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DELAY-RANGE-DEPENDENT ROBUST H_∞ FILTERING FOR SINGULAR LPV SYSTEMS WITH TIME VARIANT DELAY

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Abstract. *This paper is concerned with the problem of the robust H_∞ filtering for singular linear parameter varying (LPV) systems with time variant delays. Firstly, based on a so-called piecewise analysis method, a delay-range-dependent bounded real lemma (BRL) is presented to ensure the singular time-delay LPV systems to be admissible and satisfy a prescribed H_∞ performance level. Secondly, based on the BRL, a sufficient condition for the existence of such a filter is proposed in terms of linear matrix inequality. Finally, the effectiveness of the proposed approach is shown by several numerical examples.*
Keywords: Singular LPV systems, H_∞ filtering, BRL, Time-variant delay

1. **Introduction.** Time delays are frequently the main causes of instability and poor performance of systems, and are encountered in a variety of engineering systems such as chemical processes, nuclear reactors, biological systems [1, 2]. The delay-partitioning method has extended to deal with the analysis and synthesis problems for singular time-delay systems, since the singular time-delay system models can describe a larger class of systems than usual state space ones (see [3, 4, 5] and the references therein). As an extension of the delay-partitioning method, a piecewise analysis method (PAM) [6] is considered in the stability analysis of the time-delay systems. With the PAM, better stability criteria for linear time-varying delay systems with less conservativeness are presented in [6]. On the other hand, in the past decades, many results on the analysis and synthesis of linear LPV systems have been obtained (e.g., [7, 8]), and further some research results on LPV systems have been generalized to singular LPV systems (see [9, 10, 11, 12] and the references therein).

Very recently, many researchers have focused on designing H_∞ filter for singular time-delay systems. Xu and Lam et al. [14] concerned the problem of robust H_∞ filtering for linear systems with both discrete and distributed delays. In [15], the l_2-l_∞ filtering problem for discrete-time singular Markovian jump systems with time-varying delays is proposed. Yue and Han [16] consider the robust H_∞ filter design for uncertain singular systems with discrete and distributed delays. In [17], delay-dependent H_∞ filtering problem was solved for singular system with time-varying delay in a range and two delay-range-dependent bounded real lemmas are also proposed in terms of LMIs. Zhu and Wang et al. [18] improved two nonlinear time-varying coefficients in [17] and present a BRL for singular time-varying delay systems. Compared with [17], since the proposed treatment method ensures the two nonlinear time-varying coefficients to be estimated more tightly, the results in [18] are less conservative. And delay-dependent robust H_∞ and L_2-L_∞ filtering for LPV systems with both discrete and distributed delays were proposed in [13].

However, to the best of our knowledge, there are no results on H_∞ filtering of singular LPV system with time-variant delay.

In this paper, a delay-dependent BRL in terms of LMIs is first presented to ensure the singular time-delay system to be admissible and satisfy a prescribed H_∞ performance level. The BRL is established by combining a parameter-dependent Lyapunov-Krasovskii functional method and a PAM. Finally, several numerical examples are demonstrated to illustrate that our results have less conservativeness than those reported earlier.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript 'T' denotes matrix transposition. Let $\|\cdot\|$ denote the Euclidean norm of vectors or the spectral norm of matrices, and $\|\cdot\|_2$ denote the L_2 norm on the space $\mathcal{L}_2[-\tau,+\infty)$ of all square-integrable vector-valued functions over $[-\tau,+\infty)$. I_m and 0_m denote the m-dimensional identity matrix and zero matrix, respectively. $0_{m\times n}$ denotes the $m\times n$ zero matrices. diag$\{\cdots\}$ stands for a block-diagonal matrix.

2. **Problem Statement and Preliminaries.**

Consider the following singular LPV system with time-varying delay:

$$\begin{cases} E\dot{x}(t) = A(\theta(t))x(t) + A_h(\theta(t))x(t-\tau_t) + B(\theta(t))\omega(t), \\ y(t) = C(\theta(t))x(t) + C_h(\theta(t))x(t-\tau_t) + D(\theta(t))\omega(t), \\ z(t) = L(\theta(t))x(t), \\ x(t) = \phi(t), \quad t\in[-\tau_M, 0] \end{cases} \tag{1}$$

with $\mathcal{X}(\theta(t)) = \mathcal{X}_0 + \sum\limits_{i=1}^{p}\theta_i(t)\mathcal{X}_i$,

$$\mathcal{X}(\theta(t)) = \begin{bmatrix} A(\theta(t)) & A_h(\theta(t)) & B(\theta(t)) & C(\theta(t)) \\ C_h(\theta(t)) & D(\theta(t)) & L(\theta(t)) & 0 \end{bmatrix},$$

$$\mathcal{X}_0 = \begin{bmatrix} A_0 & A_{h0} & B_0 & C_0 \\ C_{h0} & D_0 & L_0 & 0 \end{bmatrix}, \quad \mathcal{X}_i = \begin{bmatrix} A_i & A_{hi} & B_i & C_i \\ C_{hi} & D_i & L_i & 0 \end{bmatrix},$$

where time-delay τ_t is a time-varying differentiable function that satisfies

$$0\le \tau_m \le \tau_t \le \tau_M, \quad \dot{\tau}_t \le d < 1, \tag{2}$$

where τ_m, τ_M and d are known scalars; the variation interval $[\tau_m,\tau_M]$ of time-delay can be divided into l parts with equal length. That is $[\tau_m,\tau_M] = \bigcup_{i=0}^{l-1}[\tau_i,\tau_{i+1}]$, where $\tau_i = \tau_m + i(\tau_M-\tau_m)/l$, $i=0,1,2,\cdots,l$ and $\delta = \tau_i - \tau_{i-1}$.

$x(t)$ is the m-dimensional state vector, $y(t)$ is the n-dimensional measurement, $z(t)$ is the q-dimensional signal to be estimated, $\omega(t)$ is the s-dimensional disturbance input that belongs to $\mathcal{L}_2[-\tau_M,+\infty)$, $\phi(t)$ denotes the initial function of $x(t)$, $E\in\mathbb{R}^{m\times m}$ is a constant matrix satisfying rank$E = r < m$, A_i, A_{hi}, B_i, C_i, C_{hi}, D_i and L_i $(i=0,1,\cdots,p)$ are known real constant matrices with appropriate dimensions, $\theta_i(t)$, $i=1,2,\cdots,p$, are uncertain time-variant real parameters which satisfy

$$\underline{\theta}_i \le \theta_i(t) \le \overline{\theta}_i, \quad i=1,\dots,p, \tag{3}$$

$$\underline{\nu}_i \le \dot{\theta}_i(t) \le \overline{\nu}_i, \quad i=1,\dots,p, \tag{4}$$

where $\underline{\theta}_i$, $\overline{\theta}_i$, $\underline{\nu}_i$ and $\overline{\nu}_i$ are known constants.

Conditions (3) and (4) show that the parameter vector $\theta(t)$ and its derivative $\dot{\theta}(t)$ are valued in the hyper-rectangles

$$\Omega = \left\{(\eta_1,\cdots,\eta_p) \mid \eta_i\in[\underline{\theta}_i,\overline{\theta}_i], \quad i=1,\cdots,p\right\}$$

and

$$\Omega_d = \{(\varsigma_1, \cdots, \varsigma_p) \mid \varsigma_i \in [\underline{\nu}_i, \overline{\nu}_i], \quad i = 1, \cdots, p\},$$

respectively. Obviously, the sets of vertices/corners of the hyper-rectangles Ω and Ω_d are

$$\mathcal{V} = \left\{(\omega_1, \cdots, \omega_p) \mid \omega_i \in \left\{\underline{\theta}_i, \overline{\theta}_i\right\}, \quad i = 1, \cdots, p\right\}$$

and

$$\mathcal{V}_d = \{(r_1, \cdots, r_p) \mid r_i \in \{\underline{\nu}_i, \overline{\nu}_i\}, \quad i = 1, \cdots, p\},$$

respectively. Note that both $\mathcal{V}$ and $\mathcal{V}_d$ contain 2^p elements. For convenience, we write $\theta(t)$ as θ in the following context. Throughout the paper, we will adopt the following definitions (see, e.g., [11, Definitions 2.1-2.4]).

