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NEW INTERMEDIATE QUATERNION BASED CONTROL OF SPACECRAFT: PART II – GLOBAL ATTITUDE TRACKING

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Abstract. *We have demonstrated in Part I that if the initial attitude error is not located at the unstable equilibrium or its stable manifold, then our proposed intermediate quaternion based attitude control method is able to ensure asymptotically stable attitude tracking. We now proceed in Part II to address global attitude tracking. By blending the intermediate quaternion with a shifted one, we develop a systematic approach for synthesizing a hybrid attitude control scheme capable of steering any initial attitude error toward the desired stable equilibrium, delivering a final product for globally stable attitude tracking control of spacecraft free of singularity, ambiguity and unwinding.*
Keywords: Intermediate quaternion, Attitude tracking, Unwinding free

1. **Introduction.** A new 4-element variable (quaternion) has been introduced as an intermediate unit for attitude control design for spacecraft in Part I [1], where it is shown that by building attitude control upon the intermediate quaternion, several advantageous features as compared with traditional quaternion based control method can be achieved. As a correlated portion of the work, we now further extend the results to global attitude tracking. The fundamental idea behind achieving global attitude tracking constitutes of two steps: i) introducing a shifted (and easily computable) quaternion so that the unstable equilibrium with respect to the original quaternion is no longer the equilibrium w.r.t. the shifted quaternion; and ii) constructing attitude control using both the original and the shifted quaternion variables to drive the vehicle situating at any initial attitude to the stable equilibrium (desired attitude) asymptotically, i.e., global attitude tracking is achieved, as detailed in the rest of the paper.

2. **Motivation for Global Attitude Tracking Control Design.**

2.1. **Problem statement.** For completeness and easy reference, some of the key points in Part I of this work [1] are briefly repeated here. According to Euler's eigenaxis rotation theorem [2], the attitude tracking error in terms of the direction cosine matrix (DCM) $\mathbf{R}_e$ can be parameterized by a unit vector $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T$ and an angle ϑ as follows:

$$\mathbf{R}_e(\boldsymbol{\sigma}, \vartheta) = \cos\vartheta \mathbf{I} + (1 - \cos\vartheta)\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \sin\vartheta \boldsymbol{\sigma}^{\times} \tag{1}$$

Now we introduce the following new intermediate 4-element variable p using $\boldsymbol{\sigma}$ and ϑ as

$$p = (p_0, \mathbf{p}) = (\cos\vartheta, \boldsymbol{\sigma}\sin\vartheta) \tag{2}$$

By using such quaternion (the vector part $\mathbf{p}$), the following attitude control scheme (i.e., (17) from [1]) is constructed,

$$\boldsymbol{\tau} = -k_v\boldsymbol{\omega}_e - k_p\mathbf{p} + \mathbf{F}(.) \tag{3}$$

where $\mathbf{F}(.)$ is a nonlinear term, defined as

$$\mathbf{F}(.) = (\mathbf{R}_e\boldsymbol{\omega}_d)^{\times}\mathbf{J}(\mathbf{R}_e\boldsymbol{\omega}_d) + \mathbf{J}(\mathbf{R}_e\dot{\boldsymbol{\omega}}_d) \tag{4}$$

Refer to [1] for the definitions of other variables/parameters. The stability is analyzed by examining the following Lyapunov function candidate

$$V = \frac{1}{2}\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + k_p(1 - p_0) \geq 0 \tag{5}$$

along the error dynamics

$$\mathbf{J}\dot{\boldsymbol{\omega}}_e = \boldsymbol{\tau} - \mathbf{F}(.) - \mathbf{H}(.) \tag{6}$$

$$\mathbf{H}(.) = \boldsymbol{\omega}_e^{\times}\mathbf{J}(\boldsymbol{\omega}_e + \mathbf{R}_e\boldsymbol{\omega}_d) + [(\mathbf{R}_e\boldsymbol{\omega}_d)^{\times}\mathbf{J} + \mathbf{J}(\mathbf{R}_e\boldsymbol{\omega}_d)^{\times}]\boldsymbol{\omega}_e \tag{7}$$

leading to

$$\dot{V} = -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e \leq 0 \tag{8}$$

from which it is established in [1] that the trajectory $(\boldsymbol{\omega}_e, p)$ converges to the equilibria $Q = Q^s \cup Q^u$, here Q^s and Q^u represent stable (desired) and unstable equilibria and are defined, respectively, as

$$Q^s = \{(\boldsymbol{\omega}_e, p) : \boldsymbol{\omega}_e = \mathbf{0}, p = (+1, \mathbf{0})\} = \{(\boldsymbol{\omega}_e, \mathbf{R}_e) : \boldsymbol{\omega}_e = \mathbf{0}, \mathbf{R}_e = \mathbf{I}\} \tag{9}$$

$$Q^u = \{(\boldsymbol{\omega}_e, p) : \boldsymbol{\omega}_e = \mathbf{0}, p = (-1, \mathbf{0})\} = \{(\boldsymbol{\omega}_e, \mathbf{R}_e) : \boldsymbol{\omega}_e = \mathbf{0}, \mathbf{R}_e = 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \mathbf{I}\} \tag{10}$$

It is rigorously shown in [1] that with the control scheme (3), all trajectories $(\boldsymbol{\omega}_e, p)$ attracted to the equilibria Q^u form a lower-dimensional manifold Ω^u ($\Omega^u \supset Q^u$) in $TSO(3)$ [3,4], which is a set with measure zero [5,6], thus almost globally stable attitude tracking is achieved in that any trajectory $(\boldsymbol{\omega}_e, p)$ not originating from the negligible region Ω^u always tends to the desired equilibrium Q^s.

Now here comes the interesting question: if the vehicle does initially situate at the extreme region Ω^u (possible if considering the global operation envelope), can we still achieve asymptotically stable attitude tracking with the proposed quaternion based method? This represents the global attitude tracking control problem of theoretical and practical importance to be addressed in this work.

2.2. **Some useful observations.** First note that, in view of $p_0 = \cos\vartheta$ as given in (2), the equilibria Q^s and Q^u given in (9) and (10) can be equivalently expressed in terms of $(\boldsymbol{\omega}_e, \vartheta)$ as

$$Q^s = \{(\boldsymbol{\omega}_e, \vartheta) : \boldsymbol{\omega}_e = \mathbf{0}, \vartheta = 0°\} \text{ and } Q^u = \{(\boldsymbol{\omega}_e, \vartheta) : \boldsymbol{\omega}_e = \mathbf{0}, \vartheta = 180°\} \tag{11}$$

Also the Lyapunov function candidate (5) is equivalent to

$$V = \frac{1}{2}\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + k_p(1 - \cos\vartheta) \geq 0 \tag{12}$$

which defines a potential field, where the potential energy V is monotonically decreasing since $\dot{V} \leq 0$ always holds according to (8). Now the following useful observations can be made.

