International Journal of Innovative
Computing, Information and Control
Volume 8, Number 10(B), October 2012
ICIC International ©2012 ISSN 1349-4198
pp. 7307–7319

NEW INTERMEDIATE QUATERNION BASED CONTROL OF SPACECRAFT: PART I – ALMOST GLOBAL ATTITUDE TRACKING

Yong Duan Song[1,2] and Wen Chuan Cai[2]

[1]School of Automation
Chongqing University
No. 174, Shazhengjie, Shapingba, Chongqing 400044, P. R. China
{ ydsong; wchcai }@bjtu.edu.cn

[2]School of Electronic and Information Engineering
Beijing Jiaotong University
No. 3, Shangyuancun, Haidian, Beijing 100044, P. R. China

Received September 2011; revised January 2012

Abstract. *Euler-angle based and conventional quaternion based methods have been extensively employed for spacecraft attitude control. However, the first method suffers from singularity that prohibits large orientation maneuvers, while the second exhibits ambiguity and unwinding phenomena. This work attempts to circumvent these undesirable features in spacecraft attitude control by introducing a simple notion of an intermediate 4-element variable (quaternion) that possesses several beneficial features, among which is the fact that it can be determined uniquely and readily for any attitude spacecraft orientation. This attribute allows for the development of a new attitude control free of ambiguity and unwinding associated with the conventional quaternion based method. Furthermore, since the rate of the proposed intermediate variable is bounded with bounded angular velocity, singularity inherent in the Euler-angle based method is avoided here. These attractive features are theoretically authenticated and numerically verified.*

Keywords: Intermediate quaternion, Attitude tracking, Unwinding free

1. **Introduction.** Attitude control of spacecraft has long been known as an important problem and has been the subject of many publications during the past decades. The abundance of attitude control results fall broadly into two categories: direct attitude-based method directly built upon physical attitude parameters (such as Euler angles) [1,2] and indirect attitude-based method constructed with the aid of an intermediate mechanism (such as quaternion [3-9], Rodrigues parameters and modified Rodrigues parameters [10,11].)

The method directly based on Euler angles is preferable from control design and implementation point of view since it permits the direct use of the physically measureable orientation to generate control signals without the need for "physical-to-virtual" attitude conversion. However, Euler-angle based attitude control suffers from the well-known singularity problem that arises when it comes to generating the actual attitude control action – one needs to analytically determine the rates of Euler angles (used in control) from angular velocity, and such determination involves a matrix that is not invertible for some particular attitudes [1,2].

The indirect method on the other hand is to go through an intermediate mechanism (such as quaternion, Rodrigues parameters, or modified Rodrigues parameters) to extract the physical orientation information and invoke such information indirectly to build attitude control, where the physically measureable attitude information (i.e., direction cosine matrix (DCM)) is utilized indirectly in constructing the attitude control. Among various indirect methods, Hamilton quaternion is the most popular one because of its singularity-free feature, and has been widely used for building attitude control schemes for spacecraft during the past decades [3-9].

However, it should be noted that quaternion or Rodrigues variables bear no physical meaning; thus there exists no physical sensor for direct measurement of them or any other representation of the body attitude – the quaternion/Rodrigues variables are always obtained indirectly from accompanying filters/observers that are driven by measurements from rate-gyros, sun sensors, star sensors, earth sensors, magnetometers, and a host of other sensor candidates [8,9]. As such, the implementation of most existing indirect method requires a rather time-consuming process to convert the physical measurement of orientation into proper quaternion variables.

Furthermore, most existing attitude control methods (either Hamilton quaternion-based or Euler-angle based) exhibit the unwinding phenomenon as first noticed by [12], also by [13]. This phenomenon is due to the fact that the parameters/variables used for attitude control design are not unique for a given physical attitude position (Hamilton quaternion based method) or not continuous globally (Euler-angle based method). More specifically, in the Hamilton quaternion method, there always exist two different unit quaternion vectors with opposite signs for any given orientation, while in the Euler angles method or the (modified) Rodrigues parameters method, the parameters used for control design are not continuous at some particular attitudes involving singularity. Consequently, the vehicle initially staying fairly close to the desired (destination) attitude might rotate through a larger angle (longer path) before resting at the desired attitude (i.e., unwinding). Such phenomenon is highly undesirable in spacecraft applications from the point of view of fuel consumption and vibration suppression.

It is therefore of theoretical and practical importance to develop an alternative approach for attitude control that does not involve singularity or unwinding phenomenon, and yet avoiding troublesome "physical-to-virtual" attitude conversions in control design and implementation. To our best knowledge, however, the only noticeable work that dealt with part of the issues is from [13,14], where results on almost global attitude stabilization are established without unwinding phenomenon by using rotation matrix (instead of Hamilton quaternion). The works of [15] and [16] also attempt to eliminate ambiguity in quaternion conversion.

In this work, a new intermediate 4-element variable (or quaternion for brevity) is introduced, based on which new attitude control algorithms are derived to achieve almost globally stable attitude tracking (Part I) and globally stable attitude tracking (Part II). Because of the beneficial features associated with the introduced intermediate quaternion, ambiguity and singularity are no longer present with the proposed attitude control scheme. It is shown that when integrated with the DCM for attitude representation, the proposed intermediate 4-elememt variable leads to well-defined "virtual" angular rate, allowing for arbitrary 3D rotation maneuvering without singularity. Furthermore, since the intermediate variable is uniquely determined given any physical attitude, yet continuous at any possible attitude, ambiguity and unwinding phenomena are avoided (refer to section 3 for more detail). In addition, because the intermediate variable can be trivially computed from physically measureable attitude (through the DCM) without the need for special treatment as needed in traditional quaternion based method, the design and implementation procedure for the proposed intermediate quaternion based attitude control is made simpler and more straightforward as compared with traditional one.

The remainder of the paper is organized as follows. In Section 2, the ambiguity and unwinding phenomena associated with traditional quaternion are examined, which motivates the introduction of an alternative intermediate quaternion as detailed in Section 3. Attitude control built upon the proposed intermediate quaternion and the stability analysis are presented in Section 4. The performance of the new attitude control is demonstrated and validated in Section 5, and conclusions are drawn in Section 6.

