamma - \|T(s)\|_2^2}. \tag{12}$$

It is clear that in order to obtain J^*, Q must be appropriately selected.

Lemma 3.1. *(Bound on γ)* [2] *Consider the SISO linear time-invariant networked control systems, for the feedback system to be stabilisable, the γ must satisfy*

$$\gamma > \gamma_{\text{inf}} = \sum_{i,j \in \Omega} \frac{4\text{Re}(p_j)\text{Re}(p_i)}{\bar{b}_j b_i (\bar{p}_j + p_i)} (L_z^{-1}(p_i))^H L_z^{-1}(p_j), \quad b_j = \prod_{\substack{i \in \Omega \\ i \neq j}} \frac{p_i - p_j}{p_j + \bar{p}_i},$$

where γ_{inf} is the largest lower bound of SNR on the communication channel which allows the networked control systems in this paper to be stable.

Theorem 3.1. *If $G(s)$ is factorized as in (6) and (9), and the communication channel has a given (admissible) SNR γ, then*

$$J^* > J_1^* \sigma^2 + \frac{J_2^{*2} \sigma^2}{\gamma - J_2^*}, \tag{13}$$

where

$$J_1^* = \sum_{i=1}^{n} 2\,\text{Re}(z_i) + \sum_{i,j \in \Omega} \frac{4\,\text{Re}(p_j)\,\text{Re}(p_i)}{\bar{b}_j b_i (\bar{p}_j + p_i)} (1 - L_z^{-1}(p_i))^H (1 - L_z^{-1}(p_j)),$$

$$J_2^* = \sum_{i,j \in \Omega} \frac{4\text{Re}(p_j)\text{Re}(p_i)}{\bar{b}_j b_i (\bar{p}_j + p_i)} (L_z^{-1}(p_i))^H L_z^{-1}(p_j), \quad b_j = \prod_{\substack{i \in \Omega \\ i \neq j}} \frac{p_i - p_j}{p_j + \bar{p}_i}.$$

Ω is an index set defined by $\Omega := \{i : M(p_i) = 0\}$.

Proof: From (12), we denote

$$J_1^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|S(s)\|_2^2, \quad J_2^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|T(s)\|_2^2. \tag{14}$$

From (2), (6), (7), (8) and (14), we can obtain

$$J_2^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|(Y - MQ)N\|_2^2.$$

Because L_z and B_p are the all pass factors, it follows that

$$J_2^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\|B_p^{-1} Y N_m - M_m Q N_m\right\|_2^2.$$

Based on a partial fraction procedure, we may write

$$B_p^{-1}(s) N_m(s) Y(s) = \sum_{j\in\Omega} \frac{s+\bar{p}_j}{s-p_j} \frac{N_m(p_j)Y(p_j)}{b_j} + R_1,$$

where $R_1 \in \mathbb{R}\mathcal{H}_\infty$, $b_j = \prod\limits_{\substack{i\in\Omega \\ i\neq j}} \frac{p_i-p_j}{p_j+\bar{p}_i}$.

Then,

$$J_2^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \sum_{j\in\Omega} \left(\frac{s+\bar{p}_j}{s-p_j} - 1\right) \frac{N_m(p_j)Y(p_j)}{b_j} + R_2 - M_m Q N_m \right\|_2^2,$$

where $R_2 \in \mathbb{R}\mathcal{H}_\infty$, $R_2 = R_1 + \sum\limits_{j\in\Omega} \frac{N_m(p_j)Y(p_j)}{b_j}$.

It is noted that $\sum\limits_{j\in\Omega} \left(\frac{s+\bar{p}_j}{s-p_j} - 1\right) \frac{N_m(p_j)Y(p_j)}{b_j}$ is in $\mathcal{H}_2^\perp$, while $(R_2 - M_m Q N_m)$ is in $\mathcal{H}_2$.