Observation 2.1. *It is observed from (1) and (11) that* $(\boldsymbol{\omega}_e, \mathbf{R}_e) \in (\mathbf{0}, 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \mathbf{I})$ *corresponds to the same equilibria* Q^u. *Furthermore, from (12) it is seen that all of the unstable equilibria* $(\boldsymbol{\omega}_e, \mathbf{R}_e) \in (\mathbf{0}, 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \mathbf{I})$ *bear the same potential energy* $V = 2k_p$. *As a result, any error trajectory diverging from any one of the unstable equilibria diverges actually from all the unstable equilibria.*

Observation 2.2. *As mentioned earlier, the set* Ω^u *is the stable manifold of* Q^u, *thus* *only those trajectories* $(\boldsymbol{\omega}_e, \vartheta)$ *starting from* Ω^u *converge to the equilibrium* Q^u*. Since* $\dot{V} \leq 0$ *holds all the time as ensured by (8), it can be concluded that*

$$V(\boldsymbol{\omega}_e, \vartheta) \geq V(\mathbf{0}, 180°) = 2k_p \ \ \forall (\boldsymbol{\omega}_e, \vartheta) \in \Omega^u \tag{13}$$

Namely, the unstable equilibrium Q^u *carries the minimum potential energy in* Ω^u*.*

Observation 2.3. *From (12) one easily infers that*

$$V(\boldsymbol{\omega}_e, \vartheta) < V(\mathbf{0}, 180°) = 2k_p \ \ \forall (\boldsymbol{\omega}_e, \vartheta) \in M \tag{14}$$

if the set M *is defined as*

$$M = \{(\boldsymbol{\omega}_e, \vartheta) : \boldsymbol{\omega}_e = \mathbf{0}, \vartheta \in [0°, 180°) \cup (180°, 360°]\} \tag{15}$$

Then, from (13) and (14), one can deduce that with the controller (3), any trajectory $(\boldsymbol{\omega}_e, \vartheta)$ *starting from any point* $(\boldsymbol{\omega}_e, \vartheta) \in M$ *never goes to the unstable (undesirable) equilibria* Q^u*, but rather, asymptotically converges to* Q^s *in which* $(\boldsymbol{\omega}_e, \vartheta) = (\mathbf{0}, 0°)$*, as shown in Theorem 4.1 in [1].*

Observation 2.4. *If there exists a controller* $\boldsymbol{\tau}_\eta$ *capable of driving the error trajectory* $(\boldsymbol{\omega}_e,\ \mathbf{R}_e)$ *away from a point in* Q^u *and pushing it toward the region M, then the error trajectory* $(\boldsymbol{\omega}_e,\ \mathbf{R}_e)$ *actually diverges from the entire set* Q^u *due to the same potential energy property of all the points in* Q^u *as mentioned in Observation 2.1. Therefore, if one could find such* $\boldsymbol{\tau}_\eta$ *that drives* $(\boldsymbol{\omega}_e, \vartheta)$ *towards M, and as soon as* $(\boldsymbol{\omega}_e, \vartheta)$ *enters into M, the controller (3) is activated, then with such strategy the trajectory* $(\boldsymbol{\omega}_e, \vartheta)$ *never goes to the unstable (undesirable) equilibria* Q^u*, but rather asymptotically converges to the desired stable equilibrium* $(\boldsymbol{\omega}_e, \vartheta) = (\mathbf{0}, 0°)$ *according to Observation 2.3.*

2.3. **Main idea for global attitude tracking control.** Inspired by the above observations, we construct a hybrid attitude control strategy by combining $\boldsymbol{\tau}_\eta$ with $\boldsymbol{\tau}_p$ through a signal χ as follows:

$$\boldsymbol{\tau} = \chi \boldsymbol{\tau}_p + (1 - \chi)\boldsymbol{\tau}_\eta \tag{16}$$

where $\boldsymbol{\tau}_p$ denotes the original intermediate quaternion based control (3), $\boldsymbol{\tau}_\eta$ is the control with the capability as imposed in Observation 2.4 (as designed later, see (25)), and χ is a switch signal taking 0 or 1 according to the following simple logic reasoning mechanism,

$$\begin{aligned}
&\text{Initially } \chi(0) = 1;\\
&\text{IF } \chi = 1 \text{ and } (\boldsymbol{\omega}_e, \vartheta) \in Q^u, \ \text{THEN } \chi = 0;\\
&\text{IF } \chi = 0 \text{ and } (\boldsymbol{\omega}_e, \vartheta) \in M, \ \text{THEN } \chi = 1.
\end{aligned}$$

To see how the proposed control scheme (16) would act during the entire control process so as to achieve global attitude tracking, we only need to examine how the two cases – Case 1: $(\boldsymbol{\omega}_e(0), \vartheta(0)) \in \Omega^u$ and Case 2: $(\boldsymbol{\omega}_e(0), \vartheta(0)) \notin \Omega^u$ are handled by the control scheme since any initial error $(\boldsymbol{\omega}_e(0), \vartheta(0))$ falls into either case.

First note that since $\chi(0) = 1$ the control scheme (16) initially is $\boldsymbol{\tau} = \boldsymbol{\tau}_p$. Then if Case 1 happens (i.e., initially $(\boldsymbol{\omega}_e(0), \vartheta(0)) \in \Omega^u$), then since $\chi = 1$ the control $\boldsymbol{\tau}_p$ is active from that moment on till the trajectory reaches a point in Q^u, as soon as that time instant comes, χ is updated to 0 and then, by (16), the control $\boldsymbol{\tau}_\eta$ takes over from that moment. Such control is capable of driving the system away from Q^u and pushing it towards a point $(\boldsymbol{\omega}_e, \vartheta) \in M$ as assumed in Observation 2.4. Once $(\boldsymbol{\omega}_e, \vartheta)$ enters into M, χ is updated back to 1, thus $\boldsymbol{\tau}_\eta$ is off and $\boldsymbol{\tau}_p$ becomes active again, with which the trajectory $(\boldsymbol{\omega}_e, \vartheta)$ starting from a point in M converges to the point Q^s as analyzed in Observation 2.3.

If Case 2 happens, which is equivalent to $(\boldsymbol{\omega}_e(t), \vartheta(t)) \notin \Omega^u \ \ \forall t \geq 0$, according to the control setting, $\chi(t) = 1$ always holds so that $\boldsymbol{\tau} = \boldsymbol{\tau}_p$ is activated all the time, such control, as already shown in [1], is able to drive any trajectory initially on the point $(\boldsymbol{\omega}_e, \vartheta) \notin \Omega^u$ to the desired position Q^s, i.e., $(\boldsymbol{\omega}_e, \mathbf{R}_e) \to (\mathbf{0}, \mathbf{I})$.

Based upon the above analysis, it is seen that control $\boldsymbol{\tau}_\eta$ plays a crucial role in achieving global tracking. Therefore, the problem of designing global attitude tracking control boils down to the construction of the control $\boldsymbol{\tau}_\eta$, which is addressed in next section.

3. **New Intermediate Variable Based Control for Global Tracking.** From the new intermediate quaternion defined in (2), it is not difficult to infer that $\mathbf{p} = \mathbf{0}$ leads to $\vartheta = 0°(360°)$ or $\vartheta = 180°$. Thus, if there exists a vector $\boldsymbol{\eta}$, based on which a controller $\boldsymbol{\tau}_\eta$ is designed such that the closed-loop system has the equilibria $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$ where $\vartheta \neq 180°$ (i.e., at that moment $(\boldsymbol{\omega}_e, \vartheta) \in M$), then such controller $\boldsymbol{\tau}_\eta$ is exactly the one as imposed in Observation 2.4 that can be incorporated in (16).

