International Journal of Innovative Computing, Information and Control
Volume **9**, Number **5**, May **2013**

ICIC International ©2013 ISSN 1349-4198
pp. 2167–2183

ROBUST H_∞ FILTERING FOR UNCERTAIN 2-D CONTINUOUS SYSTEMS WITH DELAYS

Abdelaziz Hmamed[1], Chakir El Kasri[1], El Haoussi Tissir[1]
Teresa Alvarez[2] and Fernando Tadeo[2]

[1]Laboratory of Electronics, Signals, Systems and Information
Faculty of Sciences Dhar El Mehraz
P.O. Box 1796, Fes-Atlas 30000, Morocco
{ hmamed_abdelaziz; chakir_elkasri; elh_tissir }@yahoo.fr

[2]Department of Systems Engineering and Automatic Control
University of Valladolid
Valladolid 47005, Spain
{ tere; fernando }@autom.uva.es

Received November 2011; revised March 2012

Abstract. *This paper deals with the problem of robust H_∞ filtering for the class of two dimensional (2-D) continuous systems that are described by a Roesser state space model with delays. A sufficient condition to have an H_∞ noise attenuation for these 2-D systems is given in terms of a linear matrix inequality. The optimal H_∞ filter is obtained by solving a convex optimization problem. A simulation example is also given to illustrate the effectiveness of the proposed result.*
Keywords: 2-D continuous systems, State delay, H_∞ filtering, Linear matrix inequality (LMI)

1. **Introduction.** The main objective of H_∞ filtering is to design an estimator that minimizes the H_∞ norm of a filtering error system, in order to ensure that the L_2-induced gain from the noise signals to the estimation error is smaller than a prescribed level. One of the most popular ways to deal with the filtering problem is the celebrated Kalman filtering approach, which provides an optimal estimation of the state variables in the sense that the covariance of the estimation error is minimized [1]. H_∞ filtering was introduced in 1989 [2], by assuming that the input signal is energy bounded. Many results on the H_∞ filtering have been proposed in the literature, for both the deterministic and stochastic contexts: see, e.g., [3-6], and references therein. When parameter uncertainties affect a system, the corresponding robust H_∞ filtering has also been investigated, and some results on this topic have been presented: see, e.g., [7,8], and references therein. Note that all these mentioned H_∞ filtering results are obtained in the context of one-dimensional (1-D) system. The study of two-dimensional (2-D) systems has received much attention in past decades: [9-13]. For example, the 2-D H_∞ filtering problem for Roesser models was solved in [9] in the absence of uncertainties and delayes. The corresponding results for the 2-D Fornasini-Marchesini second model were reported in [11,12]. We point out that the H_∞ filtering results in [9,12] are obtained for 2-D discrete parameter systems. In the study of distributed parameter systems, partial differential equations actually correspond to 2-D or n-D continuous systems [10]. Therefore, the study of 2-D continuous systems is of both practical and theoretical importance. Although many stability analysis and control results for 2-D continuous and discrete-time systems have been reported in the literature [10,14-18], for such systems, however, we can just cite [13], concerning the H_∞ filtering.

These previous results ignored the effect of delays. Unfortunately, delays of signal transmissions are frequently encountered in practical problems, specially in engineering and biological systems. Examples of 2-D systems with significant delays include the material rolling process [19] and in general, systems described by delayed lattice differential equations [20] and partial difference equations [21]. In addition, certain 2-D systems containing digital processors that need finite numerical computation time [22,23] display also the delay phenomenon. These delays are known to be a frequent source of instability and poor performance. Therefore, for one-dimensional (1-D) state-delayed systems, there have been much literature on the stability and robust filtering that have offered various schemes (see, e.g., [24-31], and references therein). In contrast, most results for the 2-D filtering problem focus on systems without delays; although for specific stability and control problems of uncertain 2-D discrete state-delayed systems research results are given in [23,32], the H_∞ filtering problem for 2-D continuous state-delayed systems has not been fully investigated, which motivates the present study. Thus, we propose in this paper a method to deal with the robust H_∞ filtering problem for 2-D continuous systems described by a Roesser model with delays. A sufficient condition for such a 2-D system to have a specified H_∞ noise attenuation is first presented via the LMI approach. Furthermore, a convex optimization problem with LMI constraints is formulated to design a 2-D filter such that, for all admissible uncertainties, the filtering error dynamics is asymptotically stable, and a prescribed H_∞-norm performance level is achieved. The simulation results demonstrate the effectiveness of the proposed method.

Notations: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscript T represents the transpose of a matrix; $diag\{...\}$ denotes a block-diagonal matrix; $her(S)$ stands for $S + S^T$. The symbol $\sigma_{\max}(.)$ denotes the spectral norm of a matrix. The symmetric term in a symmetric matrix is denoted by $*$, e.g., $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem Formulation.

Consider a 2-D continuous system described by the following Roesser state-space model with delays:

$$(\Sigma): \quad \begin{aligned} \dot{x}(t_1,t_2) &= (A + \Delta A)x(t_1,t_2) + (A_d + \Delta A_d)x(t_1 - \tau_1, t_2 - \tau_2) \\ &\quad + (B + \Delta B)\omega(t_1,t_2) \qquad (1) \\ y(t_1,t_2) &= (C_1 + \Delta C_1)x(t_1,t_2) + (C_{1d} + \Delta C_{1d})x(t_1 - \tau_1, t_2 - \tau_2) \\ &\quad + (D_1 + \Delta D_1)\omega(t_1,t_2) \qquad (2) \\ z(t_1,t_2) &= Cx(t_1,t_2) + D\omega(t_1,t_2) \qquad (3) \end{aligned}$$

with $x(0,t_2) = f(t_2)$ for $t_2 \in [-\tau_2, 0]$, $x(t_1, 0) = g(t_1)$ for $t_1 \in [-\tau_1, 0]$, $x(t_1,t_2) = \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix}$, $\dot{x}(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1,t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1,t_2) \end{bmatrix}$ and $x(t_1 - \tau_1, t_2 - \tau_2) = \begin{bmatrix} x^h(t_1 - \tau_1, t_2) \\ x^v(t_1, t_2 - \tau_2) \end{bmatrix}$, where $x^h(t_1,t_2) \in \Re^{n_h}$ and $x^v(t_1,t_2) \in \Re^{n_v}$ are the horizontal and vertical states, respectively, $y(t_1,t_2) \in \Re^p$ is the measured output, $z(t_1,t_2) \in \Re^r$ is the signal to be estimated, $w(t_1,t_2) \in \Re^m$ is the exogenous input, and τ_1, $\tau_2 > 0$ are constant time delays. Matrices A, A_d, B, C, C_1, D and D_1, are known constant real matrices. The uncertainties are assumed to be of the form