2. Motivation for New Intermediate Quaternion for Attitude Control Design.

To motivate the new quaternion for attitude control design that avoids the drawbacks associated with traditional quaternion based method, we first examine in this section how DCM is used in constructing attitude control scheme, through which we reveal the undesirable features inherent with the commonly used quaternion.

According to Euler's eigenaxis rotation theorem, the attitude of rigid-body can be determined by a rotation angle θ about an eigenaxis $\mathbf{e}$. Based on such angle θ and axis $\mathbf{e} = \{e_1, e_2, e_3\}^T$, the well-known Hamilton rotation quaternion is defined as

$$q = (q_0, \mathbf{q}) = \left(\cos\frac{\theta}{2}, \mathbf{e}\sin\frac{\theta}{2}\right) \tag{1}$$

where q_0 and $\mathbf{q} = [q_1, q_2, q_3]^T$ denote the scalar part and vector part of the quaternion q, respectively, and $q_0^2 + \mathbf{q}^T\mathbf{q} = 1$. Also, given θ and $\mathbf{e}$, the DCM is calculated as [17,18]

$$\mathbf{R}(\mathbf{e}, \theta) = \cos\theta\mathbf{I} + (1 - \cos\theta)\mathbf{e}\mathbf{e}^T - \sin\theta\mathbf{e}^\times \tag{2}$$

From (1) and (2), the relation between the DCM and the quaternion is established as follows

$$\mathbf{R}(\pm q) = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 1 - 2(q_1^2 + q_3^2) & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \tag{3}$$

and the kinematic differential equation of the quaternion is given as

$$\dot{q} = (\dot{q}_0, \dot{\mathbf{q}}) = \frac{1}{2}\left(-\boldsymbol{\omega}^T\mathbf{q}, q_0\boldsymbol{\omega} + \mathbf{q}^\times\boldsymbol{\omega}\right) \tag{4}$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ denotes the body angular velocity, "$\mathbf{q}^\times$" represents a skew symmetric matrix generated from the vector $\mathbf{q}$.

The 4-element vector defined in (1) is referred to as half rotation quaternion hereafter if no confusion is likely to occur. For aerospace vehicles navigated by inertial navigation equipment, the body rate $\boldsymbol{\omega}$ is measured by rate gyros; thus the virtual orientation of the vehicle in terms of quaternion (q) can be determined by integrating numerically the kinematic differential Equation (4). Because of its singularity-free feature, the quaternion so defined has been widely used to address the attitude control problem of spacecraft, see [3-9] (to just name a few).

However, implementation of any attitude control scheme based on traditional quaternion requires the determination of quaternion variables from physical measurement of the actual attitude/orientation of the vehicle (i.e., "physical-to-virtual" attitude conversion), and such determination process involves several undesirable issues as explained in what follows.

1) Ambiguity in deriving quaternions from physical attitude parameters. From the definition (1), it is apparent that $q(\mathbf{e}, \theta + 360°) = -q(\mathbf{e}, \theta)$ for any $(\mathbf{e}, \theta)$. However, their corresponding DCMs $\mathbf{R}(\mathbf{e}, \theta + 360°)$ and $\mathbf{R}(\mathbf{e}, \theta)$ physically represent the same attitude. In other words, with the quaternion as defined in (1), any specific attitude always has two quaternion vectors with opposite signs, such ambiguity in deriving q from a given attitude brings about additional complexity for control design and implementation because "sign selection" must be consistently made during the entire control process.

2) Extra caution required for quaternion determination. The quaternion so defined is a vector that virtually reflects the orientation of a vehicle and cannot be directly measured using any feasible sensors [9]. Therefore, implementing quaternion based attitude control scheme needs to literally determine the quaternion variables from the physical measurement of the actual attitude/orientation of the vehicle. This is usually done through the DCM (which is obtained from attitude information measured by various sensors). Note that given a DCM $\mathbf{R}$, together with the unit quaternion satisfying $q_0^2 + \mathbf{q}^T\mathbf{q} = 1$, there are ten quadratic equations to deal with in determining the four elements of q, which, unfortunately, do not lead to a direct and unique formula for computing the four elements of the quaternion q for a given rotation matrix $\mathbf{R}$. In practice, several sets of formulae valid for different rotation matrices have to be employed interchangeably during system operation in order to avoid the issues of zero division or square root of a negative number and to ensure computation accuracy [15]. Furthermore, each element of the quaternion involves "sign uncertainty" and the quaternion vector cannot be uniquely and independently determined from a given DCM. As such, determining the quaternion from a DCM is quite involved since it not only requires a series of logic selections based on the diagonal elements of the DCM, but also demands computing square root functions; all are undesirable for design and implementation.

3) Discontinuous quaternion conversion. To eliminate ambiguity in quaternion conversion, several approaches have been suggested, and one of the commonly used methods is to define q_0 on $\mathbb{R}^+$ or $\mathbb{R}^-$ only [15,16]. However, this treatment makes the quaternion discontinuous at the attitude $\mathbf{R}(\mathbf{e}, 180°)$, because from (1) with $q_0 \in \mathbb{R}^+$, the attitude quaternion vectors around $\mathbf{R}(\mathbf{e}, 180°)$ should be $q(\mathbf{e}, 180° - \delta/2)$ at one side of the particular attitude and $-q(\mathbf{e}, 180° + \delta/2)$ at the other side of that attitude for an arbitrarily small $\delta > 0$, and $q(\mathbf{e}, 180° - \delta/2) \neq -q(\mathbf{e}, 180° + \delta/2)$ as $\delta \to 0$, which leads to discontinuous control laws that are unfavorable for implementation since most attitude control actuators used practically are reaction wheels or magnetic torquers [19].

4) Unwinding phenomenon. Since q and $-q$ differ by 360°, a vehicle must rotate 360° to change its attitude coordinates from q to $-q$ (or from $-q$ to q). Thus, if a vehicle is stably settled on q, then due to improper choice of quaternion for representing this particular attitude (e.g., $-q$ is chosen), any arbitrarily small perturbation may cause the vehicle to rotate 360° about some axis to eventually converge to q. This is known as "unwinding phenomenon" as noticed by [12,13]. It is worth noting that discontinuity of quaternion conversion could also lead to unwinding phenomenon. For instance, with the constraint $q_0 \in \mathbb{R}^+$, the vector part of quaternion $\mathbf{q}$ is discontinuous at $\mathbf{R}(\mathbf{e}, 180°)$, then the vehicle has to rotate a very large angle $360° - \delta$ (rather than a small angle δ) from one side of the point $\mathbf{R}(\mathbf{e}, 180°)$ to another side, because it cannot pass through the discontinuous point $\mathbf{R}(\mathbf{e}, 180°)$ directly, even though the initial and final attitude positions are fairly close. Similarly, unwinding phenomenon also exists in Euler angles method and (modified) Rodrigues parameters method due to singularity.