Therefore, we focus on i) finding (introducing) the vector $\boldsymbol{\eta}$; ii) building $\boldsymbol{\tau}_\eta$ upon such $\boldsymbol{\eta}$ and showing that under the control of $\boldsymbol{\tau}_\eta$ the closed-loop system has the equilibria $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$ where $\vartheta \neq 180°$; and iii) proving that the hybrid control (16) consisting of $\boldsymbol{\tau}_p$ and $\boldsymbol{\tau}_\eta$ indeed ensures global and asymptotic attitude tracking in the rest of the paper.

3.1. **Shifted intermediate variable.** We first define a shifted intermediate variable $p^* \triangleq (p_0^*, \mathbf{p}^*)$, which is obtained by rotating the original intermediate quaternion p with a small angle δ along an axis $\boldsymbol{\xi}$. Mathematically, the variable p^* is computed by [2,7-10]

$$p^* \triangleq (p_0^*, \mathbf{p}^*) = p \otimes \bar{\tilde{p}} = (\tilde{p}_0 p_0 + \tilde{\mathbf{p}}^T \mathbf{p}, \tilde{p}_0 \mathbf{p} - p_0 \tilde{\mathbf{p}} - \mathbf{p}^\times \tilde{\mathbf{p}}) \tag{17}$$

where "$\otimes$" denotes quaternion multiplication, $\bar{\tilde{p}}$ is the conjugate of $\tilde{p} = (\tilde{p}_0, \tilde{\mathbf{p}})$, a known small orientation shift, defined as

$$\tilde{p} = (\tilde{p}_0, \tilde{\mathbf{p}}) = (\cos\delta, \boldsymbol{\xi}\sin\delta) \tag{18}$$

where δ is a small constant angle chosen by the designer and $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]^T$ is a given constant unit vector.

Recalling that $\dot{p}$ is given by (11) in [1], it is not difficult to show that the time derivative of p_0^* is obtained as

$$\dot{p}_0^* = -\left(\tilde{p}_0 \mathbf{p} - p_0 \tilde{\mathbf{p}} - \frac{1}{2}\tilde{\mathbf{p}} + \frac{1}{2}\mathbf{R}_e^T \tilde{\mathbf{p}}\right)^T \boldsymbol{\omega}_e \tag{19}$$

It is interesting to note that if a variable $\boldsymbol{\eta}$ is defined as

$$\boldsymbol{\eta} = \tilde{p}_0 \mathbf{p} - p_0 \tilde{\mathbf{p}} - \frac{1}{2}\tilde{\mathbf{p}} + \frac{1}{2}\mathbf{R}_e^T \tilde{\mathbf{p}} \tag{20}$$

one gets $\dot{p}_0^* = -\boldsymbol{\eta}^T \boldsymbol{\omega}_e$ from (19). Clearly, such intermediate variable $\boldsymbol{\eta}$ deriving from p^* can be straightforwardly and uniquely determined from the physically available $\mathbf{R}_e$ and the chosen $\tilde{p} = (\tilde{p}_0, \tilde{\mathbf{p}})$ as given in (18). Furthermore, the vector $\boldsymbol{\eta}$ so defined exhibits an interesting feature as stated in the following lemma.

Lemma 3.1. *Let $\boldsymbol{\eta}$ be defined as in (20). If $\boldsymbol{\sigma}^T \boldsymbol{\xi} \neq 0$ then $\boldsymbol{\eta} = \mathbf{0}$ is equivalent to either one of the following two cases*

$$(\boldsymbol{\sigma}, \vartheta) \in \{\boldsymbol{\sigma} = \boldsymbol{\xi}, \vartheta = \delta \text{ or } \vartheta = 180° + \delta\} \tag{21}$$

$$(\boldsymbol{\sigma}, \vartheta) \in \{\boldsymbol{\sigma} = \boldsymbol{\xi}, \vartheta = \delta^* \text{ or } \vartheta = 180° - \delta^*\} \tag{22}$$

where δ^* represents an angle given as

$$\delta^* = \arcsin(\tan\delta) \tag{23}$$

Proof: In view of $\mathbf{R}_e$ as defined in (1) and $\boldsymbol{\eta}$ as given in (20), we see that $\boldsymbol{\eta} = \mathbf{0}$ implies that

$$x_1\boldsymbol{\sigma} + x_2\boldsymbol{\xi} + x_3\boldsymbol{\sigma}^{\times}\boldsymbol{\xi} = 0 \tag{24}$$

where $x_1 = 2\cos\delta\sin\vartheta + \sin\delta(1-\cos\vartheta)\boldsymbol{\sigma}^T\boldsymbol{\xi}$, $x_2 = -\sin\delta(\cos\vartheta + 1)$, and $x_3 = \sin\delta\sin\vartheta$. Note that since both $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$ are unit vectors, for any $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$ one has i) $\boldsymbol{\sigma} = \boldsymbol{\xi}$, ii) $\boldsymbol{\sigma} = -\boldsymbol{\xi}$, or iii) $\boldsymbol{\sigma} \neq \pm\boldsymbol{\xi}$. Thus from (24) one has that $x_1 = -x_2$ if $\boldsymbol{\sigma} = \boldsymbol{\xi}$; and $x_1 = x_2$ if $\boldsymbol{\sigma} = -\boldsymbol{\xi}$; if $\boldsymbol{\sigma} \neq \pm\boldsymbol{\xi}$, then we have $\boldsymbol{\sigma}^{\times}\boldsymbol{\xi} \neq \mathbf{0}$, $\boldsymbol{\sigma} \perp \boldsymbol{\sigma}^{\times}\boldsymbol{\xi}$ and $\boldsymbol{\xi} \perp \boldsymbol{\sigma}^{\times}\boldsymbol{\xi}$, so that the solution of (24) is $x_1 = x_2 = x_3 = 0$. Therefore, Equation (24) would have three solutions if no additional constraint is imposed.

Solution 1: $\boldsymbol{\sigma} = \boldsymbol{\xi}$ and $x_1 = -x_2$. For this case, it follows from $x_1 = -x_2$ that $2\sin(\vartheta - \delta) = 0$ (noting that $\boldsymbol{\sigma}^T\boldsymbol{\xi} = 1$), from which one infers that $\vartheta = \delta$ or $\vartheta = 180° + \delta$, leading to (21).

Solution 2: $\boldsymbol{\sigma} = -\boldsymbol{\xi}$ and $x_1 = x_2$. Again from $x_1 = x_2$, it holds that $2\cos\delta(\sin\vartheta + \tan\delta) = 0$ (noting that $\boldsymbol{\sigma}^T\boldsymbol{\xi} = -1$), from which we obtain $\sin\vartheta + \tan\delta = 0$ since $\cos\delta \neq 0$ for a nonzero small angle δ, thus one can get $\vartheta = -\delta^*$ or $\vartheta = -180° + \delta^*$, where δ^* is defined in (23). Note that $(\boldsymbol{\sigma}, \vartheta)$ and $(-\boldsymbol{\sigma}, -\vartheta)$ induce the same $\mathbf{R}_e$ as defined in (1), so that this case is also equivalent to the situation $\boldsymbol{\sigma} = \boldsymbol{\xi}$ with $\vartheta = \delta^*$ or $\vartheta = 180° - \delta^*$, as given in (22).