$$\begin{bmatrix} \Delta A & \Delta A_d & \Delta B \\ \Delta C_1 & \Delta C_{1d} & \Delta D_1 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F \begin{bmatrix} N_1 & N_d & N_2 \end{bmatrix} \qquad (4)$$

where M_1, M_2, N_1, N_d and N_2 are known real constant matrices, and $F \in R^{k\times l}$ is an unknown real matrix, that satisfies

$$F^T F \leq I. \tag{5}$$

The uncertainties ΔA, ΔA_d, ΔB, ΔC_1, ΔC_{1d} and ΔD_1, are said to be admissible if both (4) and (5) hold. In this paper, we consider the following 2-D continuous filter, in order to estimate $z(t_1,t_2)$:

$$(\Sigma_f): \quad \dot{\hat{x}}(t_1,t_2) = A_f\hat{x}(t_1,t_2) + B_f y(t_1,t_2) \tag{6}$$
$$\hat{z}(t_1,t_2) = C_f\hat{x}(t_1,t_2) \tag{7}$$

where $\hat{x}(t_1,t_2) = \begin{bmatrix} \hat{x}^h(t_1,t_2) \\ \hat{x}^v(t_1,t_2) \end{bmatrix}$, with $\hat{x}^h(t_1,t_2) \in \Re^{n_h}$ and $\hat{x}^v(t_1,t_2) \in \Re^{n_v}$ the horizontal and vertical states of the filter, respectively, and $\hat{z}(t_1,t_2) \in \Re^r$ the estimate of $z(t_1,t_2)$. The matrices A_f, B_f and C_f, are to be selected using the procedure developed later.

Denote

$$\begin{aligned}
&\tilde{x}^h(t_1,t_2) = [x^h(t_1,t_2)^T \quad \hat{x}^h(t_1,t_2)^T]^T, && \tilde{x}^v(t_1,t_2) = [x^v(t_1,t_2)^T \quad \hat{x}^v(t_1,t_2)^T]^T,\\
&\tilde{x}^h(t_1-\tau_1,t_2) = \begin{bmatrix} x^h(t_1-\tau_1,t_2) \\ \hat{x}^h(t_1-\tau_1,t_2) \end{bmatrix}, && \tilde{x}^v(t_1,t_2-\tau_2) = \begin{bmatrix} x^v(t_1,t_2-\tau_2) \\ \hat{x}^v(t_1,t_2-\tau_2) \end{bmatrix},\\
&\tilde{z}(t_1,t_2) = z(t_1,t_2) - \hat{z}(t_1,t_2), && \tilde{\xi}(t_1,t_2) = [\tilde{x}^h(t_1,t_2)^T \quad \tilde{x}^v(t_1,t_2)^T]^T,\\
&\tilde{\xi}(t_1,t_2) = [\tilde{x}^h(t_1,t_2)^T \quad \tilde{x}^v(t_1,t_2)^T]^T, && \tilde{\xi}(t_1-\tau_1,t_2-\tau_2) = \begin{bmatrix} \tilde{x}^h(t_1-\tau_1,t_2) \\ \tilde{x}^v(t_1,t_2-\tau_2) \end{bmatrix}.
\end{aligned}$$

Then the filtering error dynamics from the systems (Σ) and (Σ_f) can be written as follows:

$$\begin{aligned}
(\Sigma_e): \quad \dot{\tilde{\xi}}(t_1,t_2) &= (\tilde{A} + \Delta\tilde{A})\tilde{\xi}(t_1,t_2) + (\tilde{A}_d + \Delta\tilde{A}_d)\tilde{\xi}(t_1-\tau_1,t_2-\tau_2)\\
&\quad + (\tilde{B} + \Delta\tilde{B})\omega(t_1,t_2) && (8)\\
\tilde{z}(t_1,t_2) &= \tilde{C}\tilde{\xi}(t_1,t_2) + \tilde{D}\omega(t_1,t_2) && (9)
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{A} = \Phi\tilde{A}_f\Phi^T, \ \ \tilde{A}_d = \Phi\tilde{A}_{df}\Phi^T, \ \ \tilde{B} = \Phi\tilde{B}_f, \ \ \tilde{C} = \tilde{C}_f\Phi^T, \ \ \tilde{D} = D,\\
&\Delta\tilde{A} = \Phi\Delta\tilde{A}_f\Phi^T, \ \ \Delta\tilde{A}_d = \Phi\Delta\tilde{A}_{df}\Phi^T, \ \ \Delta\tilde{B} = \Phi\Delta\tilde{B}_f,
\end{aligned} \tag{10}$$

and the augmented matrices are given by:

$$\begin{aligned}
&\tilde{A}_f = \begin{bmatrix} A & 0 \\ B_fC_1 & A_f \end{bmatrix}, \quad \tilde{B}_f = \begin{bmatrix} B \\ B_fD_1 \end{bmatrix},\\
&\tilde{A}_{df} = \begin{bmatrix} A_d & 0 \\ B_fC_{1d} & 0 \end{bmatrix}, \quad \tilde{C}_f = \begin{bmatrix} C & -C_f \end{bmatrix}, && (11)\\
&\Delta\tilde{A}_f = \begin{bmatrix} \Delta A & 0 \\ B_f\Delta C_1 & A_f \end{bmatrix}, \quad \Delta\tilde{B}_f = \begin{bmatrix} \Delta B \\ B_f\Delta D_1 \end{bmatrix}, \quad \Delta\tilde{A}_{df} = \begin{bmatrix} \Delta A_d & 0 \\ B_f\Delta C_{1d} & 0 \end{bmatrix}, && (12)\\
&\Phi = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{n_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{n_v} \end{bmatrix}. && (13)
\end{aligned}$$