Hence,

$$J_2^* = \left\| \sum_{j\in\Omega} \left(\frac{s+\bar{p}_j}{s-p_j} - 1\right) \frac{N_m(p_j)Y(p_j)}{b_j} \right\|_2^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|R_2 - M_m Q N_m\|_2^2.$$

Because N_m and M_m are an outer function and minimum phase, it follows that

$$\inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|(R_2 - M_m Q N_m\|_2^2 = 0.$$

In particular, a straightforward calculation can be obtained

$$\frac{s+\bar{p}_j}{s-p_j} - 1 = \frac{2\operatorname{Re} p_j}{s-p_j}.$$

Therefore, we can rewrite

$$J_2^* = \left\| \sum_{j\in\Omega} \frac{2\operatorname{Re} p_j}{s-p_j} \frac{N_m(p_j)Y(p_j)}{b_j} \right\|_2^2 = \sum_{i,j\in\Omega} \frac{4\operatorname{Re}(p_j)\operatorname{Re}(p_i)}{\bar{b}_j b_i(\bar{p}_j + p_i)} (N_m(p_i)Y(p_j))^H N_m(p_i)Y(p_j).$$

Simultaneously, according to (7) and $M(p_j) = 0$, we can obtain $Y(p_j) = -N_m^{-1}(p_j)L_z^{-1}$ (p_j).

Then,

$$N_m(p_i)Y(p_j) = -L_z^{-1}(p_j).$$

From (2), (6), (7), (8) and (14), we can obtain

$$J_1^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|(X - NQ)M\|_2^2.$$

According to the same proof of J_2^*, we can obtain

$$J_1^* = \sum_{i=1}^{n} 2\operatorname{Re}(z_i) + \sum_{i,j\in\Omega} \frac{4\operatorname{Re}(p_j)\operatorname{Re}(p_i)}{\bar{b}_j b_i(\bar{p}_j + p_i)} (1 - L_z^{-1}(p_i))^H (1 - L_z^{-1}(p_j)),$$

which completes the proof.

Remark 3.1. *Theorem 3.1 considers an additive noise channel model for the link between the sensor and the plant, and obtain explicit expressions for the optimal tracking performance in terms of the SNR constraints of a communication channel and characteristics of a given plant with one-parameter compensator. From the expression in Theorem 3.1, the optimal tracking performance consists of two parts, one depends on the nonminimum phase zeros, the unstable poles of the given plant, the reference input signal, and the other depends on the unstable poles of the given plant, as well as the SNR of a communication channel. If the communication channel does not exist, the optimal tracking performance is the similar to* [3].

The following corollary can be obtained by Theorem 3.1 directly.

Corollary 3.1. *(1) In Theorem 3.1, if the $\gamma \to \gamma_{\inf}$, then $J^* \to \infty$;*
(2) In Theorem 3.1, if the $\gamma \to \infty$, then $J^* \to J_1^*\sigma^2$;
where

$$J_1^* = \sum_{i=1}^{n} 2\operatorname{Re}(z_i) + \sum_{i,j\in\Omega} \frac{4\operatorname{Re}(p_j)\operatorname{Re}(p_i)}{\bar{b}_j b_i(\bar{p}_j + p_i)} (1 - L_z^{-1}(p_i))^H (1 - L_z^{-1}(p_j)), \quad b_j = \prod_{\substack{i\in\Omega \\ i\neq j}} \frac{p_i - p_j}{p_j + \bar{p}_i}.$$

Corollary 3.1 shows that the optimal tracking performance tends to infinity when the γ of a communication channel tends to $\gamma_{\inf}$, and the optimal tracking performance tends to $J_1^*\sigma^2$ when the γ of a communication channel tends to ∞.

3.2. **Optimal performance of two-parameter with SNR.** Consider the feedback configuration of SISO linear time-invariant systems depicted in Figure 2, which obtain the optimal tracking performance by two-parameter or two-degree-of-freedom compensator with SNR constraints of communication channel. In this setup, all the variables are the same with Section 2. The set of all stabilizing two-parameter compensators is [15]

$$\begin{aligned} \mathcal{K} := \big\{ K : K = \left[\begin{array}{cc} K_1 & K_2 \end{array}\right] = (X - RN)^{-1} \\ \cdot \left[\begin{array}{cc} Q & Y - RM \end{array}\right], \ Q \in \mathbb{R}\mathcal{H}_\infty, \ R \in \mathbb{R}\mathcal{H}_\infty \big\}. \end{aligned} \tag{15}$$