Solution 3: $\boldsymbol{\sigma} \neq \pm\boldsymbol{\xi}$ and $x_1 = x_2 = x_3 = 0$. For this case, from $x_2 = x_3 = 0$ and $\sin\delta \neq 0$, one gets $\vartheta = 180°$, which with $x_1 = 0$ leads to $\boldsymbol{\sigma}^T\boldsymbol{\xi} = 0$.

However, since $\boldsymbol{\sigma}^T\boldsymbol{\xi} \neq 0\ \forall t \geq 0$ as imposed in the lemma, the third solution to (24) does not actually exist, leading to the conclusion that only (21) and (22) are equivalent to $\boldsymbol{\eta} = \mathbf{0}$, which completes the proof.

Remark 3.1. *The important implication of this lemma is that* $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$ *ensures* $(\boldsymbol{\omega}_e, \vartheta) \in M$ *if* $\boldsymbol{\sigma}^T\boldsymbol{\xi} \neq 0$. *Therefore, if we build* $\boldsymbol{\tau}_\eta$ *upon* $\boldsymbol{\eta}$ *which can drive the system to the equilibria* $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$, *then such controller* $\boldsymbol{\tau}_\eta$ *is capable of driving the error trajectory* $(\boldsymbol{\omega}_e, \vartheta)$ *away from a point in* Q^u *and pushing it toward some point* $(\boldsymbol{\omega}_e, \vartheta) \in M$, *as will be shown shortly. Also note that the condition* $\boldsymbol{\sigma}^T\boldsymbol{\xi} \neq 0$ *must be satisfied in order to maintain the property as stated in the lemma. Interestingly, since the Euler axis* $\boldsymbol{\xi}$ *is chosen freely by the designer, literally there are infinite numbers of choices, as detailed later.*

3.2. **Shifted intermediate variable based control.** Now we are ready to present the following theorem regarding how to design $\boldsymbol{\tau}_\eta$ with $\boldsymbol{\eta}$ so that $\boldsymbol{\tau}_\eta$ is capable of driving the error trajectory $(\boldsymbol{\omega}_e, \vartheta)$ away from Q^u and pushing it toward some point in M, where $\boldsymbol{\omega}_e = \mathbf{0}$ and $\vartheta \neq 180°$.

Theorem 3.1. *Consider the error dynamic system (6). If the attitude control is designed as*

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\eta = -k_v\boldsymbol{\omega}_e - k_p\boldsymbol{\eta} + \mathbf{F}(.) \tag{25}$$

where $\boldsymbol{\eta}$ *is determined by (20) in which* $\boldsymbol{\xi}$ *is chosen properly such that* $\boldsymbol{\xi}^T\boldsymbol{\sigma}(0) \neq 0$, *then any trajectory* $(\boldsymbol{\omega}_e, \vartheta)$ *starting from* Q^u *asymptotically converges to the equilibria* $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$ *in which* $(\boldsymbol{\omega}_e, \vartheta) \in M$.

Proof: Choose a Lyapunov function candidate as

$$V^* = \frac{1}{2}\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + k_p(1 - p_0^*) \geq 0 \tag{26}$$

Under the control of (25), it follows from (6) and (26) that

$$\dot{V}^* = \boldsymbol{\omega}_e^T\mathbf{J}\dot{\boldsymbol{\omega}}_e - k_p\dot{p}_0^* = \boldsymbol{\omega}_e^T(\boldsymbol{\tau} - \mathbf{F}(.) - \mathbf{H}(.)) + k_p\boldsymbol{\omega}_e^T\boldsymbol{\eta}$$

$$= -k_v \boldsymbol{\omega}_e^T \boldsymbol{\omega}_e - \boldsymbol{\omega}_e^T \mathbf{H}(.) = -k_v \boldsymbol{\omega}_e^T \boldsymbol{\omega}_e \leq 0 \tag{27}$$

where $\boldsymbol{\omega}_e^T \mathbf{H}(.) \equiv 0$ as shown in [1], it follows from (26) and (27) that $\boldsymbol{\omega}_e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, thus $\mathbf{F}(.) \in \mathcal{L}_\infty$ and $\mathbf{H}(.) \in \mathcal{L}_\infty$ from (4) and (7) as the desired angular velocity $\boldsymbol{\omega}_d$ and its rate $\dot{\boldsymbol{\omega}}_d$ are bounded. Now it is observed that all terms in the controller (25) are bounded at any time, thus $\boldsymbol{\tau} \in \mathcal{L}_\infty$, as a result, combining $\boldsymbol{\omega}_e \in \mathcal{L}_\infty$, $\mathbf{F}(.) \in \mathcal{L}_\infty$ and $\mathbf{H}(.) \in \mathcal{L}_\infty$, one gets $\dot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$ from (6). According to Barbalat lemma, $\boldsymbol{\omega}_e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$ (i.e., $\boldsymbol{\omega}_e$ uniformly continuous) imply that $\boldsymbol{\omega}_e \to \mathbf{0}$ as $t \to \infty$, and then $\mathbf{H}(.) \to 0$ as $t \to \infty$ from (7). Taking derivative of (6) with respect to time yields

$$\mathbf{J}\ddot{\boldsymbol{\omega}}_e = \dot{\boldsymbol{\tau}} - \dot{\mathbf{F}}(.) - \dot{\mathbf{H}}(.) \tag{28}$$

In light of the fact that $\dot{\boldsymbol{\omega}}_d$ and $\ddot{\boldsymbol{\omega}}_d$ are bounded, it is not difficult to show $\dot{\mathbf{F}}(.)$ and $\dot{\mathbf{H}}(.)$ are also bounded (because $\dot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$). Thus from (25) $\dot{\boldsymbol{\tau}}$ is also bounded since from (20) $\dot{\boldsymbol{\eta}} \in \mathcal{L}_\infty$ due to $\boldsymbol{\omega}_e \in \mathcal{L}_\infty$, $\mathbf{R}_e \in \mathcal{L}_\infty$ and $\dot{\mathbf{p}} \in \mathcal{L}_\infty$ (from (11) in [1]). Then from (28) we have $\ddot{\boldsymbol{\omega}}_e \in \mathcal{L}_\infty$, implying that $\dot{\boldsymbol{\omega}}_e$ is uniformly continuous, which, together with $\boldsymbol{\omega}_e \to \mathbf{0}$, allows the Barbalat lemma to be used again so that $\dot{\boldsymbol{\omega}}_e \to 0$ as $t \to \infty$ [11-13]. Consequently, from the closed-loop error dynamics $\mathbf{J}\dot{\boldsymbol{\omega}}_e = -k_v \boldsymbol{\omega}_e - k_p \boldsymbol{\eta} - \mathbf{H}(.)$ obtained from (6) and (25), one obtains that $\boldsymbol{\eta} \to 0$ as $t \to \infty$ since $\boldsymbol{\omega}_e \to 0$, $\dot{\boldsymbol{\omega}}_e \to 0$ and $\mathbf{H}(.) \to 0$ as $t \to \infty$. In other words, the system with any initial condition asymptotically converges to the equilibrium $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$.