The robust H_∞ filtering problem to be addressed in this paper can be formulated as follows: Given a scalar $\gamma > 0$ and the 2-D continuous system with delays (Σ), find an asymptotically stable filter (Σ_f) in the form of (6) such that the filtering error system (Σ_e) is asymptotically stable and the transfer function of the error system given as

$$\begin{aligned} T_{\tilde{z}\omega}(s_1, s_2) = &\tilde{C}[I(s_1, s_2) - (\tilde{A} + \Delta\tilde{A}) \\ &- (\tilde{A}_d + \Delta\tilde{A}_d)I(e^{-s_1\tau_1}, e^{-s_2\tau_2})]^{-1}(\tilde{B} + \Delta\tilde{B}) + \tilde{D}, \end{aligned} \tag{14}$$

satisfies

$$\|T_{\tilde{z}\omega}\|_\infty < \gamma, \tag{15}$$

for all admissible uncertainties and with null initial conditions, where

$$I(\sigma_1, \sigma_2) = diag(\sigma_1 I_{n_h}, \sigma_2 I_{n_v}) \tag{16}$$

and

$$\|T_{\tilde{z}\omega}(s_1, s_2)\|_\infty = \sup_{\theta_1, \theta_2 \in R} \sigma_{\max}[T_{\tilde{z}\omega}(j\theta_1, j\theta_2)]. \tag{17}$$

3. **Main Results.** In this section, an LMI approach will be developed to solve the Robust H_∞ filtering problem formulated in the previous section. Before giving the main results, we present some stability conditions that will be used in the following developments.

3.1. **Stability issues.** This section discusses the stability of the following free 2-D continuous system:

$$\dot{x}(t_1, t_2) = Ax(t_1, t_2) + A_d x(t_1 - \tau_1, t_2 - \tau_2). \tag{18}$$

To test the stability of system (18), the following condition, based on the characteristic polynomial, can be used:

$$C(s_1, s_2) \neq 0 \quad \text{for} \quad (s_1, s_2) \in D^2, \quad D^2 = \{(s_1, s_2) : Re(s_1, s_2) \geq 0\} \tag{19}$$

where

$$C(s_1, s_2) = det[I(s_1, s_2) - A - A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2})]. \tag{20}$$

This condition is difficult to use in practice for the stability problem, so an alternative will be used, based on Linear Matrix Inequalities (LMI).

Lemma 3.1. *The 2-D continuous system with delays (18) is asymptotically stable if there exist matrices $P = diag(P_h, P_v) > 0$ and $Q = diag(Q_h, Q_v) > 0$ satisfying the following LMI:*

$$\begin{bmatrix} A^T P + PA + Q & PA_d \\ A_d^T P & -Q \end{bmatrix} < 0. \tag{21}$$

Proof: We prove Lemma 3.1 by contradiction. Suppose that although the conditions of Lemma 3.1 are satisfied, the 2-D continuous system (18) is unstable. Then, there exists $(s_1, s_2) \in D^2$ such that

$$det[I(s_1, s_2) - A - A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2})] = 0 \tag{22}$$

Hence, there exists a vector $v \neq 0$ such that

$$I(s_1, s_2)v = [A + A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2})]v. \tag{23}$$

It is easy to see that

$$v^*[PI(s_1, s_2)^* + I(s_1, s_2)P]v = 2v^* diag(Re(s_1), Re(s_2))P]v \tag{24}$$

$$\begin{aligned} &v^*[P(A + A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2})) + (A + A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2}))^* P]v \\ &\quad = v^*[A^T P + PA + PA_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2}) + I(e^{-s_1\tau_1}, e^{-s_2\tau_2})^* A_d^T P]v. \end{aligned} \tag{25}$$

From (23), (24) and (25) we obtain the following result

$$2v^*diag(Re(s_1)P_h, Re(s_2)P_v)v = v^*[A^TP + PA + PA_dI(e^{-s_1\tau_1}, e^{-s_2\tau_2}) + I(e^{-s_1\tau_1}, e^{-s_2\tau_2})^*A_d^T]v. \tag{26}$$

By applying the Schur Complement Formula to (21) we obtain

$$A^TP + PA + Q + PA_dQ^{-1}A_d^TP < 0. \tag{27}$$

It follows from $Q > 0$ and $(s_1, s_2) \in D^2$ that

$$[Q - PA_dI(e^{-s_1\tau_1}, e^{-s_2\tau_2})]Q^{-1}[Q - PA_dI(e^{-s_1\tau_1}, e^{-s_2\tau_2})]^* \geq 0,$$

which then implies

$$\begin{aligned} &PA_dI(e^{-s_1\tau_1}, e^{-s_2\tau_2}) + I(e^{-s_1\tau_1}, e^{-s_2\tau_2})^*A_d^TP \\ &\leq Q + PA_d diag(e^{-2Re(s_1)\tau_1}Q_h^{-1}, e^{-2Re(s_2)\tau_2}Q_v^{-1})A_d^TP \\ &\leq Q + PA_dQ^{-1}A_d^TP. \end{aligned} \tag{28}$$

This, together with (27) and $v \neq 0$, means that the right-hand side of (26) is negative. On the other hand, $(s_1, s_2) \in D^2$ and the positive definiteness of P_h and P_v implies that $diag(Re(s_1)P_h, Re(s_2P_v) \geq 0$; therefore, the left-hand side of (26) is nonnegative, leading to a contradiction, which completes the proof.