Definition 2.1. [19] *The matrix pair (E, A) is said to be regular if* $\det(sE - A)$ *is not identically zero, and if, in addition, its degree is equal to* rank$E$*, the matrix pair (E, A) is further said to be impulse-free.*

Definition 2.2. *The singular LPV time-delay system*

$$\begin{cases} E\dot{x}(t) = A(\theta)x(t) + A_h(\theta)x(t - \tau_t), \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{5}$$

is said to be regular and impulse-free if the pairs $(E, A(\theta))$ and $(E, A(\theta) + A_h(\theta))$ are regular and impulse-free for any θ satisfying (3) and (4).

Remark 2.1. *The regularity and the absence of impulses of the pairs $(E, A(\theta))$ and $(E, A(\theta) + A_h(\theta))$ ensure that for all compatible initial function $\phi(t)$, there exists a unique continuous solution on $[0, +\infty)$ to the system (5).*

Definition 2.3. *For given scalars $0 \le d < 1$ and $0 \le \tau_m < \tau_M$, the singular LPV system (5) is said to be robustly stable if for any $\epsilon > 0$ there exists a scalar $\delta(\epsilon) > 0$ such that, for any compatible initial function $\phi(t)$ satisfying $\sup_{-\tau_M \le t \le 0} \|\phi(t)\| \le \delta(\epsilon)$, and for any θ satisfying (3) and (4), the solution $x(t)$ to (5) satisfies that $\|x(t)\| \le \epsilon$ for any $t \ge 0$, and $x(t) \to 0$ as $t \to \infty$.*

Definition 2.4. *For given scalars $0 \le d < 1$ and $0 \le \tau_m < \tau_M$, the singular LPV system (5) is said to be admissible, if it is regular, impulse-free and robustly stable.*

In this paper, in order to estimate $z(t)$, we are interested in designing a filter of the following structure:

$$\begin{cases} E\dot{\hat{x}}(t) = A_f(\theta)\hat{x}(t) + B_f y(t), \\ \hat{z}(t) = C_f(\theta)\hat{x}(t), \end{cases} \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^m$, $\hat{z}(t) \in \mathbb{R}^q$, E is as in (1), and $A_f(\theta)$, B_f and $C_f(\theta)$ are the filter matrices with appropriate dimensions, which are to be designed. Define

$$\bar{z}(t) = z(t) - \hat{z}(t), \quad \bar{x}(t) = \begin{bmatrix} x^T(t) & \hat{x}^T(t) \end{bmatrix}^T.$$

Combining (1) and (6) we obtain the filtering error dynamics as follows:

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = \bar{A}(\theta)\bar{x}(t) + \bar{A}_h(\theta)\bar{x}(t - \tau_t) + \bar{B}(\theta)\omega(t), \\ \bar{z}(t) = \bar{L}(\theta)\bar{x}(t), \\ \bar{x}(t) = \begin{bmatrix} \phi^T(t) & 0 \end{bmatrix}^T, \quad t \in [-\tau_M, 0], \end{cases} \tag{7}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad \bar{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ B_f C(\theta) & A_f(\theta) \end{bmatrix}, \quad \bar{A}_h(\theta) = \begin{bmatrix} A_h(\theta) & 0 \\ B_f C_h(\theta) & 0 \end{bmatrix}, \quad \bar{B}(\theta) = \begin{bmatrix} B(\theta) \\ B_f D(\theta) \end{bmatrix},$$

$$\bar{L}(\theta) = \begin{bmatrix} L(\theta) & -C_f(\theta) \end{bmatrix}.$$

Definition 2.5. *For given scalars $0 \le d < 1$, $0 \le \tau_m < \tau_M$ and $\gamma > 0$, the singular time-delay LPV system (1) is said to be admissible with H_∞ performance γ if the system (1) is admissible when $\omega(t) = 0$, and under the zero initial condition, $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$ for any θ satisfying (3) and (4).*

The purpose of this paper is to design an H_∞ filter of the form (6), for the system (1) such that the filtering error systems (7) is admissible with H_∞ performance γ, where γ is a prescribed positive number.

3. **Preliminary Results.** In this section, a pair of basic conclusions are offered, which will be useful to present our main results.

Lemma 3.1. [11] *Let Ω and $\mathcal{V}$ be defined as previously. Assume that a scalar-valued quadratic function $f : \Omega \to \mathbb{R}$ is defined by*

$$f(\theta) = \alpha_0 + \sum_i \alpha_i\theta_i + \sum_{i<j} \beta_{ij}\theta_i\theta_j + \sum_i \gamma_i\theta_i^2, \quad \forall\theta = [\theta_1 \ \ \theta_2 \ \ \cdots \ \ \theta_p]^T \in \Omega.$$

If $f(\theta)$ is multi-convex, i.e., $\frac{\partial^2 f(\theta)}{\partial\theta_i^2} \ge 0$, $i = 1, 2, \cdots, p$, then $f(\theta) \le 0$ for any $\theta \in \Omega$ if and only if $f(\omega) \le 0$ for any $\omega \in \mathcal{V}$.

Lemma 3.2. *Let $x(t)$ be a vector-valued function with first-order continuous-derivative entries. Then the following descriptor integral-inequality (8) holds for any matrices E and $R = R^T > 0$, and a scalar $\tau > 0$.*

$$-\int_{t-\tau}^t \dot{x}^T(s)E^TRE\dot{x}(s)\mathrm{d}s \le -\frac{1}{\tau}\int_{t-\tau}^t \dot{x}^T(s)\mathrm{d}sE^TRE\int_{t-\tau}^t \dot{x}(s)\mathrm{d}s. \tag{8}$$

4. **Bounded Real Lemmas.** The following theorem gives a new delay-range-dependent BRL of the system (1), which is investigated by a PAM. The BRL will play a key role in solving the H_∞ filtering problem.

Theorem 4.1. *For given scalars $\gamma > 0$, $0 < \tau_m < \tau_M$ and $0 \le d < 1$, the singular time-delay system (1) is admissible with H_∞ performance γ if there exists a positive integer l, matrices $P_0, \cdots, P_p$ and real symmetric positive-definite matrices Q, Q_i and R_i, $i = 1, 2, \cdots, l+1$, such that*

$$P^T(\theta)E = E^TP(\theta) \ge 0, \tag{9}$$

$$\begin{bmatrix} \Gamma(\theta) + \Phi_k & \mathbb{I}_1^TP^T(\theta)B(\theta) & \mathbb{I}_1^TL^T(\theta) & \mathscr{A}^T(\theta)M \\ * & -\gamma^2 I & 0 & B^T(\theta)M \\ * & * & -I & 0 \\ * & * & * & -M \end{bmatrix} < 0, \quad k = 0, 1, \cdots, l-1, \tag{10}$$

$$\forall\theta \in \Omega, \quad \dot{\theta} \in \Omega_d,$$

where δ and τ_i, $i = 0, 1, 2, \cdots, l$ are defined as previously,

$$P(\theta) = P_0 + \theta_1P_1 + \cdots + \theta_pP_p,$$

$$\Gamma(\theta) = \mathscr{A}^T(\theta)P(\theta)\mathbb{I}_1 + \mathbb{I}_1^TP^T(\theta)\mathscr{A}(\theta) + \mathbb{I}_1^T\left(Q + \sum_{i=0}^l Q_{i+1} + \frac{1}{2}E^T\frac{\mathrm{d}P(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}P^T(\theta)}{\mathrm{d}t}E\right)\mathbb{I}_1$$
$$- \sum_{i=0}^l \mathbb{I}_{3i}^TQ_{i+1}\mathbb{I}_{3i} - \mathbb{I}_4^T(E^TR_1E)\mathbb{I}_4 - (1-d)\mathbb{I}_2^TQ\mathbb{I}_2,$$

$$\mathscr{A}(\theta) = \begin{bmatrix} A(\theta) & A_h(\theta) & 0_{m\times(l+1)m} \end{bmatrix}, \quad M = \tau_m^2R_1 + \sum_{i=1}^l \delta R_{i+1},$$