The tracking error of the control system is defined as

$$\tilde{e} = \tilde{r} - \tilde{y} = S_1(s)\tilde{r} + S_2(s)\tilde{n},$$

where

$$S_1(s) \triangleq \frac{1 - GK_2 - GK_1}{(1 - GK_2)}, \quad S_2(s) \triangleq \frac{GK_2}{1 - GK_2}. \tag{16}$$

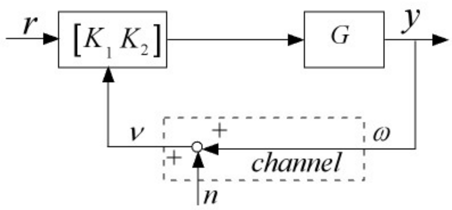

FIGURE 2. The two-parameter feedback system with communication channel

The tracking performance of the NCS system is defined as the variance of e

$$J = \sigma_e^2 = \|S_1(s)\|_2^2 \sigma^2 + \sigma_n^2 \|S_2(s)\|_2^2. \quad (17)$$

From Figure 2, we can obtain

$$\sigma_\omega^2 = \|S_o(s)\|_2^2 \sigma^2 + \sigma_n^2 \|S_2(s)\|_2^2,$$

where

$$S_o(s) \triangleq \frac{GK_1}{1 - GK_2}. \quad (18)$$

From (1), we can obtain

$$\gamma = \frac{\sigma_\omega^2}{\sigma_n^2} = \frac{\sigma^2}{\sigma_n^2} \|S_o(s)\|_2^2 + \|S_2(s)\|_2^2. \quad (19)$$

From (17) and (19), we can obtain J

$$J = \|S_1(s)\|_2^2 \sigma^2 + \frac{\|S_2(s)\|_2^2 \|S_o(s)\|_2^2 \sigma^2}{\gamma - \|S_2(s)\|_2^2}. \quad (20)$$

From (4), (6), (7), (15), (16), (18) and (20), we can rewrite J^*

$$J^* = \inf_{K \in \mathcal{K}} \left\{ \|(1 - NQ)\|_2^2 \sigma^2 + \frac{\|NQ\|_2^2 \|N(Y - MR)\|_2^2 \sigma^2}{\gamma - \|N(Y - MR)\|_2^2} \right\}. \quad (21)$$

Theorem 3.2. *If $G(s)$ is factorized as in (6) and (9), and the communication channel has a given (admissible) SNR γ, then*

$$J^* = \sigma^2 \sum_{i=1}^{n} 2\,\mathrm{Re}(z_i) + \frac{J_1^*}{\gamma} \sigma^2,$$

where

$$J_1^* = \sum_{i,j \in \Omega} \frac{4\mathrm{Re}(p_j)\mathrm{Re}(p_i)}{\bar{b}_j b_i (\bar{p}_j + p_i)} (L_z^{-1}(p_i))^H L_z^{-1}(p_j), \quad b_j = \prod_{\substack{i \in \Omega \\ i \neq j}} \frac{p_i - p_j}{p_j + \bar{p}_i}.$$

Proof: From (21), we denote

$$J_1^* = \inf_{R \in \mathbb{R}\mathcal{H}_\infty} \|N(Y - MR)\|_2^2.$$

From Theorem 3.1, we can obtain

$$J_1^* = \sum_{i,j \in \Omega} \frac{4\mathrm{Re}(p_j)\mathrm{Re}(p_i)}{\bar{b}_j b_i (\bar{p}_j + p_i)} (L_z^{-1}(p_i))^H L_z^{-1}(p_j), \quad b_j = \prod_{\substack{i \in \Omega \\ i \neq j}} \frac{p_i - p_j}{p_j + \bar{p}_i}.$$

Then,

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left[\|(1 - NQ)\|_2^2 \sigma^2 + \|NQ\|_2^2 \sigma^2 \frac{J_1^*}{\gamma - J_1^*} \right]$$

Define

$$\varepsilon = \frac{J_1^*}{\gamma - J_1^*}.$$

Therefore,

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} (1 - NQ) \\ \sqrt{\varepsilon} NQ \end{bmatrix} \right\|_2^2 \sigma^2.$$