Introduce now the following Lyapunov-Krasovski functional [24].

$$V(t_1, t_2) = V_1(t_1, t_2) + V_2(t_1, t_2), \tag{29}$$

$$V_1(t_1, t_2) = x^h(t_1, t_2)^TP_hx^h(t_1, t_2) + \int_{t_1-\tau_1}^{t_1} x^h(\theta, t_2)^TQ_hx^h(\theta, t_2)d\theta, \tag{30}$$

$$V_2(t_1, t_2) = x^v(t_1, t_2)^TP_vx^v(t_1, t_2) + \int_{t_2-\tau_2}^{t_2} x^v(t_1, \theta)^TQ_vx^v(t_2, \theta)d\theta, \tag{31}$$

with the associated unidirectional derivative of $V(t_1, t_2)$ in (29) defined as [15]

$$\dot{V}_u(t_1, t_2) := \frac{\partial V_1(t_1, t_2)}{\partial t_1} + \frac{\partial V_2(t_1, t_2)}{\partial t_2}, \tag{32}$$

where $P_h > 0$, $P_v > 0$, $Q_h > 0$ and $Q_v > 0$. Then, we have the following result:

Lemma 3.2. *The 2-D continuous system with delays (18) is asymptotically stable if*

$$\dot{V}_u(t_1, t_2) < 0. \tag{33}$$

Proof: By using the calculus given in [24], we obtain

$$\dot{V}(t_1, t_2) = \xi(t_1, t_2)^T \begin{bmatrix} her(PA) + Q & PA_d \\ A_d^TP & -Q \end{bmatrix} \xi(t_1, t_2),$$

where

$$\xi(t_1, t_2) = [x(t_1, t_2)^T \quad x(t_1 - \tau_1, t_2 - \tau_2)^T]^T.$$

Now, for any $\xi(t_1, t_2) \neq 0$, $\dot{V}_u(t_1, t_2) < 0$ requires that $\begin{bmatrix} her(PA) + Q & PA_d \\ A_d^TP & -Q \end{bmatrix} < 0$, so the proof is completed, by simply using Lemma 3.1.

Lemma 3.3. [33] *Let $\mathcal{D}$, $\mathcal{S}$ and F be real matrices of appropriate dimensions with F satisfying $F^TF \leq I$. Then, for any scalar $\epsilon > 0$*

$$\mathcal{D}F\mathcal{S} + (\mathcal{D}F\mathcal{S})^T \leq \epsilon^{-1}\mathcal{D}\mathcal{D}^T + \epsilon\mathcal{S}^T\mathcal{S}.$$

Remark 3.1. *The proof of Lemma 3.2 given in* [24] *is not justified since in 2-D systems, until the moment, there are no results proving the stability by Lyapunov-Krasovski functional. On the other hand, in this paper the stability proof is obtained as a direct consequence of the results of Lemma 3.1, which itself is established by manipulating the characteristic polynomial. Therefore, our result validates the previous unproved results in* [24].

Remark 3.2. *Lemma 3.1 provides an LMI condition for the 2-D continuous system with delays (18) to be asymptotically stable. Lemma 3.1 can be regarded as an extension of the existing results on asymptotic stability for 1-D continuous systems with delays* [34], *to the 2-D case.*

Remark 3.3. *It is clear that setting $A_d = 0$ Lemmas 3.1 and 3.2 yields precisely some results in [14,15,18,35]. Hence, Lemmas 3.1 and 3.2 here can be viewed as extensions of the existing results on asymptotic stability for 1-D continuous systems to 2-D continuous system with delays.*

3.2. **Bounded realness.** We consider now the nominal system (Σ_0) of (Σ), which is given by

$$(\Sigma_0): \quad \dot{x}(t_1,t_2) = Ax(t_1,t_2) + A_d x(t_1-\tau_1, t_2-\tau_2) + B\omega(t_1,t_2) \tag{34}$$

$$z(t_1,t_2) = Cx(t_1,t_2) + D\omega(t_1,t_2) \tag{35}$$

where the transfer function matrix is given as

$$T_{z\omega}(s_1,s_2) = C[I(s_1,s_2) - A - A_d I(e^{-s_1\tau_1}, e^{-s_2\tau_2})]^{-1}B + D. \tag{36}$$

Now, we give the bounded realness for 2-D continuous system with delays and no uncertainties (Σ_0), which plays a key role in solving the robust H_∞ filtering problem investigated in the next section.

Theorem 3.1. *Given a scalar $\gamma > 0$, the continuous system with delays (Σ_0) is asymptotically stable and satisfies the H_∞ performance $\|T_{zw}\|_\infty < \gamma$ if there exist matrices $P = diag(P_h, P_v) > 0$ and $Q = diag(Q_h, Q_v) > 0$ such that the following LMI holds:*

$$\begin{bmatrix} A^TP + PA + Q & PA_d & PB & C^T \\ * & -Q & 0 & 0 \\ * & * & -\gamma I & D^T \\ * & * & * & -\gamma I \end{bmatrix} < 0. \tag{37}$$

Proof: First, from (37), it is easy to see that

$$\begin{bmatrix} A^TP + PA + Q & PA_d \\ A_d^TP & -Q \end{bmatrix} < 0$$

which, by Lemma 3.1, gives that system (18) is asymptotically stable. Next, we show the H_∞ performance: by applying the $Schur$ Complement Formula to (37), we obtain

$$V := \gamma^2 I - D^TD > 0$$

and

$$(A^TP + PA + Q + \gamma^{-1}C^TC + PA_dQ^{-1}A_d^TP) \\ +\gamma[PB + \gamma^{-1}C^TD]V^{-1}[B^TP + \gamma^{-1}D^TC] < 0.$$

Multiplying this inequality by γI yields

$$[her(A^T(\gamma P)) + (\gamma Q) + C^TC + (\gamma P)A_d(\gamma Q)^{-1}A_d^T(\gamma P)] \\ + [(\gamma P)B + C^TD]V^{-1}[B^T(\gamma P) + D^TC] < 0. \tag{38}$$

Let $\tilde{P} = \gamma P > 0$ and $\tilde{Q} = \gamma Q > 0$; then, (38) can be rewritten as

$$[her(A^T\tilde{P}) + \tilde{Q} + C^TC + \tilde{P}A_d\tilde{Q}^{-1}A_d^T\tilde{P}] + [\tilde{P}B + C^TD]V^{-1}[B^T\tilde{P} + D^TC] < 0.$$

Therefore, there exists a matrix $U > 0$ such that

$$-her(A^T\tilde{P}) - \tilde{Q} - C^TC - \tilde{P}A_d\tilde{Q}^{-1}A_d^T\tilde{P} > [\tilde{P}B + C^TD]V^{-1}[B^T\tilde{P} + D^TC] + U. \quad (39)$$

Define

$$\Omega(j\theta_1, j\theta_2) = I(j\theta_1, j\theta_2) - A - A_dI(e^{-j\theta_1\tau_1}, e^{-j\theta_2\tau_2}),$$
$$z(j\theta_1, j\theta_2) = \tilde{P}A_dI(e^{-j\theta_1\tau_1}, e^{-j\theta_2\tau_2}).$$