To summarize, Euler-angle based attitude control method involves singularity and unwinding phenomena near the singular point, whereas the Hamilton quaternion based attitude tracking control exhibits unwinding phenomenon in that the convergence of the attitude tracking error might take two possible paths: a longer one and a shorter one. There is no feasible and systematic solution to ensure a shorter path tracking. Furthermore, since the unstable equilibrium and stable equilibrium are the same attitude in the quaternion method, any disturbance would cause 360° rotation due to improper selection of quaternion even though the desired attitude is already achieved.

The interesting issue to be addressed then is that if it is possible to introduce a new intermediate quaternion through which the real orientation information is used for control design and implementation in such a way that not only the singularity is avoided, but also the undesirable features associated with the currently widely used quaternion are removed. A solution is attempted in this work, which is detailed in the rest of the paper.

3. **New Intermediate Quaternion and Its Salient Features.** In this section, a new intermediate 4-elemet variable (or quaternion) is introduced. We show that such quaternion exhibits the salient features that, when invoked for attitude control design, not only prevent the singularity in Euler-angle based method, but also avoid the ambiguity and unwinding phenomena in traditional quaternion based method.

To describe this method, let $\mathbf{R}_b$, $\mathbf{R}_d$ and $\mathbf{R}_e$ denote the actual, desired and relative attitude matrices of a vehicle respectively, all defined in the matrix Lie group of rigid body rotation matrices $SO(3)$. The relative attitude matrix is obtained by [17,18]

$$\mathbf{R}_e = \mathbf{R}_b \mathbf{R}_d^T \tag{5}$$

Thus $\mathbf{R}_e = \mathbf{I}$ implies that $\mathbf{R}_b = \mathbf{R}_d$, i.e., the actual attitude is identical to the desired attitude. Obviously, the matrix $\mathbf{R}_e$ physically represents the attitude tracking error in terms of the DCM (or rotation matrix). According Euler's eigenaxis rotation theorem, such matrix can be parameterized by a unit vector $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T$ and an angle ϑ as follows

$$\mathbf{R}_e(\boldsymbol{\sigma}, \vartheta) = \cos\vartheta \mathbf{I} + (1 - \cos\vartheta)\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \sin\vartheta \boldsymbol{\sigma}^\times \tag{6}$$

Now we introduce the following new intermediate 4-element variable p using $\boldsymbol{\sigma}$ and ϑ as

$$p = (p_0, \mathbf{p}) = (\cos\vartheta, \boldsymbol{\sigma}\sin\vartheta) \tag{7}$$

where $\mathbf{p} = [p_1, p_2, p_3]^T$. The vector p given in (7) is also called "full" angle based intermediate quaternion since it is defined upon the full angle of ϑ, instead of the "half" angle as with the Hamilton quaternion q defined in (1).

The motivation behind using full rotation angle to define the new intermediate quaternion stems from the observation that the traditional quaternion based on half the rotation angle, as in (1), leads to the relationship between the DCM and the quaternion as in (3), which contains both square and product operations of the quaternion elements. As a result, not only square root operations are required in determining the quaternion variables from the DCM, ambiguity and unwinding phenomena are also involved, as discussed in previous section. However, it is interesting to note that if the full rotation angle is used directly to define the intermediate quaternion as in (7), a simpler and more straightforward relationship between the DCM and the intermediate quaternion can be established, as seen shortly. Moreover, the intermediate quaternion so defined naturally removes singularity, ambiguity and unwinding phenomena. Lemma 3.1 below shows how easily the new intermediate quaternion can be determined from the physical orientation information conveyed in the DCM.

Lemma 3.1. *If $\mathbf{R}_e$ denotes the relative DCM with R_{eij} being its (i,j)th element, then the new intermediate quaternion p as defined by (7) can be simply and uniquely obtained as*

$$p_0 = \frac{1}{2}(trace(\mathbf{R}_e) - 1) \tag{8}$$

$$\mathbf{p} = \frac{1}{2}[\mathrm{R}_{e23} - \mathrm{R}_{e32}, \mathrm{R}_{e31} - \mathrm{R}_{e13}, \mathrm{R}_{e12} - \mathrm{R}_{e21}]^T \tag{9}$$

Proof: (8) and (9) can be established straightforwardly from (6) and (7).

Remark 3.1. *In contrast with traditional quaternion that demands extra caution in determining its elements from DCM, the new intermediate quaternion $p = (p_0, \mathbf{p})$ as defined in (7) can be computed easily and uniquely as long as the DCM is given. Furthermore,* *the ambiguity in determining the 4-element quaternion, the likelihood of zero division, the uncertainty in positive/negative sign selection, or the square root operations (all encountered in traditional quaternion) are not involved here when computing the new intermediate quaternion from the DCM. Also note that the new intermediate quaternion* p *as determined by (8) and (9) are globally continuous since the DCM* $\mathbf{R}_e$ *is continuous at any attitude position. Thus, attitude control based on such new intermediate quaternion naturally exhibits advantageous features, as addressed in next section.*

Lemma 3.2. *Let* $\boldsymbol{\omega}_b$ *and* $\boldsymbol{\omega}_d$ *represent the actual and the desired angular velocity of the vehicle respectively, and define the relative angular velocity as*

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_b - \mathbf{R}_e \boldsymbol{\omega}_d \tag{10}$$

Then the rate of the intermediate quaternion $\dot{p}$ *can be obtained by the following kinematic differential equations*

$$\dot{p} = (\dot{p}_0, \dot{\mathbf{p}}) = \left(-\mathbf{p}^T \boldsymbol{\omega}_e, \left[p_0 \mathbf{I} + \frac{1}{2}(\mathbf{I} - \mathbf{R}_e) \right] \boldsymbol{\omega}_e \right) \tag{11}$$

Proof: As the relative angular velocity $\boldsymbol{\omega}_e$ is defined by (10), the attitude error rotation matrix $\mathbf{R}_e$ satisfies the following well-known Poisson differential equation [18]

$$\dot{\mathbf{R}}_e(t) + \boldsymbol{\omega}_e^{\times}(t)\mathbf{R}_e(t) = 0 \tag{12}$$

from which it is not difficult to obtain (11) by making use of (8) and (9).