$$\Phi_k = -\frac{1}{\delta}\sum_{\substack{i=1\\ i\neq k+1}}^{l} \mathbb{I}_{5i}^T E^T R_{i+1} E \mathbb{I}_{5i} - \frac{1}{\delta}\mathbb{I}_{6k}^T E^T R_{k+2} E \mathbb{I}_{6k} - \frac{1}{\delta}\mathbb{I}_{7k}^T E^T R_{k+2} E \mathbb{I}_{7k}, \quad k = 0, \cdots, l-1$$

$$\mathbb{I}_1 = \begin{bmatrix} I_m & 0_{m\times(l+2)m}\end{bmatrix}, \quad \mathbb{I}_2 = \begin{bmatrix} 0_m & I_m & 0_{m\times(l+1)m}\end{bmatrix},$$

$$\mathbb{I}_{3i} = \begin{bmatrix} 0_{m\times(i+2)m} & I_m & 0_{m\times(l-i)m}\end{bmatrix}, \quad i = 0, 1, 2, \cdots, l,$$

$$\mathbb{I}_4 = \begin{bmatrix} I_m & 0_m & -I_m & 0_{m\times lm}\end{bmatrix},$$

$$\mathbb{I}_{5i} = \begin{bmatrix} 0_{m\times(i+1)m} & I_m & -I_m & 0_{m\times(l-i)m}\end{bmatrix}, \quad i = 1, 2, \cdots, l,$$

$$\mathbb{I}_{6k} = \begin{bmatrix} 0_m & I_m & 0_{m\times km} & -I_m & 0_{m\times(l-k)m}\end{bmatrix}, \quad k = 0, 1, \cdots, l-1,$$

$$\mathbb{I}_{7k} = \begin{bmatrix} 0_m & I_m & 0_{m\times(k+1)m} & -I_m & 0_{m\times(l-k-1)m}\end{bmatrix}, \quad k = 0, 1, \cdots, l-1.$$

Proof: We first show that the system (1) is regular and impulse-free when $\omega(t) = 0$. It follows from (10) that $\Gamma(\theta) + \Phi_1 < 0$, and hence, the $2m \times 2m$ principal matrix of $\Gamma(\theta) + \Phi_1$ is negative-definite, that is,

$$\begin{aligned}\tilde{\Gamma}(\theta) := &\begin{bmatrix} A^T(\theta) \\ A_h^T(\theta)\end{bmatrix} P(\theta) \begin{bmatrix} I_m & 0_m \end{bmatrix} + \begin{bmatrix} I_m \\ 0_m \end{bmatrix} P^T(\theta) \begin{bmatrix} A(\theta) & A_h(\theta)\end{bmatrix} \\ &+ \begin{bmatrix} I_m \\ 0_m \end{bmatrix}\left(Q + \sum_{i=0}^{l} Q_{i+1} + \frac{1}{2}E^T\frac{\mathrm{d}P(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}P^T(\theta)}{\mathrm{d}t}E - E^T R_1 E\right)\begin{bmatrix} I_m & 0_m \end{bmatrix} \\ &- \begin{bmatrix} 0_m \\ I_m \end{bmatrix}\left((1-d)Q + \frac{2}{\delta}E^T R_3 E\right)\begin{bmatrix} 0_m & I_m \end{bmatrix} < 0.\end{aligned}$$

Furthermore, $\begin{bmatrix} I_m & I_m \end{bmatrix}\tilde{\Gamma}(\theta)\begin{bmatrix} I_m \\ I_m\end{bmatrix} < 0$. Using $0 \le d \le 1$, $Q > 0$ and $Q_i > 0$, $i = 1, \cdots, l+1$, we obtain that

$$\begin{aligned}&(A(\theta) + A_h(\theta))^T P(\theta) + P^T(\theta)(A(\theta) + A_h(\theta)) \\ &+ \frac{1}{2}E^T\frac{\mathrm{d}P(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}P^T(\theta)}{\mathrm{d}t}E - E^T R_1 E - \frac{2}{\delta}E^T R_3 E < 0.\end{aligned} \tag{11}$$

On the other hand, it follows from $\tilde{\Gamma}(\theta) < 0$, $Q > 0$ and $Q_i > 0$, $i = 1, 2, \cdots, l+1$, that

$$A^T(\theta)P(\theta) + P^T(\theta)A(\theta) + \frac{1}{2}E^T\frac{\mathrm{d}P(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}P^T(\theta)}{\mathrm{d}t}E - E^T R_1 E < 0. \tag{12}$$

Due to (11) and (12), by an argument similar to that in the proof of [11, Theorem 1], one can derive that the pairs $(E, A(\theta))$ and $(E, A(\theta) + A_h(\theta))$ are regular and impulse-free for any $\theta \in \Omega$, i.e., the system (1) is regular and impulse-free when $\omega(t) = 0$.

Secondly, we will show that the system (1) is robustly stable when $\omega(t) = 0$.

Define the following Lyapunov-Krasovskii functional:

$$\begin{aligned}V(x_t, \theta) = &x^T(t)E^T P(\theta)x(t) + \int_{t-\tau_t}^{t} x^T(s)Qx(s)\mathrm{d}s + \sum_{i=0}^{l}\int_{t-\tau_i}^{t} x^T(s)Q_{i+1}x(s)\mathrm{d}s \\ &+ \tau_m \int_{t-\tau_m}^{t}\int_{s}^{t} \dot{x}^T(\nu)E^T R_1 E\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}s \\ &+ \sum_{i=1}^{l}\int_{t-\tau_i}^{t-\tau_{i-1}}\int_{s}^{t}\dot{x}^T(\nu)E^T R_{i+1}E\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}s.\end{aligned}$$

It is clear that $V(x_t,\theta) > 0$ for any $t \geq 0$. The derivative of $V(x_t,\theta)$ along the trajectories of the system (1) when $\omega(t) = 0$ satisfies

$$\begin{aligned}\dot{V}(x_t,\theta) \leq & 2\xi^T(t)\mathscr{A}^T(\theta)P(\theta)\mathbb{I}_1\xi(t) + \xi^T(t)\mathbb{I}_1^T\left(Q + E^T\frac{\mathrm{d}P(\theta)}{\mathrm{d}t}\right)\mathbb{I}_1\xi(t) \\ & + \xi^T(t)\mathscr{A}^T(\theta)M\mathscr{A}(\theta)\xi(t) - (1-d)\xi^T(t)\mathbb{I}_2^TQ\mathbb{I}_2\xi(t) \\ & + \xi^T(t)\mathbb{I}_1^T\left(\sum_{i=0}^{l}Q_{i+1}\right)\mathbb{I}_1\xi(t) - \sum_{i=0}^{l}\xi^T(t)\mathbb{I}_{3i}^TQ_{i+1}\mathbb{I}_{3i}\xi(t) \\ & - \tau_m\int_{t-\tau_m}^{t}\dot{x}^T(s)E^TR_1E\dot{x}(s)\mathrm{d}s - \sum_{i=1}^{l}\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^T(s)E^TR_{i+1}E\dot{x}(s)\mathrm{d}s,\end{aligned} \tag{13}$$

where $\xi^T(t) = [x^T(t) \ \ x^T(t-\tau_t) \ \ x^T(t-\tau_m) \ \ x^T(t-\tau_1) \ \ \cdots \ \ x^T(t-\tau_{l-1}) \ \ x^T(t-\tau_M)]$.