Recalling that for any matrices K_1, K_2 and K_3 of appropriate dimension, with $K_2 > 0$,

$$K_1^*K_3 + K_3^*K_1 \le K_1^*K_2K_1 + K_3^*K_2^{-1}K_3. \quad (40)$$

Therefore,

$$z(j\theta_1, j\theta_2) + z(j\theta_1, j\theta_2)^* \le \tilde{P}A_d\tilde{Q}^{-1}A_d^T\tilde{P} + \tilde{Q}. \quad (41)$$

Then, it can be verified that

$$\tilde{P}I(j\theta_1, j\theta_2) + I(-j\theta_1, -j\theta_2)^T\tilde{P} = 0. \quad (42)$$

By (39), (41) and (42), we have that

$$\begin{aligned} &\Omega(-j\theta_1, -j\theta_2)^T\tilde{P} + \tilde{P}\Omega(j\theta_1, j\theta_2) - C^TC \\ &= -her(A^T\tilde{P}) - z(j\theta_1, j\theta_2) - z(j\theta_1, j\theta_2)^* - C^TC \\ &> (\tilde{P}B + C^TD)V^{-1}(B^T\tilde{P} + D^TC) + U. \end{aligned} \quad (43)$$

Since system (Σ) is asymptotically stable, we have that

$$det(I(j\theta_1, j\theta_2) - A - A_dI(e^{-j\theta_1}, e^{-j\theta_2})) \ne 0,$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. Therefore, $\Omega(j\theta_1, j\theta_2)^{-1}$ is well defined for all $\theta_1, \theta_1 \in \mathbb{R}$. Now, pre- and post-multiplying (43) by $B^T\Omega(j\theta_1, j\theta_2)^{-T}$ and $\Omega(j\theta_1, j\theta_2)^{-1}B$, respectively, we have that for all $\theta_1, \theta_2 \in \mathbb{R}$ the following holds:

$$\begin{aligned} B^T\Omega(j\theta_1, j\theta_2)^{-T}&[\Omega(-j\theta_1, -j\theta_2)^T\tilde{P} + \tilde{P}\Omega(j\theta_1, j\theta_2) - C^TC]\Omega(j\theta_1, j\theta_2)^{-1}B \\ &\ge B^T\Omega(j\theta_1, j\theta_2)^{-T}\Lambda\Omega(j\theta_1, j\theta_2)^{-1}B \end{aligned} \quad (44)$$

where

$$\Lambda = (\tilde{P}B + C^TD)V^{-1}(B^T\tilde{P} + D^TC) + U.$$

Then, by noting (36), we have

$$\begin{aligned} &\gamma^2I - T_{z\omega}(-j\theta_1, -j\theta_2)^TT_{z\omega}(j\theta_1, j\theta_2) \\ &= \gamma^2I - [B^T\Omega(-j\theta_1, -j\theta_2)^{-T}C^T + D^T][C\Omega(j\theta_1, j\theta_2)^{-1}B + D] \\ &= \gamma^2I - D^TD + B^T\Omega(-j\theta_1, -j\theta_2)^{-T}[\tilde{P}\Omega(j\theta_1, j\theta_2) \\ &\quad + \Omega(-j\theta_1, -j\theta_2)^T\tilde{P} - C^TC]\Omega(j\theta_1, j\theta_2)^{-1}B \\ &\quad - B^T\Omega(-j\theta_1, -j\theta_2)^{-T}(\tilde{P}B + C^TD) - (B^T\tilde{P} + D^TC)\Omega(j\theta_1, j\theta_2)^{-1}B \\ &\ge V + B^T\Omega(-j\theta_1, -j\theta_2)^{-T}\Lambda\Omega(j\theta_1, j\theta_2)^{-1}B \\ &\quad - B^T\Omega(-j\theta_1, -j\theta_2)^{-T}(\tilde{P}B + C^TD) - (B^T\tilde{P} + D^TC)\Omega(j\theta_1, j\theta_2)^{-1}B \end{aligned} \quad (45)$$

By using the relation (40), we obtain

$$\gamma^2I - T_{z\omega}(-j\theta_1, -j\theta_2)^TT_{z\omega}(j\theta_1, j\theta_2) \ge V - (B^T\tilde{P} + D^TC)\Lambda^{-1}(\tilde{P}B + C^TD). \quad (46)$$

Now, observe that

$$\Lambda - (\tilde{P}B + C^T D)V^{-1}(B^T \tilde{P} D^T C) = U > 0$$

Then, by the Schur Complement Formula, we have that

$$\begin{bmatrix} V & B^T\tilde{P} + D^T C \\ \tilde{P}B + C^T D & \Lambda \end{bmatrix} > 0,$$

which, using again the Schur Complement Formula, gives

$$V - [B^T\tilde{P} + D^T C]\Lambda^{-1}[\tilde{P}B + C^T D] > 0. \tag{47}$$

Then, it follows from (46) and (47) that for all $\theta_1, \theta_2 \in \mathbb{R}$

$$\gamma^2 I - T_{z\omega}(-j\theta_1, -j\theta_2)^T T_{z\omega}(j\theta_1, j\theta_2) > 0. \tag{48}$$

Hence, by (48), we have $\|T_{z\omega}\|_\infty \leq \gamma$, which completes the proof.

Remark 3.4. *The delay is a source of instability and poor performance; Theorem 3.1 provides a sufficient condition for the 2-D continuous system with delay* (Σ_0) *to be asymptotically stable and satisfy a specified* H_∞ *performance level. This condition can be tested easily by resorting to some standard numerical algorithms, with no tuning of parameters involved* [36]. *Moreover, taking* $A_d = 0$ *in Theorem 3.1, our results reduce to Theorem 3.1 in* [13]. *Hence, Theorem 3.1 here can be viewed as an extension of the existing results on the bounded realness condition for 2-D continuous systems to 2-D continuous systems with delays. It is also worth pointing out that in the case when system* (Σ) *reduce to a 1-D continuous system with delays, Theorem 3.1 coincides with the bounded realness condition for 1-D continuous systems with delays* [34]. *Therefore, Theorem 3.1 can be regarded as the extension of existing results on bounded realness for 1-D continuous systems with delays to the 2-D case.*

Now, we are in a position to present the solvability condition for the robust H_∞ filtering problem with delay.