To complete the proof, we need to show that any trajectory $(\boldsymbol{\omega}_e, \vartheta)$ starting from Q^u converges to the equilibria $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$ (where $\vartheta \neq 180°$) asymptotically. Note that at the initial position $(\boldsymbol{\omega}_e(0), \vartheta(0)) \in Q^u$, we have $\vartheta(0) = 180°$, which, together with $\boldsymbol{\xi}^T \boldsymbol{\sigma}(0) \neq \mathbf{0}$, infers from Lemma 3.1 that $\boldsymbol{\eta}(0) \neq 0$, which implies that initially $(\boldsymbol{\omega}_e(0), \boldsymbol{\eta}(0)) \neq (\mathbf{0}, \mathbf{0})$, thus the trajectory $(\boldsymbol{\omega}_e, \vartheta)$ must diverge from the initial position $(\boldsymbol{\omega}_e(0), \vartheta(0)) \in Q^u$. Furthermore, from (17) it is obtained that $p_0^* = -\tilde{p}_0$ as $(p_0, \mathbf{p}) = (-1, 0)$, thus from (26) we have

$$V^*(\boldsymbol{\omega}_e, \vartheta) = k_p(1 + \tilde{p}_0) \;\; \forall (\boldsymbol{\omega}_e, \vartheta) \in Q^u \tag{29}$$

which implies that once the system diverges from the initial position $(\boldsymbol{\omega}_e(0), \mathbf{R}_e(0)) \in Q^u$, it diverges from all points in Q^u (because of the same potential energy property as indicated in (29)). Therefore, with the controller (25), any trajectory $(\boldsymbol{\omega}_e, \vartheta)$ originating in Q^u diverges from Q^u, then converges to the equilibria $(\boldsymbol{\omega}_e, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0})$, in which $\vartheta \neq 180°$ (i.e., $(\boldsymbol{\omega}_e, \vartheta) \in M$), more specifically, $\vartheta \in \{\delta, \delta^*, 180° - \delta^*, 180° + \delta\}$ (where δ^* is given in (23)) according to Lemma 3.1, and the proof completes.

From this theorem, the control scheme $\boldsymbol{\tau}_\eta$ that possesses the property as described in Observation 2.4 has been constructed as in (25). By making use of such $\boldsymbol{\tau}_\eta$, global attitude tracking results can be readily established, as presented in next subsection.

3.3. **Globally asymptotic attitude control design.** We have shown in previous subsection that $\boldsymbol{\tau}_\eta$ constructed as in (25) exhibits the property as imposed in Observation 2.4. It then remains to show that the controller (16) consisting of $\boldsymbol{\tau}_\eta$ and $\boldsymbol{\tau}_p$ ensures globally and asymptotically stable attitude tracking, as formally presented in the following theorem.

Theorem 3.2. *Consider the error dynamic system (6). If the hybrid control scheme (16) is applied, in which $\boldsymbol{\tau}_p$ is designed as in (3) and $\boldsymbol{\tau}_\eta$ is generated by (25). Then it is ensured that i) the error trajectory never rests on the point Q^u given (10) or (11); and ii) globally and asymptotically stable attitude tracking is ensured in that the error trajectory starting from any initial position converges to Q^s asymptotically, i.e., $(\boldsymbol{\omega}_e, \mathbf{R}_e) \to (\mathbf{0}, \mathbf{I})$ as $t \to \infty$.*

Proof: Since initially $\chi(0) = 1$, the hybrid control (16) initially is

$$\boldsymbol{\tau} = \boldsymbol{\tau}_p = -k_v\boldsymbol{\omega}_e - k_p\mathbf{p} + \mathbf{F}(.) \tag{30}$$

Choose a Lyapunov function candidate

$$V = \frac{1}{2}\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + k_p(1 - p_0) \geq 0 \tag{31}$$

Considering the error dynamic system (6) with the control (30), it has been shown in [1] that $\dot{V} = -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e \leq 0$, from which it is concluded that the error trajectory starting from any position asymptotically converges to the equilibria $Q = Q^s \cup Q^u$, as detailed in [1].

According to the rule for updating χ as described in (16), if $(\boldsymbol{\omega}_e(0), \vartheta(0)) \notin \Omega^u$, then $(\boldsymbol{\omega}_e(t), \vartheta(t)) \notin \Omega^u\ \forall t \geq 0$ because only those trajectories starting from Ω^u converge to Q^u, as described in Observation 2.2, therefore, $\chi = 1$ holds for any time instance, and only controller (30) is in action during the entire control process. Since (30) is identical to that as developed in [1], it has been shown that (30) is able to drive the system attitude errors asymptotically to the stable equilibrium Q^s as long as $(\boldsymbol{\omega}_e(0), \vartheta(0)) \notin \Omega^u$.

If, however, $(\boldsymbol{\omega}_e, \vartheta)$ hits Q^u at some time instant (t_{c1}), χ is updated to 0 according to the control setting, and then the hybrid controller (16) becomes

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\eta = -k_v\boldsymbol{\omega}_e - k_p\boldsymbol{\eta} + \mathbf{F}(.) \tag{32}$$

in which $\boldsymbol{\eta}$ is given as (20), where $\tilde{p} = (\cos\delta, \boldsymbol{\xi}\sin\delta)$, here $\boldsymbol{\xi}$ is properly chosen such that $\boldsymbol{\xi}^T\boldsymbol{\sigma}(t_{c1}) \neq 0$ and $\delta \neq 0$ is a constant small angle (e.g., $\delta = 2°$ or $\delta = 5°$).

Choose a Lyapunov function candidate

$$V^* = \frac{1}{2}\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + k_p(1 - p_0^*) \geq 0 \tag{33}$$

with p_0^* defined as in (17). It follows from (6), (32) and (33) that $\dot{V}^* = -k_v\boldsymbol{\omega}_e^T\boldsymbol{\omega}_e \leq 0$. Therefore, as shown in the proof of Theorem 3.1, under the control of $\boldsymbol{\tau}_\eta$ the error trajectory $(\boldsymbol{\omega}_e, \vartheta)$ (already hitting Q^u as assumed, thus $(\boldsymbol{\omega}_e, \vartheta) \in Q^u$) asymptotically goes to the position $(\boldsymbol{\omega}_e, \vartheta) \in M$ (i.e., $\boldsymbol{\omega}_e = \mathbf{0}$ and $\vartheta \neq 180°$) and never approaches any other point $(\boldsymbol{\omega}_e, \vartheta) \in Q^u$. Note that at the moment (t_{c2}) that the trajectory $(\boldsymbol{\omega}_e, \vartheta)$ enters into the region M, χ changes back to 1 and the controller (32) changes to (30), at this time the potential energy $V(\boldsymbol{\omega}_e(t_{c2}), \vartheta(t_{c2}))$ defined by (31) is bounded as

$$V(\boldsymbol{\omega}_e(t_{c2}), \vartheta(t_{c2})) < V(\mathbf{0}, 180°) = 2k_p \tag{34}$$

for $\boldsymbol{\omega}_e(t_{c2}) = \mathbf{0}$ and $\vartheta(t_{c2}) \in \{\delta, \delta^*, 180° - \delta^*, 180° + \delta\}$, namely, $\boldsymbol{\omega}_e(t_{c2}), \vartheta(t_{c2}) \notin \Omega^u$, thus under the control of (30), the error trajectory $(\boldsymbol{\omega}_e, \vartheta)$ starting from $\boldsymbol{\omega}_e(t_{c2}), \vartheta(t_{c2}) \notin \Omega^u$ asymptotically converges to Q^s, as shown in Theorem 3.1 in [1].