Because L_z is the all pass factors, it follows that

$$J^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} (L_z^{-1} - N_m Q) \\ \sqrt{\varepsilon} N_m Q \end{bmatrix} \right\|_2^2 \sigma^2 = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} (L_z^{-1} - 1 + 1 - N_m Q) \\ \sqrt{\varepsilon} N_m Q \end{bmatrix} \right\|_2^2 \sigma^2.$$

Because $(L_z^{-1} - 1)$ is in $\mathcal{H}_2^\perp$, conversely, and $(1 - N_m Q)$ is in $\mathcal{H}_2$. Hence,

$$J^* = \left\|L_z^{-1} - 1\right\|_2^2 \sigma^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} (1 - N_m Q) \\ \sqrt{\varepsilon} N_m Q \end{bmatrix} \right\|_2^2 \sigma^2.$$

It is straightforward to see that

$$J^* = \left\|L_z^{-1} - 1\right\|_2^2 \sigma^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ \sqrt{\varepsilon} \end{bmatrix} N_m Q \right\|_2^2 \sigma^2.$$

From [5], we introduce an inner-outer factorization

$$\begin{bmatrix} -1 \\ \sqrt{\varepsilon} \end{bmatrix} N_m = \Delta_i \Delta_0.$$

Because N_m is a minimun phase, it is easy to see that

$$N_m \sqrt{1+\varepsilon} = \Delta_0,$$

and

$$\Delta_i = \frac{1}{\sqrt{1+\varepsilon}} \begin{bmatrix} -1 \\ \sqrt{\varepsilon} \end{bmatrix}.$$

To find the optimal Q, introduce

$$\psi \triangleq \begin{bmatrix} \Delta_i^T \\ I - \Delta_i \Delta_i^T \end{bmatrix}.$$

It follows that

$$\begin{aligned} J^* &= \left\|L_z^{-1} - 1\right\|_2^2 \sigma^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \psi \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ \sqrt{\varepsilon} \end{bmatrix} N_m Q \right) \right\|_2^2 \sigma^2 \\ &= \left\|L_z^{-1} - 1\right\|_2^2 \sigma^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \Delta_i^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Delta_0 Q \right\|_2^2 \sigma^2 + \left\| (I - \Delta_i \Delta_i^T) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2^2 \sigma^2. \end{aligned}$$

According to $Q \in \mathbb{R}\mathcal{H}_\infty$, it follows that

$$\inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \Delta_i^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Delta_0 Q \right\|_2^2 \sigma^2 = 0.$$

It then follows via the same arguments as in paper [3] that

$$\left\|L_z^{-1} - 1\right\|_2^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} 2\mathrm{Re}(z_i).$$

By an easy calculation, we can obtain

$$J^* = \left\| (I - \Delta_i \Delta_i^T) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2^2 \sigma^2$$

$$= \left\| \begin{bmatrix} \dfrac{\varepsilon}{1+\varepsilon} & \dfrac{\sqrt{\varepsilon}}{1+\varepsilon} \\ \dfrac{\sqrt{\varepsilon}}{1+\varepsilon} & \dfrac{1}{1+\varepsilon} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2^2 \sigma^2$$

$$= \frac{\varepsilon\sigma^2}{1+\varepsilon}.$$

This completes the proof.

Remark 3.2. *From the expression in Theorem 3.2, the optimal tracking performance consists of two parts, one depends on the nonminimum phase zeros of the given plant, the reference input signal, and the other depends on the unstable poles of the given plant, as well as the SNR of a communication channel. The tracking performance is improved by two-parameter compensator scheme. It is also shown that when the communication channel do not exist, the optimal tracking performance reduces to the existing normal tracking performance of the control system. The results show how the optimal tracking performance is limited by the SNR of communication channel.*

4. **Illustrative Example.** In this section, an example is given to illustrate the theoretical results.

We use the model described in the paper [13]

$$\gamma = \frac{3}{\alpha^2}(2^b - 1),$$

where b is the number of bits of the quantizes and α is overload factor. In this paper, we use $\alpha = 4$ and $b \in \begin{pmatrix} 8 & 16 \end{pmatrix}$.