3.3. **Robust H_∞ filtering.** Extending Theorem 3.1 to the uncertain 2-D system (Σ_e) yields the following robust H_∞ filtering criterion.

Theorem 3.2. *Given a scalar* $\gamma > 0$, *the 2-D robust* H_∞ *filtering problem is solvable if the 2-D system* (Σ) *is asymptotically stable with* γ *performance. That is, if there exist a scalar* $\epsilon > 0$ *and matrices* Z, Θ, Ψ, $X = diag(X_h, X_v) > 0$, *and* $Y = diag(Y_h, Y_v) > 0$ *with* X_h, $Y_h \in \mathbb{R}^{n_h}$ *and* X_v, $Y_v \in \mathbb{R}^{n_v}$ *satisfying the following LMIs:*

$$\begin{bmatrix} J_{11} & J_{12} & YA_d + \epsilon N_1^T N_d & YA_d + \epsilon N_1^T N_d & YB + \epsilon N_1^T N_2 & J_{16} & YM_1 \\ * & J_{22} & J_{23} & J_{23} & J_{25} & C^T & XM_1 + \Psi M_2 \\ * & * & -Y + \epsilon N_d^T N_d & -Y + \epsilon N_d^T N_d & N_d^T N_2 & 0 & 0 \\ * & * & * & -S + \epsilon N_d^T N_d & \epsilon N_d^T N_2 & 0 & 0 \\ * & * & * & * & \epsilon N_2^T N_2 - \gamma I & D^T & 0 \\ * & * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{49}$$

$$X - Y > 0, \tag{50}$$

$$S - Y > 0, \tag{51}$$

where

$$\begin{array}{ll} J_{11} = her(YA) + Y + \epsilon N_1^T N_1, & J_{12} = YA + A^T X + C_1^T \Psi^T + Z^T + Y + \epsilon N_1^T N_1, \\ J_{16} = C^T - \Theta^T, & J_{22} = her(XA + \Psi C_1) + S + \epsilon N_1^T N_1, \\ J_{23} = XA_d + \Psi C_{1d} + \epsilon N_1^T N_d, & J_{25} = XB + \Psi D_1 + \epsilon N_1^T N_2. \end{array}$$

In this case, a desired 2-D continuous filter in the form of (Σ_f) *can be chosen with parameters as follows:*

$$A_f = X_{12}^{-1} Z Y^{-1} Y_{12}^{-T} \tag{52}$$

$$B_f = X_{12}^{-1}\Psi \tag{53}$$

$$C_f = \Theta Y^{-1} Y_{12}^{-T} \tag{54}$$

where

$$X_{12} = \begin{bmatrix} X_{h12} & 0 \\ 0 & X_{v12} \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} Y_{h12} & 0 \\ 0 & Y_{v12} \end{bmatrix}, \quad S_{12} = \begin{bmatrix} S_{h12} & 0 \\ 0 & S_{v12} \end{bmatrix}, \tag{55}$$

in which X_{h12}, X_{v12}, Y_{h12},Y_{v12}, S_{h12} *and* S_{v12} *are nonsingular matrices satisfying*

$$X_{12}Y_{12}^T = I - XY^{-1}, \quad S_{12}Y_{12}^T = I - SY^{-1}, \tag{56}$$

Proof: Let $\bar{Y}_h = Y_h^{-1}$, $\bar{Y}_v = Y_v^{-1}$, $\bar{Y} = Y^{-1}$. Then, the relations (50) and (51) can be written as

$$\begin{bmatrix} X & I \\ I & \bar{Y} \end{bmatrix} > 0, \quad \begin{bmatrix} S & I \\ I & \bar{Y} \end{bmatrix} > 0. \tag{57}$$

By the Schur Complement Formula, it follows from (57) that

$$\bar{Y} - X^{-1} > 0, \quad \bar{Y} - S^{-1} > 0$$

which implies that $I - X\bar{Y}$ and $I - S\bar{Y}$ are nonsingular. Therefore, by noting the structure of X, Y and S we have that there always exist nonsingular matrices X_{h12}, X_{v12}, Y_{h12}, Y_{v12}, S_{h12} and S_{v12} such that (56) is satisfied; that is,

$$X_{h12}Y_{h12}^T = I - X_h\bar{Y}_h, \quad S_{h12}Y_{h12}^T = I - S_h\bar{Y}_h, \tag{58}$$

$$X_{v12}Y_{v12}^T = I - X_v\bar{Y}_v, \quad S_{v12}Y_{v12}^T = I - S_v\bar{Y}_v, \tag{59}$$

Set

$$\Pi_{h1} = \begin{bmatrix} \bar{Y}_h & I \\ Y_{h12}^T & 0 \end{bmatrix}, \quad \Pi_{v1} = \begin{bmatrix} \bar{Y}_v & I \\ Y_{v12}^T & 0 \end{bmatrix}, \quad \Pi_{h2} = \begin{bmatrix} I & X_h \\ 0 & X_{h12}^T \end{bmatrix},$$

$$\Pi_{v2} = \begin{bmatrix} I & X_v \\ 0 & X_{v12}^T \end{bmatrix}, \quad \Pi_{h3} = \begin{bmatrix} I & S_h \\ 0 & S_{h12}^T \end{bmatrix}, \quad \Pi_{v3} = \begin{bmatrix} I & S_v \\ 0 & S_{v12}^T \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} \Pi_{h1} & 0 \\ 0 & \Pi_{v1} \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} \Pi_{h2} & 0 \\ 0 & \Pi_{v2} \end{bmatrix}, \quad \Pi_3 = \begin{bmatrix} \Pi_{h3} & 0 \\ 0 & \Pi_{v3} \end{bmatrix}.$$