Remark 3.2. *As clearly indicated in (11), a bounded* $\boldsymbol{\omega}_e$ *leads to a bounded* $\dot{p}$ *since* $\mathbf{p}$ *and* $\mathbf{R}_e$ *are always bounded. Therefore, no singularity is involved in determining the rates of the new intermediate quaternion (*$\dot{p}_0$ *and* $\dot{\mathbf{p}}$*) from the relative angular velocity* $\boldsymbol{\omega}_e$*.*

Remark 3.3. *In light of (5) and (10), it is apparent that* $\mathbf{R}_b \to \mathbf{R}_d$ *and* $\boldsymbol{\omega}_b \to \boldsymbol{\omega}_d$ *are achieved if* $\mathbf{R}_e \to \mathbf{I}$ *and* $\boldsymbol{\omega}_e \to \mathbf{0}$*. Thus, the objective of attitude tracking can be addressed by regulating the attitude tracking error* $\mathbf{R}_e$ *and angular velocity tracking error* $\boldsymbol{\omega}_e$ *so that* $\lim_{t\to\infty}(\boldsymbol{\omega}_e, \mathbf{R}_e) = (\mathbf{0}, \mathbf{I})$*. Meanwhile,* $\mathbf{R}_e = \mathbf{I}$ *is equivalent to* $p = (1, \mathbf{0})$ *according to (8) and (9); therefore, the attitude tracking is achieved as* $(\boldsymbol{\omega}_e, p) = (\mathbf{0}, (1, \mathbf{0}))$*.*

To close this section, it is worth recapping the crucial feature associated with the new intermediate quaternion from attitude tracking control point of view – the intermediate quaternion so defined, not yet reported in the literature, allows for the virtual attitude error (p) and angular velocity error ($\dot{p}$) to be determined from the physically obtainable $\mathbf{R}_e$ and $\boldsymbol{\omega}_e$ trivially and uniquely. This attribute is useful and sufficient in synthesizing attitude tracking control for spacecraft without involving singularity-ambiguity-unwinding phenomena, as developed in next section.

4. New Intermediate Quaternion Based Attitude Tracking Control.

Our objective now is to use the new intermediate quaternion for attitude control design. We show in detail that by utilizing the new quaternion one can develop an attitude control scheme that not only avoids singularity and unwinding phenomena, but also allows for the straightforward use of the readily computable or physically measurable signals in its design and implementation.

4.1. New intermediate quaternion based control.

For a rigid-body, its attitude dynamics can be described by [18]

$$\mathbf{J}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\mathbf{J}\boldsymbol{\omega} + \boldsymbol{\tau} \tag{13}$$

where $\mathbf{J} \in \mathbb{R}^{3\times3}$ is a symmetric and positive definite inertial matrix, i.e., $\mathbf{J} = \mathbf{J}^T > 0$, and $\boldsymbol{\tau} \in \mathbb{R}^3$ denotes the driving torque to be specified for attitude tracking. By virtue of the definition of the angular velocity tracking error as given in (10), we can re-express (13) as

$$\mathbf{J}\dot{\boldsymbol{\omega}}_e = \boldsymbol{\tau} - \mathbf{F}(.) - \mathbf{H}(.) \tag{14}$$

where $\mathbf{F}(.)$ and $\mathbf{H}(.)$ are two nonlinear terms, defined as

$$\mathbf{F}(.) = (\mathbf{R}_e\boldsymbol{\omega}_d)^\times \mathbf{J}(\mathbf{R}_e\boldsymbol{\omega}_d) + \mathbf{J}(\mathbf{R}_e\dot{\boldsymbol{\omega}}_d) \tag{15}$$

$$\mathbf{H}(.) = \boldsymbol{\omega}_e^\times \mathbf{J}(\boldsymbol{\omega}_e + \mathbf{R}_e\boldsymbol{\omega}_d) + [(\mathbf{R}_e\boldsymbol{\omega}_d)^\times \mathbf{J} + \mathbf{J}(\mathbf{R}_e\boldsymbol{\omega}_d)^\times]\boldsymbol{\omega}_e \tag{16}$$

By making use of $\boldsymbol{\omega}_e$ (computed through measured $\boldsymbol{\omega}_b$ and $\boldsymbol{\omega}_d$ via (10)) and the vector part of the intermediate quaternion $\mathbf{p}$ (computed simply from the attitude error DCM $\mathbf{R}_e$ with the formulae given in (8) and (9)), the following non-traditional quaternion based attitude control scheme can be constructed.

Theorem 4.1. *Consider the error dynamic system (14). If the control scheme is built upon the new intermediate quaternion p as (only the vector part of p needed)*

$$\boldsymbol{\tau} = -k_v\boldsymbol{\omega}_e - k_p\mathbf{p} + \mathbf{F}(.) \tag{17}$$

where $k_v > 0$ and $k_p > 0$ are control parameters chosen by the designer, $\mathbf{F}(.)$ is given as in (15), then asymptotically stable attitude tracking in the sense that $\lim_{t\to\infty}(\boldsymbol{\omega}_e, \mathbf{R}_e) = (\mathbf{0}, \mathbf{I})$ is almost globally achieved.