By Lemma 3.2, we have

$$\begin{aligned}-\tau_m\int_{t-\tau_m}^{t}\dot{x}^T(s)E^TR_1E\dot{x}(s)\mathrm{d}s \leq & -\left(\int_{t-\tau_m}^{t}\dot{x}^T(s)\mathrm{d}s\right)E^TR_1E\left(\int_{t-\tau_m}^{t}\dot{x}(s)\mathrm{d}s\right) \\ = & \,\xi^T(t)\mathbb{I}_4^T(-E^TR_1E)\mathbb{I}_4\xi(t).\end{aligned} \tag{14}$$

The combination of (13) and (14) gives that

$$\dot{V}(x_t,\theta) \leq \xi^T(t)\left(\Gamma(\theta) + \mathscr{A}^T(\theta)M\mathscr{A}(\theta)\right)\xi(t) - \sum_{i=1}^{l}\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^T(s)E^TR_{i+1}E\dot{x}(s)\mathrm{d}s. \tag{15}$$

When $\tau_t \in [\tau_k, \tau_{k+1}]$ for some positive integer k with $0 \leq k \leq l-1$. It follows by Lemma 3.2 that

$$-\sum_{\substack{i=1\\ i\neq k+1}}^{l}\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^T(s)E^TR_{i+1}E\dot{x}(s)\mathrm{d}s \leq -\frac{1}{\delta}\sum_{\substack{i=1\\ i\neq k+1}}^{l}\xi^T(t)\mathbb{I}_{5i}^TE^TR_{i+1}E\mathbb{I}_{5i}\xi(t) \tag{16}$$

and

$$\begin{aligned}& -\int_{t-\tau_{k+1}}^{t-\tau_k}\dot{x}^T(s)E^TR_{k+2}E\dot{x}(s)\mathrm{d}s \\ = & -\int_{t-\tau_t}^{t-\tau_k}\dot{x}^T(s)E^TR_{k+2}E\dot{x}(s)\mathrm{d}s - \int_{t-\tau_{k+1}}^{t-\tau_t}\dot{x}^T(s)E^TR_{k+2}E\dot{x}(s)\mathrm{d}s \\ \leq & -\frac{1}{\delta}\xi^T(t)\mathbb{I}_{6k}^TE^TR_{k+2}E\mathbb{I}_{6k}\xi(t) - \frac{1}{\delta}\xi^T(t)\mathbb{I}_{7k}^TE^TR_{k+2}E\mathbb{I}_{7k}\xi(t).\end{aligned} \tag{17}$$

So, from (15)-(17),

$$\dot{V}(x_t,\theta) \leq \xi^T(t)\left(\Gamma(\theta) + \mathscr{A}^T(\theta)M\mathscr{A}(\theta) + \Phi_k\right)\xi(t).$$

This, together with (10) and Schur complement lemma, implies that $\dot{V}(x_t,\theta) < 0$. Again by an argument similar to that in the proof of [11, Theorem 1], we obtain that the system (1) with $\omega(t) = 0$ is robustly stable.

Finally, we prove that the system (1) has the H_∞ performance γ under the zero initial conditions. The derivative of $V(x_t,\theta)$ along the trajectories of the system (1) satisfies

$$\dot{V}(x_t,\theta) \leq \tilde{\xi}^T(t)\left(\Pi_k(\theta) + \mathscr{C}^T(\theta)M\mathscr{C}(\theta)\right)\tilde{\xi}(t), \quad k = 0,1,2,\cdots,l-1,$$

with

$$\tilde{\xi}(t) = \begin{bmatrix}\xi(t)\\ \omega(t)\end{bmatrix}, \quad \Pi_k(\theta) = \begin{bmatrix}\Gamma(\theta)+\Phi_k & \mathbb{I}_1^TP^T(\theta)B(\theta)\\ * & 0\end{bmatrix}, \quad k = 0,1,\cdots,l-1,$$

$$\mathscr{C}(\theta) = \begin{bmatrix} \mathscr{A}(\theta) & B(\theta) \end{bmatrix}.$$

Employing (10) and Schur complement lemma, we have

$$\Theta_k(\theta) := \tilde{\Psi}_k(\theta) + \mathscr{C}^T(\theta))M\mathscr{C}(\theta) + \Gamma_2^T(\theta)\Gamma_2(\theta) < 0, \quad k = 0, 1, \cdots, l-1,$$

where

$$\tilde{\Psi}_k(\theta) = \begin{bmatrix} \Gamma(\theta) + \Phi_k & \mathbb{I}_1^T P^T(\theta)B(\theta) \\ * & -\gamma^2 I \end{bmatrix}, \quad \Gamma_2(\theta) = \begin{bmatrix} L(\theta)\mathbb{I}_1 & 0 \end{bmatrix}.$$

So,

$$\dot{V}(x_t, \theta) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) = \xi^T(t)\Theta_k(\theta)\xi(t) \le 0,$$

and hence,

$$J = \int_0^\infty [z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t)]\mathrm{d}t \le \int_0^\infty [z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_t, \theta)]\mathrm{d}t \le 0$$

when $\phi(t) \equiv 0$, $t \in [-\tau_M, 0]$, i.e., $\|z(t)\|_2 \le \gamma\|\omega(t)\|_2$. This completes the proof.

Remark 4.1. *From the proof of Theorem 4.1, we can see that the conditions for guaranteeing $\dot{V}(x_t, t) < 0$ are derived by checking the delay variation in the subintervals. This makes us employ more information on time delay and thus may derive criteria with less conservativeness.*

When $\theta \neq 0$, it is difficult to solve the inequalities (9) and (10), since they contain infinite number of inequalities. The following theorem reduces infinite number of inequalities in Theorem 4.1 to finitely many ones, which will be simple to solve.

Theorem 4.2. *For given scalars $\gamma > 0$, $0 < \tau_m < \tau_M$ and $0 \le d < 1$, the singular time-delay LPV system (1) is admissible with H_∞ performance γ if there exists a positive integer l, matrices $P_0, \cdots, P_p$ and real symmetric positive-definite matrices Q, Q_i and R_i, $i = 1, 2, \cdots, l+1$, such that*

$$E^T P(\omega) \ge 0, \quad \forall \omega \in \mathcal{V}, \tag{18}$$

$$E^T P_i = P_i^T E, \quad i = 1, \cdots, p, \tag{19}$$

$$\begin{bmatrix} A_i^T P_i + P_i^T A_i & P_i^T A_{hi} & P_i^T B_i \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \ge 0, \quad i = 1, \cdots, p, \tag{20}$$

$$\begin{bmatrix} \Gamma(\omega) + \Phi_k & \mathbb{I}_1^T P^T(\omega)B(\omega) & \mathbb{I}_1^T L^T(\omega) & \mathscr{A}^T(\omega)M \\ * & -\gamma^2 I & 0 & B^T(\omega)M \\ * & * & -I & 0 \\ * & * & * & -M \end{bmatrix} < 0, \quad k = 0, 1, \cdots, l-1, \tag{21}$$

$$\forall \omega \in \mathcal{V}, \quad r \in \mathcal{V}_d,$$

where

$$P(\omega) = P_0 + \omega_1 P_1 + \cdots + \omega_p P_p, \quad \mathscr{A}(\omega) = \begin{bmatrix} A(\omega) & A_h(\omega) & 0_{m\times(l+1)m} \end{bmatrix},$$

$$\Gamma(\omega) = \mathscr{A}^T(\omega)P(\omega)\mathbb{I}_1 + \mathbb{I}_1^T P^T(\omega)\mathscr{A}(\omega) + \mathbb{I}_1^T\Bigg(Q + \sum_{i=0}^{l} Q_{i+1} + \frac{1}{2}E^T(P(r) - P_0)$$
$$+ \frac{1}{2}(P(r) - P_0)^T E\Bigg)\mathbb{I}_1 - \sum_{i=0}^{l} \mathbb{I}_{3i}^T Q_{i+1}\mathbb{I}_{3i} - \mathbb{I}_4^T(E^T R_1 E)\mathbb{I}_4 - (1-d)\mathbb{I}_2^T Q\mathbb{I}_2$$

and $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_{3i}$, $\mathbb{I}_4$, M and Φ_k, $k = 0, 1, \cdots, l-1$, are defined as in Theorem 4.1.

As a by-product of the BRL obtained above, the following corollary can be easily derived by following the proof of Theorems 4.1 and 4.2, which offer a less conservative stability criterion for the systems (5).

Corollary 4.1. *For given scalars d, $0 < \tau_m < \tau_M$ and $0 \le d < 1$, the singular system (5) with τ_t satisfying (2) is admissible if there exist matrices $P_0, \cdots, P_p$, $Q > 0$, $Q_i > 0$ and $R_i > 0$, $i = 1, 2, \cdots, l+1$ with appropriate dimensions satisfying (18)-(20) and*

$$\Gamma(\omega) + \Phi_k + \mathscr{A}(\omega)^T M \mathscr{A}(\omega) < 0, \quad k = 0, 1, \cdots, l-1, \quad \forall \omega \in \mathcal{V}, \quad r \in \mathcal{V}_d, \tag{22}$$

where $\Gamma(\omega)$, $\mathscr{A}(\omega)$, M and Φ_k, $k = 0, 1, \cdots, l-1$, are defined as in Theorem 4.2.