Then, by some calculation, it can be verified that

$$P := \Pi_2\Pi_1^{-1} = \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix}, \quad Q := \Pi_3\Pi_1^{-1} = \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \tag{60}$$

where

$$P_h = \begin{bmatrix} X_h & X_{h12} \\ X_{h12}^T & X_{h12}^T(X_h - Y_h)^{-1}X_{h12} \end{bmatrix}, \quad P_v = \begin{bmatrix} X_v & X_{v12} \\ X_{v12}^T & X_{v12}^T(X_v - Y_v)^{-1}X_{v12} \end{bmatrix},$$

$$Q_h = \begin{bmatrix} S_h & S_{h12} \\ S_{h12}^T & S_{h12}^T(S_h - Y_h)^{-1}S_{h12} \end{bmatrix}, \quad P_v = \begin{bmatrix} S_v & S_{v12} \\ S_{v12}^T & S_{v12}^T(S_v - Y_v)^{-1}S_{v12} \end{bmatrix},$$

Observe that

$$X_h - X_{12}[X_{h12}^T(X_h - Y_h)^{-1}X_{h12}]^{-1}X_{h12}^T = Y_h > 0,$$
$$S_h - S_{12}[S_{h12}^T(S_h - Y_h)^{-1}S_{h12}]^{-1}S_{h12}^T = Y_h > 0,$$
$$X_v - X_{12}[X_{v12}^T(X_v - Y_v)^{-1}X_{v12}]^{-1}X_{v12}^T = Y_v > 0,$$
$$S_v - S_{12}[S_{v12}^T(S_v - Y_v)^{-1}S_{v12}]^{-1}S_{v12}^T = Y_v > 0.$$

Therefore, it is easy to see that $P_h > 0$ and $P_v > 0$. Now, pre- and post-multiplying (49) by $diag\{\bar{Y}, I, \bar{Y}, I, I, I, I\}$, we obtain

$$\begin{bmatrix} \bar{Y}J_{11}\bar{Y} & \bar{Y}J_{12} & \bar{Y}(YA_d+\epsilon N_1^TN_d)\bar{Y} & \bar{Y}(YA_d+\epsilon N_1^TN_d) & \bar{Y}(YB+\epsilon N_1^TN_2) & \bar{Y}J_{16} & M_1 \\ * & J_{22} & J_{23}\bar{Y} & J_{23} & J_{25} & C^T & J_{27} \\ * & * & \bar{Y}(-Y+\epsilon N_d^TN_d)\bar{Y} & \bar{Y}(-Y+\epsilon N_d^TN_d) & \epsilon\bar{Y}N_d^TN_2 & 0 & 0 \\ * & * & * & -S+\epsilon N_d^TN_d & \epsilon N_d^TN_2 & 0 & 0 \\ * & * & * & * & \epsilon N_2^TN_2-\gamma I & D^T & 0 \\ * & * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0 \quad (61)$$

which, by the Schur Complement Formula, implies

$$\begin{bmatrix} her(A\bar{Y})+\bar{Y} & M_{12} & A_d\bar{Y} & A_d & B & \bar{Y}C^T - Y_{12}C_f^T \\ * & M_{22} & M_{23} & XA_d+X_{12}B_fC_{1d} & XB+X_{12}B_fD_1 & C^T \\ * & * & -\bar{Y} & -I & 0 & 0 \\ * & * & * & -S & 0 & 0 \\ * & * & * & * & -\gamma I & D^T \\ * & * & * & * & * & -\gamma I \end{bmatrix}$$
$$+\epsilon^{-1}\begin{bmatrix} M_1 \\ XM_1+X_{12}B_fM_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} M_1 \\ XM_1+X_{12}B_fM_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon\begin{bmatrix} \bar{Y}N_1^T \\ N_1^T \\ \bar{Y}N_d^T \\ N_d^T \\ N_2 \\ 0 \end{bmatrix}\begin{bmatrix} \bar{Y}N_1^T \\ N_1^T \\ \bar{Y}N_d^T \\ N_d^T \\ N_2 \\ 0 \end{bmatrix}^T < 0 \quad (62)$$

where

$$\begin{aligned} M_{12} &= A+\bar{Y}A^TX+\bar{Y}C_1^TB_f^TX_{12}^T+Y_{12}A_f^TX_{12}^T+I, \\ J_{27} &= XM_1+\Psi M_2, \\ M_{22} &= XA+A^TX+X_{12}B_fC_1+C_1^TB_f^TX_{12}^T+S, \\ M_{23} &= XA_d\bar{Y}+\Psi C_{1d}\bar{Y}, \end{aligned}$$

and A_f, B_f, C_f are given in (52)-(54). By (60), the inequality (61) can be rewritten as

$$\begin{bmatrix} her(\Phi^T\Pi_1^TP\Phi\tilde{A}_f\Phi^T\Pi_1\Phi)+\Phi^T\Pi_1^TQ\Pi_1\Phi & \Phi^T\Pi_1^TP\Phi\tilde{A}_{df}\Phi^T\Pi_1\Phi & \Phi^T\Pi_1^TP\Phi\tilde{B}_f & \Phi^T\Pi_1^T\tilde{C}_f\Phi^T \\ * & -\Phi^T\Pi_1^TQ\Pi_1\Phi & 0 & 0 \\ * & * & -\gamma I & D^T \\ * & * & * & -\gamma I \end{bmatrix}$$
$$+\ \epsilon^{-1}\begin{bmatrix} \Phi^T\Pi_1^T\tilde{P}\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \Phi^T\Pi_1^T\tilde{P}\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon\begin{bmatrix} \Phi^T\Pi_1^T\Phi\tilde{N}_1^T \\ \Phi^T\Pi_1^T\Phi\tilde{N}_d^T \\ \tilde{N}_2 \\ 0 \end{bmatrix}\begin{bmatrix} \Phi^T\Pi_1^T\Phi\tilde{N}_1^T \\ \Phi^T\Pi_1^T\Phi\tilde{N}_d^T \\ \tilde{N}_2 \\ 0 \end{bmatrix}^T < 0, \quad (63)$$

where Φ is given in (13), and

$$\tilde{M}_1 = \begin{bmatrix} M_1 \\ B_fM_2 \end{bmatrix}, \quad \tilde{N}_1 = \begin{bmatrix} N_1 & 0 \end{bmatrix},$$
$$\tilde{N}_d = \begin{bmatrix} N_d & 0 \end{bmatrix}, \quad \tilde{N}_2 = N_2.$$