Proof: Define a Lyapunov function candidate as

$$V(t) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + k_p(1 - p_0) \geq 0 \tag{18}$$

where $V(t)$ obviously is a positive definite function for any $\boldsymbol{\omega}_e$ and p_0 since $|p_0| \leq 1$ always holds from (7). By taking time derivative of (18) and using (11), (14) and (17), it follows that

$$\begin{aligned} \dot{V}(t) &= \boldsymbol{\omega}_e^T \mathbf{J}\dot{\boldsymbol{\omega}}_e - k_p\dot{p}_0 = \boldsymbol{\omega}_e^T[\boldsymbol{\tau} - \mathbf{F}(.) - \mathbf{H}(.)] - k_p\boldsymbol{\omega}_e^T\mathbf{p} \\ &= -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e - \boldsymbol{\omega}_e^T\mathbf{H}(.) \end{aligned} \tag{19}$$

Note that the matrix $(\mathbf{R}_e\boldsymbol{\omega}_d)^\times\mathbf{J} + \mathbf{J}(\mathbf{R}_e\boldsymbol{\omega}_d)^\times$ is skew symmetric due to $\mathbf{J} = \mathbf{J}^T$, then it can be shown that $\boldsymbol{\omega}_e^T\mathbf{H}(.) \equiv \mathbf{0}$ for the term $\mathbf{H}(.)$ given in (16). Thus, we get

$$\dot{V}(t) = -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e \leq 0 \tag{20}$$

from which it is readily obtained that

$$V(\infty) \leq V(0) < \infty, \quad \int_0^\infty k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e dt \leq V(0) < \infty \tag{21}$$

which implies that $\boldsymbol{\omega}_e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and then we have $\mathbf{F}(.) \in \mathcal{L}_\infty$ and $\mathbf{H}(.) \in \mathcal{L}_\infty$ from (15) and (16) as the desired angular velocity $\boldsymbol{\omega}_d$ and its rate $\dot{\boldsymbol{\omega}}_d$ are bounded (true for any real attitude operation). Thus, the controller (17) is bounded at any time, i.e., $\boldsymbol{\tau} \in \mathcal{L}_\infty$ since $\boldsymbol{\omega}_e, \mathbf{H}, \mathbf{F} \in \mathcal{L}_\infty$ (already shown) and $||\mathbf{p}|| \leq 1$ from (7). Then from (14) we have $\dot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$, i.e., $\boldsymbol{\omega}_e$ is uniformly continuous, thus by Barbalat lemma, one can infer that $\boldsymbol{\omega}_e \to \mathbf{0}$ as $t \to \infty$, and then $\mathbf{H}(.) \to 0$ as $t \to \infty$ from (16). Taking derivative of (14) with respect to time yields

$$\mathbf{J}\ddot{\boldsymbol{\omega}}_e = \dot{\boldsymbol{\tau}} - \dot{\mathbf{F}}(.) - \dot{\mathbf{H}}(.) \tag{22}$$

In light of the fact that $\dot{\boldsymbol{\omega}}_d$ and $\ddot{\boldsymbol{\omega}}_d$ are bounded, it is not difficult to show $\dot{\mathbf{F}}(.)$ and $\dot{\mathbf{H}}(.)$ are also bounded (because $\dot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$). Thus from (17) $\dot{\boldsymbol{\tau}}$ is also bounded since $\dot{\mathbf{p}} \in \mathcal{L}_\infty$ as $\boldsymbol{\omega}_e \in \mathcal{L}_\infty$ according to (11). Then from (22) we have $\ddot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$, implying that $\dot{\boldsymbol{\omega}}_e$ is uniformly continuous, which, together with the already proven fact that $\lim_{t\to\infty}\boldsymbol{\omega}_e =$

$\mathbf{0}$, allows the Barbalat lemma to be used again to conclude that $\dot{\boldsymbol{\omega}}_e \to 0$ as $t \to \infty$. Consequently, from the closed-loop error dynamics $\mathbf{J}\dot{\boldsymbol{\omega}}_e = -k_v\boldsymbol{\omega}_e - k_p\mathbf{p} - \mathbf{H}(.)$ obtained from (14) and (17), it is straightforward to show that $\mathbf{p} \to \mathbf{0}$ as $t \to \infty$. Thus we have established that $(\boldsymbol{\omega}_e, \mathbf{p}) \to (\mathbf{0}, \mathbf{0})$ as $t \to \infty$, i.e., $(\boldsymbol{\omega}_e, p)$ converges to the set $Q = Q^s \cup Q^u$, where

$$Q^s = \{(\boldsymbol{\omega}_e, p) : \boldsymbol{\omega}_e = \mathbf{0}, p = (+1, \mathbf{0})\} \tag{23}$$

$$Q^u = \{(\boldsymbol{\omega}_e, p) : \boldsymbol{\omega}_e = \mathbf{0}, p = (-1, \mathbf{0})\} \tag{24}$$

To complete the proof, we still need to show that Q^u as defined in (24) is unstable and its corresponding stable manifold is a set of Lebesgue measure zero so that we can establish the almost global tracking result.

To this end, define a nontrivial set

$$W^u = \{(\boldsymbol{\omega}_e, p) : 0 < V(\boldsymbol{\omega}_e, p) \le l\} \tag{25}$$

where $V(\boldsymbol{\omega}_e, p)$ is given by (18) and $l = 2k_p - \epsilon$ for any small $\epsilon > 0$. It is easily verified that for any $(\boldsymbol{\omega}_e, p) \in W^u$

$$V(\boldsymbol{\omega}_e, p) < V(\mathbf{0}, (-1, \mathbf{0})) = 2k_p \tag{26}$$

from which two facts are deduced: i) $(\mathbf{0}, (-1, \mathbf{0})) \notin W^u$; and ii) any trajectory $(\boldsymbol{\omega}_e, p)$ starting from W^u diverges from the equilibrium $(\mathbf{0}, (-1, \mathbf{0}))$ because $\dot{V}(\boldsymbol{\omega}_e, p) \le 0$ for any $(\boldsymbol{\omega}_e, p) \in W^u$ from (20). Note that in defining W^u, the parameter ϵ could be arbitrarily small, W^u thus could be arbitrarily close to Q^u; this fact, together with Facts i) and ii), leads to the conclusion that the equilibria as defined in Q^u are unstable, consequently, the set W^u defined in (25) is an unstable manifold of such equilibria [20].

Let W^s denote the stable manifold of the equilibria Q^u. Then any trajectory $(\boldsymbol{\omega}_e, p)$ not originating from the set

$$\Omega^u = W^s \cup Q^u \tag{27}$$

always diverges from Q^u. Since Q^u has a nontrivial unstable manifold W^u, the dimension of Ω^u is then less than the dimension of the tangent bundle $TSO(3)$ [20]; hence Ω^u is a set of Lebesgue measure zero [21,22]. Namely, the likelihood that $(\boldsymbol{\omega}_e, p)$ initially lies in Ω^u is almost zero. Therefore, it is established that the proposed control ensures that $(\boldsymbol{\omega}_e, p) \to (\mathbf{0}, (1, \mathbf{0}))$ or equivalently $(\boldsymbol{\omega}_e, \mathbf{R}_e) \to (\mathbf{0}, \mathbf{I})$ almost globally [13,14].