Consequently, the error dynamic system under the controller (16) never rests on Q^u (i.e., unstable equilibrium w.r.t. $\boldsymbol{\tau}_p$) and furthermore globally and asymptotically converges to the desired equilibrium $(\boldsymbol{\omega}_e, \mathbf{R}_e) = (\mathbf{0}, \mathbf{I})$, which completes the proof.

Remark 3.2. *Note that the controller (16) is able to drive the error trajectory from any point to the desired position automatically without the need for judging where the error trajectory is initially situated. Furthermore, with the proposed intermediate quaternion based control (3), the closed-loop error dynamic system has unstable equilibria with exactly the same potential energy, thus the attitude error trajectory diverging from any one of the unstable equilibria will diverge from all the unstable equilibria, it is this feature that makes it possible to construct the hybrid controller (16) to achieve global attitude tracking, where the sub-controller* $\boldsymbol{\tau}_\eta$ *is triggered at most once during the entire control process. On the contrary, if the rotational matrix is directly used for attitude control design as in [4,14],* *it leads to three unstable equilibria with different potential energy, as a result, the system might shift from one of their unstable equilibria to another repeatedly. As such only almost global results are obtained with those methods as it is difficult to develop a systematic yet graceful solution like the one introduced here.*

4. **Implementation Consideration and Feature Analysis.**

4.1. **Implementation consideration.** It should be stressed that in order to establish the global tracking result of the proposed control scheme rigorously, the analysis and stability proof are somewhat involved; however, the resultant control scheme itself and the concept/fundamental idea behind the control scheme are quite simple. In fact, the following simple steps are needed in designing and programming the proposed control scheme:

Step 1: Set $\chi = 1$;
Step 2: Calculate $\mathbf{F}(.)$ from (4);
Step 3: IF $\chi = 1$
  if $\boldsymbol{(\omega_e, p_0)} \in \boldsymbol{N_Q}$, then $\chi = 0$, select a small angle δ,
    and determine $\boldsymbol{\xi}$ (i.e., $\boldsymbol{\tilde{p} = (\cos, \xi \sin)}$), go to Step 6
  else, go to Step 4
ELSE
  if $\boldsymbol{(\omega_e, p_0)} \in \boldsymbol{N_M}$, then $\chi = 1$, go to Step 4
  else, go to Step 6
Step 4: Calculate $\mathbf{p}$ from (9) in [1];
Step 5: Calculate $\boldsymbol{\tau}$ from (30); END
Step 6: Calculate $\boldsymbol{\eta}$ from (20);
Step 7: Calculate $\boldsymbol{\tau}$ from (32); END

where the sets N_Q and N_M are used to replace Q^u and M for practical application of the proposed controller (16), and defined by

$$N_Q = \{(\boldsymbol{\omega}_e, p_0) : \| \boldsymbol{\omega}_e \| \leq \epsilon_w, | 1 + p_0 | \leq \epsilon_p\} \tag{35}$$

$$N_M = \{(\boldsymbol{\omega}_e, p_0) : \| \boldsymbol{\omega}_e \| \leq \epsilon_w, | 1 + p_0 | > \epsilon_p\} \tag{36}$$

where $\epsilon_w > 0$ and $\epsilon_p > 0$ are small numbers chosen by the designer, which significantly facilitates the real-time implementation.

Also, at Step 3, one needs to specify the vector $\boldsymbol{\xi}$ to satisfy the condition $\boldsymbol{\xi}^T\boldsymbol{\sigma}(t_c) \neq 0$ at the moment (t_c) that the controller $\boldsymbol{\tau}_\eta$ is initiated. There are many possible choices for such $\boldsymbol{\xi}$ and the simplest one is to choose $\boldsymbol{\xi}$ such that it is parallel to the vector $\boldsymbol{\sigma}(t_c)$, namely $\boldsymbol{\xi} = \pm\boldsymbol{\sigma}(t_c)$, such that $\boldsymbol{\xi}^T\boldsymbol{\sigma}(t_c) = \pm 1$, where the vector $\boldsymbol{\sigma}(t_c)$ is derived from $\mathbf{R}_e(t_c)$. Note that according to the setting for the control (16), $\boldsymbol{\tau}_\eta$ is switched on at the time instant t_c that $(\boldsymbol{\omega}_e(t_c), \vartheta(t_c)) = (\mathbf{0}, 180°)$, where $\mathbf{R}_e(t_c) = 2\boldsymbol{\sigma}(t_c)\boldsymbol{\sigma}^T(t_c) - \mathbf{I}$, thus $\boldsymbol{\sigma}(t_c)$ can be determined from $\mathbf{R}_e(t_c)$ as

$$\sigma_i(t_c) = \pm\sqrt{(R_{eii(t_c)} + 1)/2} \quad (i = 1, 2, 3) \tag{37}$$

$$\sigma_i(t_c)\sigma_j(t_c) = R_{eij}(t_c)/2 \quad (i, j = 1, 2, 3 \text{ and } i \neq j) \tag{38}$$

Note that (37) is well defined due to $|\mathrm{R}_{eii}(.)| < 1$ for any DCM. The sign for σ_i can be easily determined. For example, if the sign of σ_1 is chosen (either positive or negative), then the signs of σ_2 and σ_3 can be determined from $\sigma_1(t_c)\sigma_2(t_c) = \mathrm{R}_{e12}(t_c)/2$ and $\sigma_1(t_c)\sigma_3(t_c) = \mathrm{R}_{e13}(t_c)/2$, which implies from $\mathbf{R}_e(t_c)$ that one gets two possible Euler axes with opposite directions, and either of which can be adopted, thus no ambiguity for the selection of $\boldsymbol{\xi}$ exists, though "+" "−" signs are involved here. Furthermore, such $\boldsymbol{\xi}$ needs to be determined and computed only once since $\boldsymbol{\tau}_\eta$ is triggered at most once (see Remark

3.2) during the entire control process, thus there is no need for repetitively computing (37).