Remark 4.2. *It will be shown by several numerical examples in Section 6 that, the criteria above can lead to less conservative results as l increases. However, it can be seen that, the stability conditions becomes more complicated as l increases, since the number of LMIs in Theorem 4.2 and Corollary 4.1 to be solved equals to l.*

5. $\boldsymbol{H_\infty}$ **Filtering.** In this section, we will design an H_∞ filtering of the form (6) for the system (1). The following theorem presents a sufficient condition of the existence of the filter for systems (1).

Theorem 5.1. *For given scalars $\gamma > 0$, $0 < \tau_m < \tau_M$ and $0 \le d < 1$, the filtering error system (7) is admissible with H_∞ performance γ if there exists a positive integer l, matrices X_i, $\bar{A}_{fi}$, C_{fi}, $i = 0, 1, \cdots, p$, Y with $Y \ne 0$, $\bar{B}_f$ and real symmetric positive-definite matrices $\tilde{Q}$, $\tilde{Q}_i$ and $\tilde{R}_i$, $i = 1, 2, \cdots, l+1$, such that*

$$E^T Y = Y^T E \ge 0, \tag{23}$$

$$E^T (X(\theta) - Y) = (X(\theta) - Y)^T E \ge 0, \tag{24}$$

$$\begin{bmatrix} \tilde{\Gamma}(\theta) + \tilde{\Phi}_k & \mathbb{I}_1^T \tilde{\Omega}_{12}(\theta) & \mathbb{I}_1^T \tilde{\Omega}_{13}(\theta) & \hat{\mathbb{I}}_1^T \tilde{\Omega}_{14}(\theta) \\ * & -\gamma^2 I & 0 & B^T(\theta)\tilde{M} \\ * & * & -I & 0 \\ * & * & * & -\tilde{M} \end{bmatrix} < 0, \quad k = 0, \cdots, l-1, \tag{25}$$

$$\forall \theta \in \Omega, \quad \dot{\theta} \in \Omega_d,$$

where

$$\tilde{\Gamma}(\theta) = \left[\begin{array}{cccc|c} \Omega_{11}(\theta) & \Omega_{12}(\theta) & \Omega_{13}(\theta) & E^T \tilde{R}_1 E & 0 \\ * & \Omega_{22}(\theta) & \Omega_{23}(\theta) & E^T \tilde{R}_1 E & \vdots \\ \vdots & & -(1-d)\tilde{Q} & 0 & \vdots \\ * & \cdots & * & -E^T \tilde{R}_1 E - \tilde{Q}_1 & 0 \\ \hline * & \cdots & \cdots & * & -\mathrm{diag}(\tilde{Q}_2, \cdots, \tilde{Q}_{l+1}) \end{array}\right],$$

$$\Omega_{11}(\theta) = A^T(\theta)\left(X(\theta) - Y\right) + \left(X(\theta) - Y\right)^T A(\theta) + \frac{1}{2} E^T \frac{\mathrm{d}X(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}X^T(\theta)}{\mathrm{d}t} E + \tilde{Q} + \sum_{i=1}^{l} \tilde{Q}_{i+1} - E^T \tilde{R}_1 E,$$

$$\Omega_{12}(\theta) = A^T(\theta)X(\theta) - C^T(\theta)\bar{B}_f^T + \left(X(\theta) - Y\right)^T A(\theta) - \bar{A}_f^T(\theta) - E^T \tilde{R}_1 E + E^T \frac{\mathrm{d}X(\theta)}{\mathrm{d}t},$$

$$\Omega_{22}(\theta) = A^T(\theta)X(\theta) + X^T(\theta)A(\theta) - C^T(\theta)\bar{B}_f^T - \bar{B}_f C(\theta) - E^T \tilde{R}_1 E + \frac{1}{2} E^T \frac{\mathrm{d}X(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}X^T(\theta)}{\mathrm{d}t} E,$$

$$\Omega_{13}(\theta) = (X(\theta) - Y)^T A_h(\theta), \quad \Omega_{23}(\theta) = X^T(\theta) A_h(\theta) - \bar{B}_f C_h(\theta),$$

$$\tilde{\Omega}_{12}(\theta) = \begin{bmatrix} (X(\theta) - Y)^T B(\theta) \\ X^T(\theta) B(\theta) - \bar{B}_f D(\theta) \end{bmatrix}, \quad \tilde{\Omega}_{13}(\theta) = \begin{bmatrix} L^T(\theta) - C_f^T(\theta) \\ L^T(\theta) \end{bmatrix},$$

$$\tilde{\Omega}_{14}(\theta) = \begin{bmatrix} \tilde{M}^T A(\theta) & \tilde{M}^T A(\theta) & \tilde{M}^T A_h(\theta) & \tilde{M}^T A_h(\theta) \end{bmatrix}^T, \quad \tilde{M} = \tau_m^2 \tilde{R}_1 + \sum_{i=1}^{l} \delta \tilde{R}_{i+1},$$

$$\tilde{\Phi}_k = -\frac{1}{\delta} \sum_{\substack{i=1 \\ i \neq k+1}}^{l} \tilde{\mathbb{I}}_{5i}^T E^T \tilde{R}_{i+1} E \tilde{\mathbb{I}}_{5i} - \frac{1}{\delta} \tilde{\mathbb{I}}_{6k}^T E^T \tilde{R}_{k+2} E \tilde{\mathbb{I}}_{6k} - \frac{1}{\delta} \tilde{\mathbb{I}}_{7k}^T E^T \tilde{R}_{k+2} E \tilde{\mathbb{I}}_{7k},$$

$$k = 0, 1, \cdots, l-1,$$

$$\tilde{\mathbb{I}}_1 = \begin{bmatrix} I_m & I_m & 0_{m \times (l+2)m} \end{bmatrix}, \quad \hat{\mathbb{I}}_1 = \begin{bmatrix} I_m & I_m & I_m & I_m & 0_{m \times lm} \end{bmatrix},$$

$$\tilde{\mathbb{I}}_2 = \begin{bmatrix} 0_{m \times 2m} & I_m & 0_{m \times (l+1)m} \end{bmatrix}, \quad \tilde{\mathbb{I}}_{5i} = \begin{bmatrix} 0_{m \times (i+2)m} & I_m & -I_m & 0_{m \times (l-i)m} \end{bmatrix},$$

$$\tilde{\mathbb{I}}_{6k} = \begin{bmatrix} 0_{m \times 2m} & I_m & 0_{m \times km} & -I_m & 0_{m \times (l-k)m} \end{bmatrix}, \quad k = 0, 1, \cdots, l-1,$$

$$\tilde{\mathbb{I}}_{7k} = \begin{bmatrix} 0_{m \times 2m} & I_m & 0_{m \times (k+1)m} & -I_m & 0_{m \times (l-k-1)m} \end{bmatrix}, \quad k = 0, 1, \cdots, l-1,$$

$$X(\theta) = X_0 + \theta_1 X_1 + \cdots + \theta_p X_p, \quad \bar{A}_f(\theta) = \bar{A}_{f0} + \theta_1 \bar{A}_{f1} + \cdots + \theta_p \bar{A}_{fp},$$

$$C_f(\theta) = \bar{C}_{f0} + \theta_1 \bar{C}_{f1} + \cdots + \theta_p C_{fp}.$$

In this case, the desired filter is given by

$$\begin{cases} E\dot{\hat{x}}(t) = Y^{-T} \bar{A}_f(\theta) \hat{x}(t) + Y^{-T} \bar{B}_f y(t) \\ \hat{z}(t) = C_f(\theta) \hat{x}(t) \end{cases}$$