Remark 4.1. *The proposed attitude control is not based on traditional Hamilton quaternion, Euler angles, or Rodrigues parameters. Instead, it is based on the new intermediate quaternion p (its vector part $\mathbf{p}$ actually). Since the vector part of the intermediate quaternion p is uniquely and trivially determined from $\mathbf{R}_e$ through (8) and (9), the troublesome procedure to consistently judge the value of the diagonal elements of $\mathbf{R}_e$ (to prevent zero division or square root of a negative number, etc.) as with the traditional quaternion is not involved. Furthermore, one does not need to deal with the issue of uncertain selection of "+" or "−" in control design and implementation.*

Remark 4.2. *Because of the aforementioned features associated with the new intermediate quaternion, it is naturally desirable to develop an attitude control with such quaternion that ensures global (rather than just almost global) tracking. This is formally addressed in the second part of the work [23].*

4.2. **Analysis on unwinding free feature.** As discussed in the previous section, traditional quaternion based attitude control cannot guarantee the shorter path tracking, or prevent 360° rotation about some axis for any small disturbance (i.e., the unwinding phenomenon), whereas the new intermediate quaternion based control scheme as given in

(17) bears the important feature that the unwinding phenomenon is avoided, as stated in the following theorem.

Theorem 4.2. *Consider the error dynamic system as described in (14). With the controller (17), the attitude/orientation error always converges to zero automatically along the shorter path, so as to prevent the unwinding phenomenon.*

Proof: "*Unwinding phenomenon*" represents the situation that the vehicle resting (i.e., zero angular velocity) at a point arbitrarily close to the desired final attitude might rotate along the longer path (rather the shorter one) to final rest at the desired attitude [12]. Thus, to verify that the proposed intermediate quaternion based attitude control is free of unwinding phenomenon, we only need to show that for the error dynamic system (14) initially resting (i.e., $\boldsymbol{\omega}_e(0) = \mathbf{0}$) at any attitude error position, the control scheme (17) is always capable of driving the orientation error to zero along a shorter path.

To show this, note that the Lyapunov function $V(t)$ as defined in (18) is equivalent to

$$V(t) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + k_p(1 - \cos\vartheta) \geq 0 \tag{28}$$

where ϑ denotes the orientation angle tracking error. As shown before, the proposed control scheme (17) leads to $\dot{V}(t) = -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e \leq 0$, so that one has

$$V(\boldsymbol{\omega}_e(0), \vartheta(0)) \geq V(\boldsymbol{\omega}_e(t), \vartheta(t)) \geq k_p(1 - \cos\vartheta(t)) \quad \forall t > 0 \tag{29}$$

Meanwhile, from (28), it is obvious that

$$V(\boldsymbol{\omega}_e(0), \vartheta(0)) = k_p(1 - \cos\vartheta(0)) \tag{30}$$

for $\boldsymbol{\omega}_e(0) = \mathbf{0}$. Thus it is obtained from (29) and (30) that $\cos\vartheta(t) \geq \cos\vartheta(0)\ \forall t > 0$, from which we get

$$\vartheta(t) \leq \vartheta(0)\ \forall t > 0 \text{ if } \vartheta(0) \in [0^\circ, 180^\circ) \tag{31}$$

$$\vartheta(t) \geq \vartheta(0)\ \forall t > 0 \text{ if } \vartheta(0) \in (180^\circ, 360^\circ] \tag{32}$$

On the other hand, we have already shown in the proof of Theorem 4.1 that control (17) ensures that $\vartheta(t) \to 0^\circ$ (or 360°) as $t \to \infty$ if $\vartheta(0) \neq 180^\circ$. Therefore, the relation (31) and (32) imply that as time goes by the tracking takes place without passing through $\vartheta(0)$.

The above analysis leads to the conclusion that unwinding phenomenon no longer exists with the proposed new intermediate quaternion based attitude control (17).

Remark 4.3. *It should be mentioned that the control scheme (17) is based on the assumption that initially the system is not located in the particular region Ω^u defined in (27). The results are therefore almost global in the spirit of [13] and [23]. Interestingly, all the unstable equilibria in our method bear the same potential energy as reflected in (28); hence, any error trajectory diverging from any one of the unstable equilibria will actually diverge from all the unstable equilibria. This property allows for the final product of global attitude tracking free of singularity, ambiguity and unwinding, as presented sequentially in [23].*

5. **Simulation Study.** To validate the effectiveness of the new intermediate quaternion based control scheme developed in the previous section, we conduct numerical simulation on a spacecraft with the inertia matrix and the initial conditions

$$\mathbf{J} = \begin{bmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 17 & 1.4 \\ 0.9 & 1.4 & 15 \end{bmatrix}, \ \mathbf{R}_b(0) = \begin{bmatrix} -0.9397 & 0.3420 & 0 \\ -0.3420 & -0.9397 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{\omega}_b(0) = \mathbf{0}$$

The desired trajectory is updated by $\dot{\mathbf{R}}_d(t) = -\boldsymbol{\omega}_d^{\times}(t)\mathbf{R}_d(t)$ with the initial condition $\mathbf{R}_d(0) = \mathrm{diag}(-1, -1, 1)$ and the desired angular velocity is given by $\boldsymbol{\omega}_d(t) = [-5, 10, -15]^T$ (deg/sec). To validate the advantageous features of the proposed control, this simulation on the intermediate quaternion based control (17) is conducted with comparison with the performance of the traditional quaternion based control of the form

$$\boldsymbol{\tau} = -k_v\boldsymbol{\omega}_e - k_p\mathbf{q}_e + \mathbf{F}(.) \tag{33}$$

where $q_e = (q_{e0}, \mathbf{q}_e)$ denotes the attitude tracking error in terms of the traditional quaternion, with particular attention to the ambiguity and unwinding issues. The control parameters for both control schemes are chosen as $k_p = k_v = 10$.