4.2. **Control magnitude variation due to switch.** Since a switch control is involved in the proposed control (16), it is worth examining the control torque variation before and after the switch. From (30) and (32), it is seen that the variation of the control action due to switch is

$$\Delta\boldsymbol{\tau} = \boldsymbol{\tau}_p - \boldsymbol{\tau}_\eta = -k_p(\mathbf{p} - \boldsymbol{\eta}) \tag{39}$$

At the moment that $\boldsymbol{\tau}_\eta$ is switched on, and $\vartheta = 180°$. By virtue of $\tilde{\mathbf{p}} = \sin\delta\boldsymbol{\xi}$, the vector $\boldsymbol{\eta}$ can be determined from (20) as $\boldsymbol{\eta} = \sin\delta\boldsymbol{\xi}$, leading to

$$\|\Delta\boldsymbol{\tau}\| = k_p\|\boldsymbol{\eta}\| = k_p \sin\delta \tag{40}$$

At the time instant that $\boldsymbol{\tau}_\eta$ is turned off, $\boldsymbol{\eta} = \mathbf{0}$ and $\vartheta \in \{\delta, \delta^*, 180° - \delta^*, 180° + \delta\}$ as given in Lemma 3.1, then from (2), we have $\mathbf{p} = \boldsymbol{\sigma}\sin\vartheta$. Thus, from (39) one gets

$$\|\Delta\boldsymbol{\tau}\| = k_p\|\mathbf{p}\| = k_p \sin\delta_0 \tag{41}$$

where $\delta_0 \in \{\delta, \arcsin(\tan\delta)\}$. It is seen that although a switch control is involved, the control variation as reflected in (40) and (41) during switch is fair small since δ can be chosen arbitrarily small. Similar conclusion can be drawn if the switch signal is determined according to (35) and (36). Also note that control $\boldsymbol{\tau}_\eta$ is triggered only once during the whole control process, thus no consistent discontinuity is involved in the control action.

5. **Numerical Simulation and Verification.** A series of numerical simulation studies are conducted in this section to validate the effectiveness of the global tracking control scheme (16). The same dynamic model of the spacecraft as in [1] is considered. The control parameters are chosen as $k_v = 10$, $k_p = 10$.

5.1. **Attitude tracking control performance.** In this subsection, seven cases are simulated and analyzed to validate the effectiveness of the proposed controller (16). The seven initial simulation conditions are listed in Table 1, where the Euler eigenaxis $\mathbf{e}_b(0)$ and the rotation angle $\theta_b(0)$ are used to calculate the actual initial attitude $\mathbf{R}_b(0)$ of the vehicle according to Formula (1).

Under these conditions, the control (16) is simulated and the results are presented in Figure 1, where (a), (b) and (c) are the attitude tracking errors in terms of the intermediate quaternion (error) and angular velocity error. It is seen that $(\boldsymbol{\omega}_e, p) \to (\mathbf{0}, (1, \mathbf{0}))$ as $t \to \infty$. Also from (d), one can observe that either $\vartheta \to 0°$ or $\vartheta \to 360°$ as $t \to \infty$, both correspond to zero orientation error. It is interesting to note that the orientation angle error ϑ for all cases does not cross the position $\vartheta = 180°$ during the control process. In other words, the results validate that tracking along the shorter path is ensured and the unwinding phenomenon [15] is prevented with the proposed quaternion based method.

Also note from (c) and (d) that $(\boldsymbol{\omega}_e, \vartheta) \to (\mathbf{0}, 180°)$ does not occur, thus no need for triggering $\boldsymbol{\tau}_\eta$ for all the simulation cases during the entire control process – only the intermediate quaternion based controller $\boldsymbol{\tau}_p$ is activated, as shown in (f), as a result, the control actions generated by (16) for all cases are bounded and smooth, as shown in (e).

5.2. **Globally stable tracking performance.** To test and validate the global tracking capability of the proposed control scheme (16), we purposely choose the initial desired angular velocity and attitude $(\boldsymbol{\omega}_d(0), \mathbf{R}_d(0))$ and initial actual angular velocity and attitude $(\boldsymbol{\omega}_b(0), \mathbf{R}_b(0))$ in such a way that the initial tracking error $(\boldsymbol{\omega}_e(0), \mathbf{R}_e(0))$ situates at unstable equilibrium Q^u. More specifically, we consider five simulation cases as shown in Table 2, where it is easy to examine $(\boldsymbol{\omega}_e(0), \mathbf{R}_e(0)) \in Q^u$ for all the five cases, thus according to the control setting, controller (32) is activated initially, in which $\tilde{p}$ is chosen

TABLE 1. The initial conditions simulated

Simulation Cases	$\mathbf{e}_b(0)$ (chosen randomly)	$\theta_b(0)$ (deg)	$\boldsymbol{\omega}_b(0)$ (rad/sec)	$\mathbf{R}_d(0)$	$\boldsymbol{\omega}_b(t)$ (rad/sec)
Case 1	[0.2270, –0.2426, –0.9432]	120	[0, 0, 0]	**I**	[0.2, –0.25, 0.2]
Case 2	[–0.6099, 0.7617, 0.2190]	140	[0, 0, 0]	**I**	[0.15, 0.2, –0.2]
Case 3	[–0.5855, 0.7924, 0.1712]	160	[0, 0, 0]	**I**	[0.15, –0.1, 0.2]
Case 4	[0.7441, 0.5726, –0.3442]	180	[0, 0, 0]	**I**	[–0.2, 0.2, 0.25]
Case 5	[0.9673, –0.0546, –0.2475]	200	[0, 0, 0]	**I**	[0.2, –0.25, 0.2]
Case 6	[–0.4734, 0.2794, 0.8353]	220	[0, 0, 0]	**I**	[0.15, 0.2, –0.2]
Case 7	[0.5267, –0.8162, –0.2374]	240	[0, 0, 0]	**I**	[0.15, –0.1, 0.2]

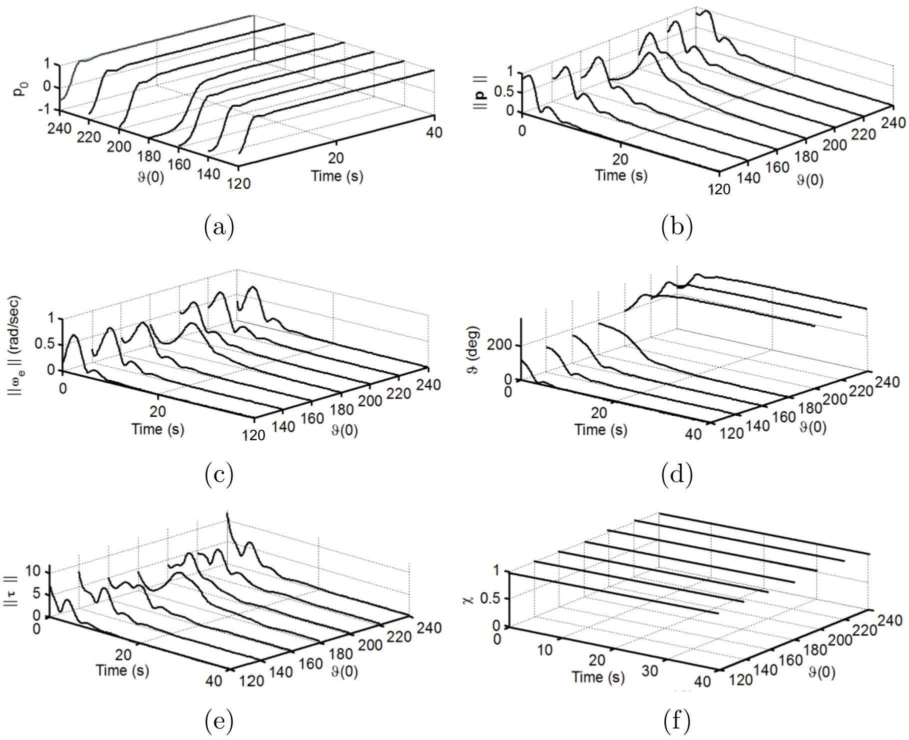

FIGURE 1. Control performance of the control (16) under the seven operation conditions. (a) Quaternion tracking error (scalar part), (b) quaternion tracking error (vector part), (c) angular velocity tracking error, (d) angular position tracking error, (e) control torque demanded, (f) switch control signal.