Proof: Let $J = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}$. Then, the systems (7) can be written as

$$\begin{cases} \tilde{E}\dot{\tilde{x}}(t) = \tilde{A}(\theta)\tilde{x}(t) + \tilde{A}_h(\theta)\tilde{x}(t - \tau_t) + \bar{B}(\theta)\omega(t), \\ \bar{z}(t) = \tilde{L}(\theta)\tilde{x}(t), \end{cases} \tag{26}$$

where

$$\tilde{E} = \begin{bmatrix} E & E \\ E & 0 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \hat{x}(t) \\ x(t) - \hat{x}(t) \end{bmatrix}, \quad \tilde{A}(\theta) = \begin{bmatrix} A(\theta) & A(\theta) \\ B_f C(\theta) + A_f(\theta) & B_f C(\theta) \end{bmatrix},$$

$$\tilde{A}_h(\theta) = \begin{bmatrix} A_h(\theta) & A_h(\theta) \\ B_f C_h(\theta) & B_f C_h(\theta) \end{bmatrix}, \quad \tilde{L}(\theta) = \begin{bmatrix} L(\theta) - C_f(\theta) & L(\theta) \end{bmatrix}.$$

Define $\tilde{P}(\theta) = \begin{bmatrix} X(\theta) - Y & X(\theta) \\ 0 & -Y \end{bmatrix}$, from (23) and (24), we have

$$\tilde{E}^T \tilde{P}(\theta) = \begin{bmatrix} E^T(X(\theta) - Y) & E^T(X(\theta) - Y) \\ E^T(X(\theta) - Y) & E^T X(\theta) \end{bmatrix} = \tilde{P}^T(\theta) \tilde{E} \geq 0, \quad \forall \theta \in \Omega. \tag{27}$$

On the other hand, it from (25) that

$$\begin{bmatrix} \Upsilon_1(\theta) + \tilde{\Phi}_k & \tilde{\mathbb{I}}_1^T \tilde{P}^T(\theta) \bar{B}(\theta) & \tilde{\mathbb{I}}_1^T \tilde{L}^T(\theta) & \tilde{\mathbb{I}}_1^T \Upsilon_2(\theta) \\ * & -\gamma^2 I & 0 & B^T(\theta) \tilde{M} \\ * & * & -I & 0 \\ * & * & * & -\tilde{M} \end{bmatrix} < 0, \quad k = 0, \cdots, l-1, \tag{28}$$

$$\forall \theta \in \Omega, \quad \dot{\theta} \in \Omega_d,$$

where

$$\Upsilon_1(\theta) = \left[\begin{array}{ccc|c} \Upsilon_{11}(\theta) & \Upsilon_{12}(\theta) & \tilde{E}^T H^T \tilde{R}_1 E & 0 \\ \vdots & -(1-d)\tilde{Q} & 0 & \vdots \\ * & * & -E^T \tilde{R}_1 E - \tilde{Q}_1 & 0 \\ \hline * & \cdots & * & -\mathrm{diag}(\tilde{Q}_2, \cdots, \tilde{Q}_{l+1}) \end{array}\right],$$

$$\Upsilon_{11}(\theta) = \tilde{A}^T(\theta)\tilde{P}(\theta) + \tilde{P}^T(\theta)\tilde{A}(\theta) - \tilde{E}^T H^T \tilde{R}_1 H \tilde{E} + \frac{1}{2}\tilde{E}^T \frac{\mathrm{d}\tilde{P}(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}\tilde{P}^T(\theta)}{\mathrm{d}t}\tilde{E} + H^T\left(\tilde{Q} + \sum_{i=1}^{l}\tilde{Q}_{i+1}\right)H,$$

$$\Upsilon_{12}(\theta) = \tilde{P}^T(\theta)\tilde{A}_h(\theta)H^T, \quad \Upsilon_2(\theta) = \begin{bmatrix} \tilde{A}^T(\theta)H^T\tilde{M} \\ \tilde{A}_h^T(\theta)H^T\tilde{M} \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Clearly, from (28), one can always find a scalar $\sigma > 0$ such that

$$\begin{bmatrix} \hat{\Gamma}(\theta) + \Phi_k & \mathbb{I}_1^T \tilde{P}^T(\theta)\bar{B}(\theta) & \mathbb{I}_1^T \tilde{L}^T(\theta) & \tilde{\mathscr{A}}^T M \\ * & -\gamma^2 I & 0 & \bar{B}^T(\theta) M \\ * & * & -I & 0 \\ * & * & * & -M \end{bmatrix} < 0, \quad k = 0, 1, \cdots, l-1, \tag{29}$$

$$\forall \theta \in \Omega, \quad \dot{\theta} \in \Omega_d,$$

where

$$\hat{\Gamma}(\theta) = \tilde{\mathscr{A}}^T(\theta)\tilde{P}(\theta)\mathbb{I}_1 + \mathbb{I}_1^T\tilde{P}^T(\theta)\tilde{\mathscr{A}}(\theta) + \mathbb{I}_1^T\left(Q + \sum_{i=0}^{l} Q_{i+1} + \frac{1}{2}\tilde{E}^T\frac{\mathrm{d}\tilde{P}(\theta)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}\tilde{P}^T(\theta)}{\mathrm{d}t}\tilde{E}\right)\mathbb{I}_1 - \sum_{i=0}^{l}\mathbb{I}_{3i}^T Q_{i+1}\mathbb{I}_{3i} - \mathbb{I}_4^T(\tilde{E}^T R_1 \tilde{E})\mathbb{I}_4 - (1-d)\mathbb{I}_2^T Q \mathbb{I}_2,$$

$$\tilde{\mathscr{A}}(\theta) = \begin{bmatrix} \tilde{A}(\theta) & \tilde{A}_h(\theta) & 0_{2m\times 2(l+1)m} \end{bmatrix},$$

$$Q = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \sigma I \end{bmatrix} > 0, \quad Q_i = \begin{bmatrix} \tilde{Q}_i & 0 \\ 0 & \sigma I \end{bmatrix} > 0, \quad R_i = \begin{bmatrix} \tilde{R}_i & 0 \\ 0 & \sigma I \end{bmatrix} > 0, \quad i = 1, 2, \cdots, l+1,$$

and Φ_k and M are defined in Theorem 4.1. Therefore, by (27) and (29), the error system (26) (i.e., (7)) is admissible with H_∞ performance γ for any $\theta(t)$ satisfying (3) and (4). This completes the proof.

Theorem 5.2. *For given scalars $\gamma > 0$, $0 < \tau_m < \tau_M$ and $0 \le d < 1$, the filtering error system (7) is admissible with H_∞ performance γ if there exists a positive integer l, matrices X_i, $\bar{A}_{fi}$, C_{fi}, $i = 0, 1, \cdots, p$, Y with $Y \neq 0$, $\bar{B}_f$ and real symmetric positive-definite matrices $\tilde{Q}$, $\tilde{Q}_i$ and $\tilde{R}_i$, $i = 1, 2, \cdots, l+1$, such that*

$$E^T(X(\omega) - Y) \ge 0, \quad \forall \omega \in \mathcal{V} \tag{30}$$

$$E^T Y = Y^T E \ge 0, \tag{31}$$

$$E^T(X_i - Y) = (X_i - Y)^T E, \quad i = 1, 2, \cdots, p, \tag{32}$$

$$\begin{bmatrix} A_i^T X_i + X_i^T A_i & X_i^T A_{hi} & X_i^T B_i \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \ge 0, \quad i = 1, \cdots, p, \tag{33}$$

$$\begin{bmatrix} \tilde{\Gamma}(\omega)+\tilde{\Phi}_k & \tilde{\mathbb{I}}_1^T\tilde{\Omega}(\omega)_{12} & \tilde{\mathbb{I}}_1^T\tilde{\Omega}(\omega)_{13} & \tilde{\mathbb{I}}_1^T\tilde{\Omega}(\omega)_{14} \\ * & -\gamma^2 I & 0 & B^T(\omega)\tilde{M} \\ * & * & -I & 0 \\ * & * & * & -\tilde{M} \end{bmatrix} < 0, \quad k=0,1,\cdots,l-1, \tag{34}$$
$$\forall\omega\in\mathcal{V}, \quad r\in\mathcal{V}_d,$$