Ambiguity: When using the traditional quaternion based control (33), one has to determine/compute $q_e = (q_{e0}, \mathbf{q}_e)$ from $\mathbf{R}_b$ and $\mathbf{R}_d$ first. For instance, $\mathbf{R}_b(0)$ and $\mathbf{R}_d(0)$ lead to $q_b(0) = \pm[0.1736, 0, 0, 0.9848]^T$ and $q_d(0) = \pm[0, 0, 0, 1]^T$, respectively. As a matter of fact, with the traditional quaternion setting, there exist four possible combinations for $q_d(t)$ and $q_b(t)$ at every time instant during the system operation, which are $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$. Obviously such ambiguity is highly undesirable in design, programming and implementation. On the other hand, however, none of the abovementioned issues is involved when using the proposed quaternion based control (17). More specifically, one does not need to check the diagonal element condition of $\mathbf{R}_e = \mathbf{R}_b\mathbf{R}_d^T$ in determining the new 4-element variable $p = (p_0, \mathbf{p})$ because no "division" or "square root" operation is involved therein as seen clearly in (8) and (9). Also, no "+" "−" choice is involved because the determination of p is uniquely done with Formulae (8) and (9).

Unwinding: As mentioned earlier, we need to make the initial choice for the sign pattern. If an improper choice is made, for example, the pattern $(+, -)$ is used initially, i.e., $q_d(0) = [0, 0, 0, 1]^T$, $q_b(0) = -[0.1736, 0, 0, 0.9848]^T$ then the initial attitude tracking error is $q_e(0) = [-0.9848, 0, 0, 0.1736]^T$. With such initial condition, the simulation results under the control of the traditional quaternion based scheme (33) are presented in Figure 1 (Case 2). For comparison, the results corresponding to new intermediate quaternion based control (17) are presented in Figure 1 (Case 1). It is seen from (a), (b) and (c) that both methods ensure $(\boldsymbol{\omega}_e, q_e) \to (\mathbf{0}, (1, \mathbf{0}))$ and $(\boldsymbol{\omega}_e, p) \to (\mathbf{0}, (1, \mathbf{0}))$, i.e., asymptotic attitude tracking and velocity tracking. However, the traditional quaternion based method achieves the tracking along the longer path (bigger angle) and larger angular velocity as shown in (e) and (c) of Case 2, as a result, larger torque and more energy are involved as shown in (d) and (f) of Case 2. The reason is that $q_e(0) = [-0.9848, 0, 0, 0.1736]^T$ represents a 340° relative orientation angle (due to $\vartheta(0) = 2\arccos(-0.9848) = 340°$), thus the vehicle has to rotate a quite large angle 340° to reach the desired attitude due to the improper sign pattern selection for the traditional quaternion, although physically there exists a much shorter path between the actual and desired attitudes, which is only 20° relative orientation angle ($\vartheta(0) = 2\arccos(0.9848) = 20°$). In contrast, with the proposed intermediate quaternion based method, the tracking is always ensured to take place along the shorter path (rather than the longer one) as already analyzed previously; this is also confirmed with the simulation as shown in (e) of Case 1. Therefore, no unwinding phenomenon is involved in the new intermediate quaternion based control method. This simulation also verifies that the unwinding phenomenon is highly undesirable in spacecraft applications because it causes extra energy consumption, as shown in (f) of Case 1 and Case 2.

6. **Conclusions.**

To close this paper, it is worth making the following comments on the features of the proposed new quaternion based attitude control design:

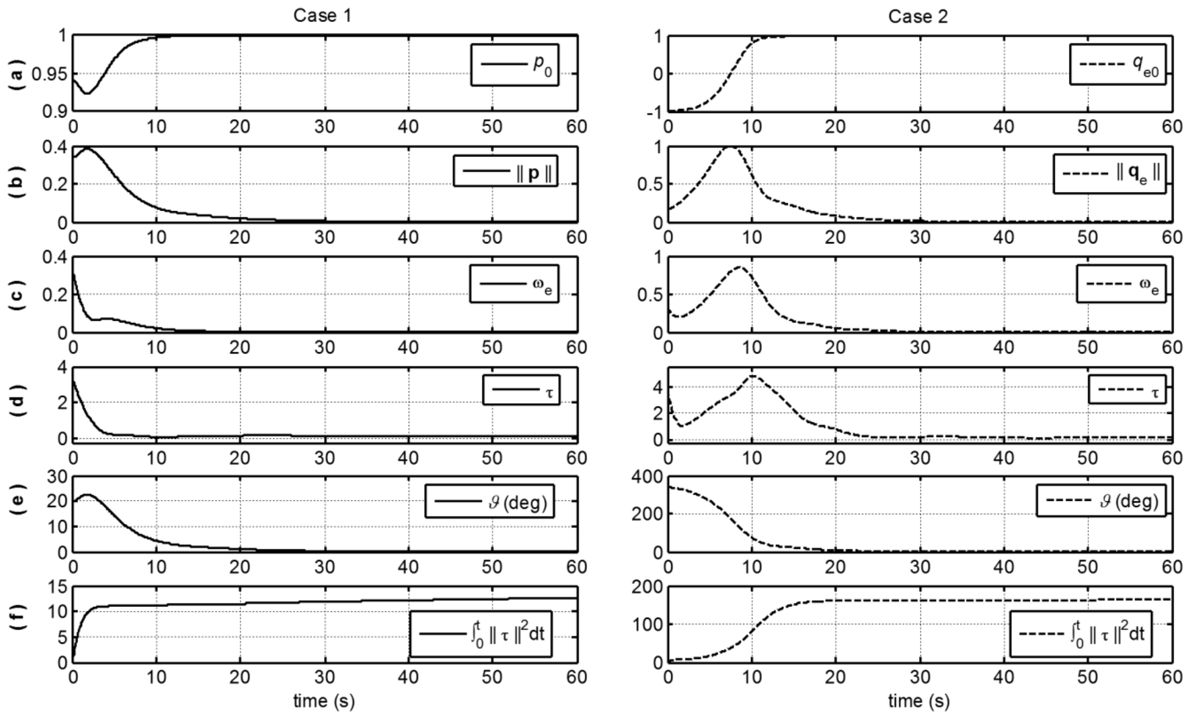

FIGURE 1. Performance comparison. The Case 1 is the results from the new intermediate quaternion based method (17) and the Case 2 is the traditional quaternion based method (33). (a) Quaternion tracking error (scalar part), (b) quaternion tracking error (norm of the vector part), (c) angular velocity tracking error, (d) control torque demanded, (e) angular position tracking error and (f) total "generalized energy" consumed.