as $\tilde{p} = (\cos 2°, \boldsymbol{\xi} \sin 2°)$ with $\boldsymbol{\xi} = \boldsymbol{\sigma}(0)$ (where $\boldsymbol{\sigma}(0)$ is determined by (37) and (38)), and $\epsilon_w = \epsilon_p = 0.0001$ is chosen to define the sets N_Q and N_M (35) and (36), respectively.

Under the five extreme operation conditions, the controller (16) with the above setting is applied and the results are presented in Figure 2, where it is seen from (a), (b) and (c) that $(\boldsymbol{\omega}_e, p) \to (\mathbf{0}, (1, \mathbf{0}))$ is achieved as $t \to \infty$. From (d) it is observed that the orientation error ϑ diverges from $\vartheta = 180°$ quickly and then converges to $\vartheta = 0°$ asymptotically. These results validate that globally and asymptotically stable attitude tracking is achieved by the proposed intermediate quaternion based control (16). As $\tilde{p} = (\cos 2°, \boldsymbol{\xi} \sin 2°)$ is used in this simulation, the variation of control action due to switch is imperceptible, as shown in (e). Also note that control (32) is triggered only once, as shown in (f), thus no consistent discontinuity is involved in the control action, as confirmed in (e).

Table 2. Simulation conditions

Simulation Cases	$\mathbf{e}_b(0)$ (chosen randomly)	$\theta_b(0)$ (deg)	$\boldsymbol{\omega}_d(0)$ (rad/sec)	$\mathbf{R}_d(0)$	$\boldsymbol{\omega}_d(t)$ (rad/sec)
Case 1	[–0.4709, 0.3733, –0.7993]	180	–[0.4075, 0.0355, 0.0521]	**I**	[0.2, 0.2, –0.30]
Case 2	[0.5193, 0.1172, –0.8465]	180	[–0.2984, 0.2278, –0.0396]	**I**	[0.2, –0.25, 0.2]
Case 3	[–0.5627, 0.8207, –0.0997]	180	[–0.2622, –0.0364, 0.1801]	**I**	[0.15, 0.2, –0.2]
Case 4	[0.2006, 0.2251, –0.9535]	180	[–0.2235, 0.0176, 0.1492]	**I**	[0.15, –0.1, 0.2]
Case 5	[0.5046, 0.5941, –0.6265]	180	[0.0600, –0.3648, –0.0762]	**I**	[–0.2, 0.2, 0.25]

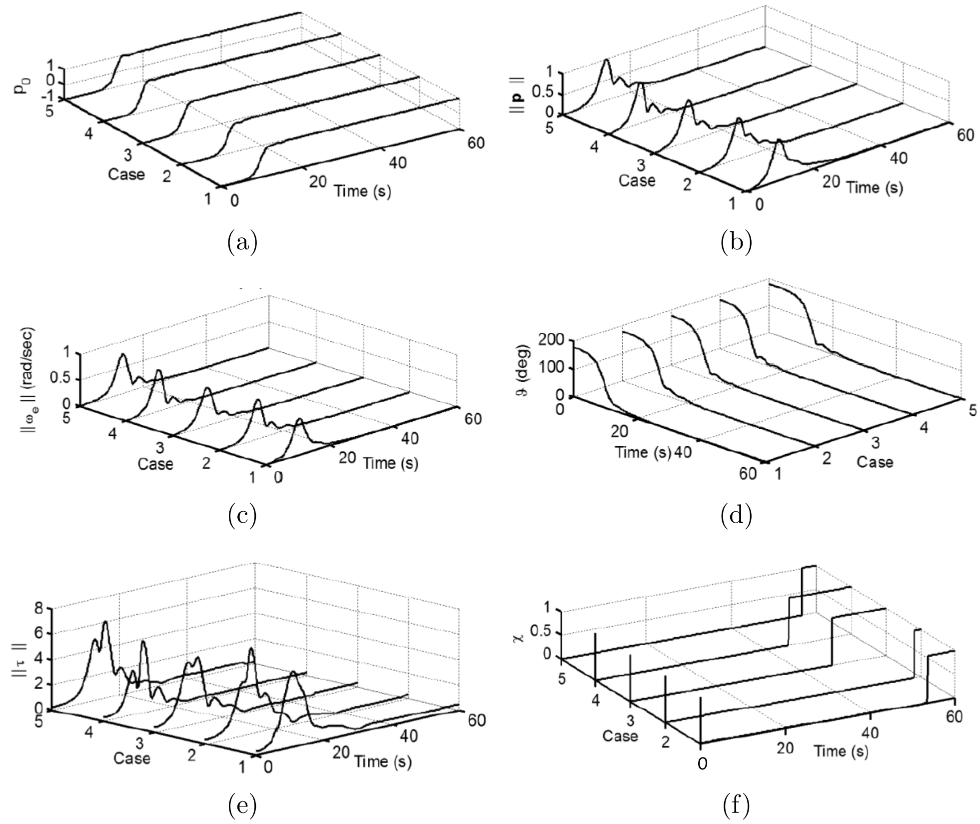

Figure 2. Control performance of the control (16) under the five extreme initial conditions. (a) Quaternion tracking error (scalar part), (b) quaternion tracking error (vector part), (c) angular velocity tracking error, (d) angular position tracking error, (e) control torque demanded, (f) switch control signal.

6. **Conclusions.** Euler angle-based attitude control method is not suitable for large orientation maneuver due to its singularity, whereas Hamilton quaternion based method is free of singularity, and has thus gained wide applications for attitude control of spacecraft. However, such a quaternion is unable to distinguish attitudes that differ by a 360° rotation about a principal axis, resulting in undesirable unwinding phenomenon. This means that attitude tracking along the shortest path is not guaranteed to take place in that even when a vehicle reaches the desired orientation, any small perturbation may cause 360° rotation about some axis. Removal of unwinding and ambiguity while achieving truly global tracking is highly desirable yet technically challenging. In this work a new intermediate quaternion is introduced and invoked for spacecraft attitude control design that ensures globally stable attitude tracking without singularity, ambiguity and

unwinding phenomena. Moreover, it is rather straightforward to determine the elements of the new intermediate quaternion from the real orientation information via the DCM, thus simplifying the design and implementation procedures. Extension of the new intermediate quaternion based method to robust adaptive (inertia-independence, disturbance rejection) attitude control represents an interesting topic for future work.

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