Consider the unstable plant model and the reference models described by

$$G(s) = \frac{0.3(s-0.5)}{s(s-1)(s+3)}, \qquad \sigma^2 = 0.02.$$

From Theorem 3.1, the optimal tracking performance is obtained

$$J^* > 0.18 + \frac{6.48}{\gamma - 18}.$$

From Theorem 3.1, if the communication channel does not exist, the optimal tracking performance is obtained

$$J^* = 0.18.$$

From Theorem 3.2, the optimal tracking performance is obtained

$$J^* = 0.02 + \frac{0.36}{\gamma}.$$

From Theorem 3.2, if the communication channel does not exist, the optimal tracking performance is obtained

$$J^* = 0.02.$$

The optimal tracking performance about SISO networked control systems with different SNR constraints is shown in Figure 3. It can be seen from Figure 3 that the optimal tracking performance has been worsened because of the limiting the SNR of the communication channel in feedback control system. The bigger the value of SNR (the allowed channel capacity), the better the optimal tracking performance. It can also be observed from Figure 3 that the tracking performance is improved by two-parameter scheme.

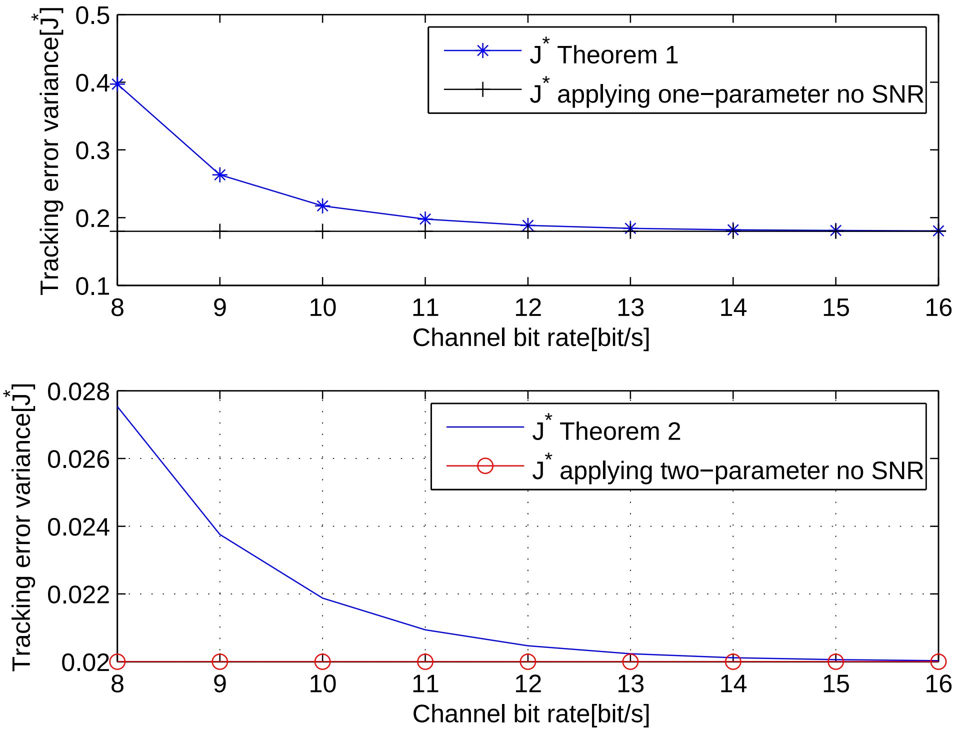

Figure 3. Optimal performance of the system

5. **Conclusions.** In this paper, the optimal tracking performance problem has been discussed for SISO networked control systems with SNR constraints. Explicit expressions of the minimal tracking error have been obtained for systems with or without SNR constraints in the feedback path. In particular, we have provided a characterization of the optimal tracking performance which is subjected to SNR constraints in two architectures of interest. The main results are obtained by H_2 criterion and spectral factorization technique. It is shown that the optimal performance is constrained by the nonminimum phase zeros, the unstable poles of a given plant, a given reference signal and the SNR of a communication channel. The results clearly demonstrate how the SNR constraints may fundamentally degrade a control system's tracking capability. An illustrative example has been given to demonstrate the proposed results.

Possible future extensions to this work include study on more general plants such as continuous-time plants with time delays, more complex control structure, and more communication channel constraints.

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