- "Ambiguity" associated with traditional quaternion based attitude control method arises when synthesizing and generating the attitude control action – one needs to analytically determine the quaternion elements (needed in control) from physically measured attitude (through Directions Cosine Matrix), but such determination is not unique because "+/−" sign needs to be consistently selected during the entire control process. While in synthesizing the proposed attitude control, the new quaternion elements can be trivially and uniquely determined from DCM without ambiguity.
- "Singularity" inherent in Euler-angle based attitude control method arises when synthesizing/generating the attitude control action – one needs to analytically determine the rate of the change in the Euler angles "$\dot{\theta}$" (used in control) from angular velocity "$\boldsymbol{\omega}$", but the matrix that converts $\boldsymbol{\omega}$ into $\dot{\theta}$ is not invertible for some particular attitudes. Whereas using the new quaternion based attitude control, one can readily determine the rate of the change in the new quaternion from angular velocity without involving singularity.
- "Unwinding" stems from the conversion ambiguity in traditional quaternion based method and the discontinuity/singularity in Euler-angle or (modified) Rodrigues parameters based method. Whereas the conversion from physical-to-virtual attitudes involved in the proposed intermediate quaternion is continuous and unique, and is free of singularity, thus avoiding unwinding.

Although unstable equilibria exist within the proposed control setting, they all bear exactly the same potential energy so that departure of the error trajectory from any one of the unstable equilibria means departure from all the unstable equilibria, which allows for the development of global attitude tracking control for spacecraft (refer to Part II of the work for details).

Acknowledgment. This work was supported in part by the Major State Basic Research Development Program 973 (No. 2012CB215202),the National Natural Science Foundation of China (No. 60974052 and 61134001), and the Beijing Jiaotong University Research Program (No. 2010RC006 and 2012JBM009).

REFERENCES

[1] P. Crouch, Spacecraft attitude control and stabilization: Applications of geometric control theory to rigid body models, *IEEE Transactions on Automatic Control*, vol.29, no.4, pp.321-331, 1984.
[2] S. N. Singh and A. Iyer, Nonlinear decoupling sliding mode control and attitude control of spacecraft, *IEEE Transactions on Aerospace and Electronic Systems*, vol.25, no.5, pp.621-633, 1989.
[3] J. T.-Y. Wen and K. Kreutz-Delgado, The attitude control problem, *IEEE Transactions on Automatic Control*, vol.36, no.10, pp.1148-1162, 1991.
[4] S. M. Joshi, A. G. Kellar and J. T.-Y. Wen, Robust attitude stabilization of spacecraft using nonlinear quaternion feedback, *IEEE Transactions on Automatic Control*, vol.40, no.10, pp.1800-1803, 1995.
[5] B. T. Costic, D. M. Dawson, M. S. de Queiroz and V. Kapila, Quaternion-based adaptive attitude tracking controller without velocity measurements, *Journal of Guidance, Control and Dynamics*, vol.24, no.6, pp.1214-1222, 2001.
[6] R. J. Wallsgrove and M. R. Akella, Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances, *Journal of Guidance, Control and Dynamics*, vol.28, no.5, pp.957-963, 2005.
[7] W. C. Cai, X. H. Liao and Y. D. Song, Indirect robust adaptive fault-tolerant control for attitude tracking of spacecraft, *Journal of Guidance, Control and Dynamics*, vol.31, no.5, pp.1465-1463, 2008.
[8] D. Seo and M. R. Akella, Separation property for the rigid-body attitude tracking control problem, *Journal of Guidance, Control and Dynamics*, vol.30, no.6, pp.1569-1576, 2007.
[9] Y. D. Song and W. C. Cai, Quaternion observer-based model-independent tracking control of spacecraft, *Journal of Guidance, Control and Dynamics*, vol.32, no.5, pp.1476-1482, 2009.
[10] R. Sharma and A. Tewari, Optimal nonlinear tracking of spacecraft attitude maneuvers, *IEEE Transactions on Control Systems Technology*, vol.12, no.5, pp.677-682, 2004.
[11] H. Schaub, M. R. Akella and J. L. Junkins, Adaptive control of nonlinear attitude motions realizing linear closed loop dynamics, *Journal of Guidance, Control and Dynamics*, vol.24, no.1, pp.95-100, 2001.
[12] S. P. Bhat and D. S. Bernstein, A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon, *Systems and Control Letters*, vol.39, no.1, pp.63-70, 2000.
[13] A. K. Sanyal, A. Fosbury, N. Chaturvedi and D. S. Bernstein, Inertia-free spacecraft attitude tracking with disturbance rejection and almost global stabilization, *Journal of Guidance, Control, and Dynamics*, vol.32, no.4, pp.1167-1178, 2009.
[14] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*, Springer, 2004.
[15] A. R. Klumpp, Singularity-free extraction of a quaternion from a direct-cosine matrix, *Journal of Spacecraft*, vol.13, no.12, pp.754-755, 1976.
[16] R. Kristiansen, A. Loria, A. Chaillet and P. J. Nicklasson, Spacecraft relative rotation tracking without angular velocity measurements, *Automatica*, vol.45, no.3, pp.750-756, 2009.
[17] M. D. Shuster, A survey of attitude representations, *Journal of the Astronautical Sciences*, vol.41, no.4, pp.439-517, 1993.
[18] B. Wie, *Space Vehicle Dynamics and Control*, AIAA Education Series, 1998.
[19] H. Yoon and P. Tsiotras, Singularity analysis of variable speed control moment gyros, *Journal of Guidance, Control, and Dynamics*, vol.27, no.3, pp.374-386, 2004.
[20] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dyn. Sys., and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
[21] P. R. Haimos, *Measure Theory*, Van Nostrand, Princeton, NJ, 1950.
[22] M. Krstic, Invariant manifolds and asymptotic properties of adaptive nonlinear stabilizers, *IEEE Transactions on Automatic Control*, vol.41, no.6, pp.817-829, 1996.

[23] W. C. Cai and Y. D. Song, New intermediate quaternion based control of spacecraft: Part II – Global attitude tracking, *International Journal of Innovative Computing, Information and Control*, to appear.