where

$$\tilde{\Gamma}(\omega) = \left[\begin{array}{cccc|c} \Omega(\omega)_{11} & \Omega(\omega)_{12} & \Omega(\omega)_{13} & E^T\tilde{R}_1E & 0 \\ * & \Omega(\omega)_{22} & \Omega(\omega)_{23} & E^T\tilde{R}_1E & \vdots \\ \vdots & \ddots & -(1-d)\tilde{Q} & 0 & \vdots \\ * & \cdots & * & -E^T\tilde{R}_1E-\tilde{Q}_1 & 0 \\ \hline * & \cdots & \cdots & * & -\mathrm{diag}(\tilde{Q}_2,\cdots,\tilde{Q}_{l+1}) \end{array}\right],$$

$$\begin{aligned} \Omega(\omega)_{11} &= A^T(\omega)\left(X(\omega)-Y\right)+\left(X(\omega)-Y\right)^T A(\omega)+\frac{1}{2}E^T\left(X(r)-X_0\right) \\ &\quad +\frac{1}{2}\left(X(r)-X_0\right)^T E+\tilde{Q}+\sum_{i=1}^{l}\tilde{Q}_{i+1}-E^T\tilde{R}_1E, \\ \Omega(\omega)_{12} &= A^T(\omega)X(\omega)-C^T(\omega)\bar{B}_f^T+\left(X(\omega)-Y\right)^T A(\omega) \\ &\quad -\bar{A}_f^T(\omega)+E^T\left(X(r)-X_0\right)-E^T\tilde{R}_1E, \\ \Omega(\omega)_{22} &= A^T(\omega)X(\omega)+X^T(\omega)A(\omega)-C^T(\omega)\bar{B}_f^T-\bar{B}_fC(\omega) \\ &\quad +\frac{1}{2}E^T(X(r)-X_0)+\frac{1}{2}(X(r)-X_0)^TE-E^T\tilde{R}_1E, \end{aligned}$$

$$\Omega(\omega)_{13} = \left(X(\omega)-Y\right)^T A_h(\omega), \quad \Omega(\omega)_{23} = X^T(\omega)A_h(\omega)-\bar{B}_fC_h(\omega),$$

$$\tilde{\Omega}(\omega)_{12} = \begin{bmatrix} \left(X(\omega)-Y\right)^T B(\omega) \\ X(\omega)^TB(\omega)-\bar{B}_fD(\omega) \end{bmatrix}, \quad \tilde{\Omega}(\omega)_{13} = \begin{bmatrix} L^T(\omega)-C_f^T(\omega) \\ L^T(\omega) \end{bmatrix},$$

$$\tilde{\Omega}(\omega)_{14} = \begin{bmatrix} \tilde{M}^TA(\omega) & \tilde{M}^TA(\omega) & \tilde{M}^TA_h(\omega) & \tilde{M}^TA_h(\omega) \end{bmatrix}^T$$

and $\tilde{\Phi}_k$, $\tilde{M}$ are defined as in Theorem 5.1.

Remark 5.1. *Based on the method given in Theorem 5.2, the minimal disturbance level γ can be obtained by solving the following optimization problem:*

$$\begin{cases} \min\limits_{\substack{X_i,Y,\bar{A}_{fi},C_{fi},i=1,2,\cdots,p,\\ \bar{B}_f,Q,Q_i \text{ and } R_i,i=1,2,\cdots,l+1,}} \mu \\ \text{s.t. } \textit{LMIs (30)-(34) with } \mu=\gamma^2. \end{cases}$$

Remark 5.2. *The positive semi-definite conditions (30), (31) and (33) can be solved by using YALMIP toolbox of MATLAB without matrix decompositions.*

6. **Numerical Examples.** In this section, we give some examples to demonstrate the effectiveness of the proposed method.

Example 6.1. *Consider the following singular time-delay system:*

$$\begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} -13.1 & -13.7 \\ -15.4 & -23.8 \end{bmatrix}x(t)+\begin{bmatrix} -18.6 & -10.4 \\ -25.2 & -16.8 \end{bmatrix}x(t-\tau_t).$$

In this example, we choose $d=0.2$. The maximum of allowed time-delay τ_M for different $\tau_m>0$ as judged by criteria in [17, 18] *and Corollary 4.1 in this paper, are shown in Table 1. The numerical results show that the approach proposed in this paper may be less conservative than ones in* [17, 18].

TABLE 1. Allowable upper bound of τ_M for different τ_m with $d = 0.2$

τ_m	0.5	1	1.5	2
[17, Theorem 1]	3.040	3.041	3.044	3.049
[18, Corollary 1]	3.199	3.362	3.439	3.407
Corollary 4.1 ($l = 3$)	4.353	4.356	4.361	4.367
Corollary 4.1 ($l = 4$)	4.360	4.364	4.370	4.379
Corollary 4.1 ($l = 5$)	4.366	4.372	4.379	4.390
Corollary 4.1 ($l = 6$)	4.373	4.380	4.388	4.401

TABLE 2. Allowable upper bound of τ_M for different d

	$d = 0.1$	$d = 0.3$
[20, Theorem 1]	0.9745	0.9378
[11, Corollary 4.1]	0.9745	0.9378
Corollary 4.1 ($l = 3$)	0.9761	0.9552
Corollary 4.1 ($l = 4$)	0.9761	0.9564
Corollary 4.1 ($l = 5$)	0.9762	0.9573

TABLE 3. The achieved H_∞ performance γ for different τ_m with $d = 0.5$ for Example 6.3

τ_m	0.1	0.3	0.5	0.8	1
[17, Theorem 1]	6.25	5.73	5.19	4.33	3.74
[18, Corollary 1]	2.17	1.54	1.37	1.32	1.36
Theorem 4.2 ($l = 3$)	0.70	0.59	0.49	0.38	0.32

Example 6.2. *Consider the following singular time-delay system:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1.1 & 1 \\ -0.1 & -0.7 \end{bmatrix} x(t - \tau_t).$$

In this example, we choose $\tau_m = 0$. The maximum of allowed time-delay τ_M for different $d > 0$ as judged by the criteria in [11, 20] *and Corollary 4.1 in this paper, are shown in Table 2. The numerical results show that the approach proposed in this paper may be less conservative than ones in* [11, 20].

Example 6.3. *Consider the following singular time-delay system:*

$$\begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -13.1 & -13.7 \\ -15.4 & -23.8 \end{bmatrix} x(t) + \begin{bmatrix} -18.6 & -10.4 \\ -25.2 & -16.8 \end{bmatrix} x(t - \tau_t) + \begin{bmatrix} 1.9 \\ 1.8 \end{bmatrix} \omega(t),$$
$$z(t) = \begin{bmatrix} -1 & -1.1 \end{bmatrix} x(t).$$

In this example, let $d = 0.5$ and $\tau_M = 2$, the achieved H_∞ performance γ, obtained by the criteria in [17, 18] *and Theorem 4.2 in this paper for different τ_m, are shown in Table 3. The numerical results show that the approach proposed in this paper are less conservative than ones in* [17, 18].

Example 6.4. *To demonstrate the effectiveness of Theorem 5.1, consider the singular time-varying delay LPV systems (1) with parameters as follows:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(\theta(t)) = \begin{bmatrix} -1 + 0.3\theta(t) & \theta(t) \\ 0.1 & -2 - \theta(t) \end{bmatrix}, \quad A_h(\theta(t)) = \begin{bmatrix} 0 & 0.3 \\ 0 & -0.5 \end{bmatrix},$$

$$B(\theta(t)) = \begin{bmatrix} -1 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}, \quad C(\theta(t)) = \begin{bmatrix} -0.5 + 0.3\theta(t) & 1 \\ 0 & -1 + 0.1\theta(t) \end{bmatrix},$$
$$C_h(\theta(t)) = \begin{bmatrix} 0.1\theta(t) & 0.3 \\ 0.1\theta(t) & 0.5 \end{bmatrix}, \quad D(\theta(t)) = \begin{bmatrix} 1 + \theta(t) & -1 + 0.1\theta(t) \\ 0 & 0.2 + 0.2\theta \end{bmatrix},$$
$$L(\theta(t)) = \begin{bmatrix} 0.1\theta(t) & -1 \\ -0.1 + 0.4\theta(t) & 0.2 \end{bmatrix},$$

$\theta(t) \in [-1, 1]$, $\dot{\theta}(t) \in [-1, 1]$, $\tau_m = 0.1$, $\tau_M = 0.3$, $d = 0.3$, and initial function $\phi(t) = [1 \;\; 0.1]^T$, $t \in [-0.3, 0]$. When $\gamma = 1.21$ we can obtain feasible solutions to the inequalities in Theorem 5.2 as follows:

$$X_0 = \begin{bmatrix} 0.6158 & 0 \\ 0.3018 & 2.2855 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0.4559 & 0 \\ 0.3163 & 0.4364 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.4559 & 0 \\ 0.3163 & 0.5314 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.2009 & 0.0152 \\ 0.0152 & 0.4928 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1.2736 & 0 \\ 0 & 1.2707 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0125 & 0.0039 \\ 0.0039 & 0.0022 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 0.0135 & 0.0039 \\ 0.0039 & 0.0022 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 2.1297 & 1.6119 \\ 1.6119 & 21.3191 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.0004 & 0.0003 \\ 0.0003 & 2.0102 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 0.0005 & 0.0004 \\ 0.0004 & 2.0102 \end{bmatrix}, \quad \bar{A}_{f0} = \begin{bmatrix} -1.9762 & -0.7368 \\ -0.1113 & -4.5504 \end{bmatrix}, \quad \bar{A}_{f1} = \begin{bmatrix} -1.2360 & -0.3090 \\ -0.1113 & -4.0920 \end{bmatrix},$$
$$\bar{B}_f = \begin{bmatrix} 0.7357 & 0.6206 \\ -0.0050 & 0.5606 \end{bmatrix}, \quad \bar{C}_{f0} = \begin{bmatrix} -2.1526 & -9.8209 \\ -1.3927 & -0.2233 \end{bmatrix}, \quad \bar{C}_{f1} = \begin{bmatrix} 2.2526 & 8.8209 \\ 1.6927 & 0.4233 \end{bmatrix},$$

and a desired filter by Theorem 5.2, is obtained as

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}}(t) &= \begin{bmatrix} -4.3351 - 2.71150\theta(t) & -1.6164 - 0.6778\theta(t) \\ 2.3707 + 1.4043\theta(t) & -7.6007 - 7.2967\theta(t) \end{bmatrix} \hat{x}(t) \\ &\quad + \begin{bmatrix} 1.6139 & 1.3614 \\ -0.9700 & 0.2446 \end{bmatrix} y(t), \qquad (35) \\ \hat{z}(t) &= \begin{bmatrix} -4.7221 + 4.94140\theta(t) & -21.5441 + 19.35040\theta(t) \\ 0.1897 + 0.2443\theta(t) & 12.4020 - 10.72000\theta(t) \end{bmatrix} \hat{x}(t). \end{aligned}$$

When $\theta(t) = \sin t$ and $\tau_t = 0.1 + 0.2|\sin t|$, and the disturbance input is given as $\omega(t) = \begin{bmatrix} 0.1\sin t & 0.1e^{0.5\sin t} \end{bmatrix}^T$, the state response of the consider singular time-delay LPV system (1) are shown in Figure 1, which indicates that the considered singular system is robustly stable. Figure 2 shows the simulation result of the state response of the designed filter (35). Figure 3 is the simulation results of the error response of $z(t) - \hat{z}(t)$. From these simulation results, it can be seen that the designed H_∞ filter satisfies the specified requirements.

7. **Conclusions.** In this paper, the problem of the robust H_∞ filtering for singular linear parameter varying systems with time variant delays is established. A delay-range-dependent bounded real lemma is presented to ensure the singular time-delay LPV systems to be admissible and satisfy a prescribed H_∞ performance level. The BRL is derived by constructing a delay-dependent and parameter-dependent Lyapunov-Krasovskii functional. With a piecewise analysis method, the results reduce the conservativeness by considering the delay varies in the subintervals. Furthermore, a sufficient condition for the existence of such a filter is proposed in terms of linear matrix inequalities. Finally, several numerical examples show the effectiveness of the approach proposed in this paper.

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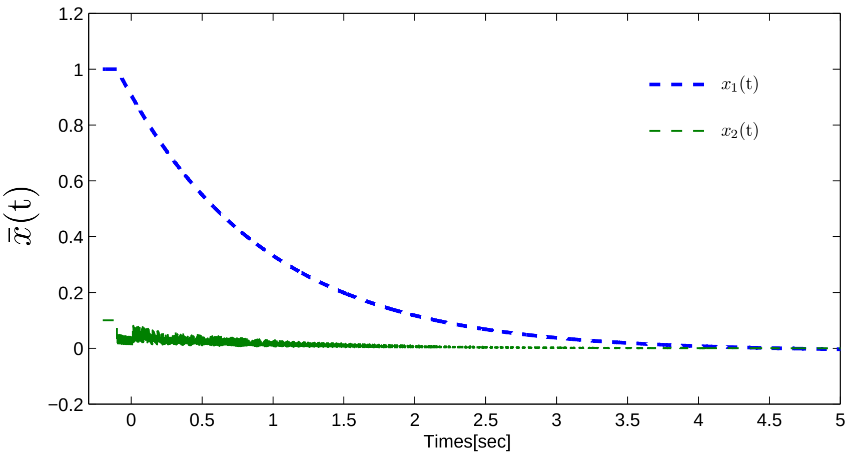

FIGURE 1. State responses of $x_1(t)$ and $x_2(t)$

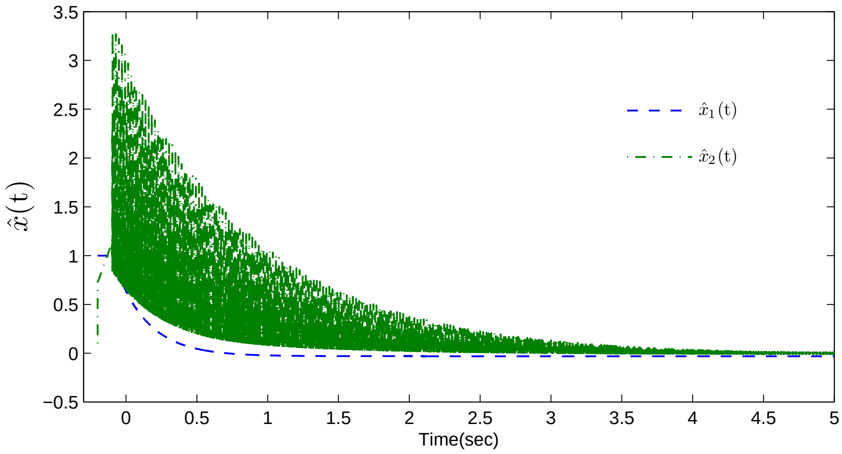

FIGURE 2. State responses of $\hat{x}_1(t)$ and $\hat{x}_2(t)$

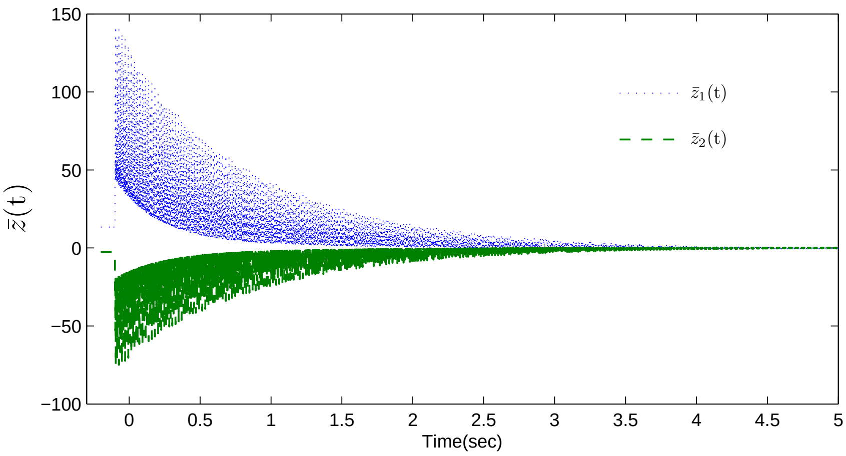

FIGURE 3. Error response of $\bar{z}(t)$

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