Pre- and post-multiplying (63) by $diag(\Pi_1^{-T}\Phi^{-T}, \Pi_1^{-T}\Phi^{-T}, I, I)$ and $diag(\Phi^{-1}\Pi_1^{-1}, \Phi^{-1}\Pi_1^{-1}, I, I)$ results in

$$\begin{bmatrix} \begin{matrix} her(P\tilde{A}) \\ +Q \end{matrix} & * & * & * \\ \tilde{A}_d^T P & -Q & * & * \\ \tilde{B}^T P & 0 & -\gamma I & * \\ \tilde{C} & 0 & \tilde{D} & -\gamma I \end{bmatrix} + \epsilon^{-1} \begin{bmatrix} P\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon \begin{bmatrix} \Phi\tilde{N}_1^T \\ \Phi\tilde{N}_d^T \\ \tilde{N}_2^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi\tilde{N}_1^T \\ \Phi\tilde{N}_d^T \\ \tilde{N}_2^T \\ 0 \end{bmatrix}^T < 0 \tag{64}$$

where the relationship $\Phi^T = \Phi^{-1}$ is used, and $\tilde{A}$, $\tilde{A}_d$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are given in (10). Now, noting that

$$\begin{bmatrix} \Delta\tilde{A}_f & \Delta\tilde{A}_{df} & \Delta\tilde{B}_f \end{bmatrix} = \tilde{M}_1 F \begin{bmatrix} \tilde{N}_1 & \tilde{N}_d & \tilde{N}_2 \end{bmatrix},$$

and using Lemma 3.3, we have

$$\begin{bmatrix} her(P\Phi\Delta\tilde{A}_f\Phi^T) & * & * & * \\ \Phi\Delta\tilde{A}_{df}^T\Phi^T P & 0 & * & * \\ \Delta\tilde{B}_f^T\Phi^T P & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \le \epsilon^{-1} \begin{bmatrix} P\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\Phi\tilde{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon \begin{bmatrix} \Phi\tilde{N}_1^T \\ \Phi\tilde{N}_d^T \\ \tilde{N}_2^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi\tilde{N}_1^T \\ \Phi\tilde{N}_d^T \\ \tilde{N}_2^T \\ 0 \end{bmatrix}^T < 0.$$

This, together with (64), gives

$$\begin{bmatrix} P(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T P + Q & * & * & * \\ (\tilde{A}_d + \Delta\tilde{A}_d)^T P & -Q & * & * \\ (\tilde{B} + \Delta\tilde{B})^T P & 0 & -\gamma I & * \\ \tilde{C} & 0 & D & -\gamma I \end{bmatrix} < 0.$$

Finally, by Theorem 3.1, it follows that the error system (Σ_e) is asymptotically stable, and the transfer function of the error system satisfies (15), which completes the proof.

Remark 3.5. *Theorem 3.2 provides a sufficient condition for the solvability of the robust H_∞ filtering for 2-D continuous systems with delays. A desired filter can be constructed by solving the LMIs in (49)-(51) and computing X_{12} and Y_{12} such that (56). These LMIs can be solved efficiently, and no tuning of parameters will be involved* [36].

Remark 3.6. *From Theorem 3.2, it is easy to see that the minimal value of the H_∞ norm $\gamma > 0$, which satisfies the LMIs in (49)-(51), can be determined by solving the following optimization problem:*

$$\underset{\epsilon, S, X, Y, Z, \Theta, \Psi}{\text{minimize}}\ \gamma$$

subject to

$$\epsilon > 0, \quad S > 0, \quad X > 0, \quad Y > 0, \quad (49), \quad (50), \quad (51).$$

In the case when there is no parameter uncertainty in system (Σ), by Theorem 3.2, we have the following corollary.

Corollary 3.1. *Consider the 2-D continuous system when there is no parameter uncertainty in system* (Σ)*. Then, the* H_∞ *filtering problem for this system is solvable if there exist matrices* Z, Θ, Ψ, $S = diag(S_h, S_v) > 0$, $X = diag(X_h, X_v) > 0$, $Y = diag(Y_h, Y_v) > 0$ *with* $S_h, X_h, Y_h \in \mathbb{R}^{n_h}$*, and* $S_v, X_v, Y_v \in \mathbb{R}^{n_v}$ *satisfying the LMIs in (65)-(67) with the desired 2-D continuous filter* (Σ_f) *chosen with parameters as given in (52)-(54).*

$$\begin{bmatrix} YA + A^TY + Y & * & * & * & * & * \\ A^TY + XA + \Psi C_1 + Z + Y & her(XA + \Psi C_1) + S & * & * & * & * \\ A_d^TY & A_d^TX + C_{1d}^T\Psi^T & -Y & * & * & * \\ A_d^TY & A_d^TX + C_{1d}^T\Psi^T & -Y & -S & * & * \\ B^TY & B^TX + D_1^T\Psi^T & 0 & 0 & -\gamma I & * \\ C - \Theta & C & 0 & 0 & D & -\gamma I \end{bmatrix} < 0, \quad (65)$$

$$X - Y > 0, \quad (66)$$

$$S - Y > 0 \quad (67)$$

Remark 3.7. *Theorem 3.2 provides an LMI technique to investigate the robust stability and* H_∞ *performance of the filtering error system* (Σ_e)*. When there are no state delays in the system, this theorem reduces to Theorem 2 in* [13]*. The condition (51) implies that* $\begin{bmatrix} S & Y \\ Y & Y \end{bmatrix} > 0$, *which can be rewritten as* $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} S & Y \\ Y & Y \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} > 0$, *that implies* $\begin{bmatrix} Y & Y \\ Y & S \end{bmatrix} > 0$, *pre- and post multiplying (49) by* φ *and* φ^T, *with* $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}$, $\varphi_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}$, $\varphi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$, $\varphi_3 = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}$ *and using the Schur Complement Formula, we reproduce the result in Theorem 2 of* [13].

Remark 3.8. *Suppose that the LMIs (49)-(51) admit a feasible solution given by* X, Y, Z, Ψ, *and* Θ*. Then, the computation of an* H_∞ *filter that solves the* H_∞ *filtering problem of the 2-D continuous system* (Σ) *can be carried out by following these steps:*

1. *By (58) and (59) we have that*

$$\begin{bmatrix} X_{h12} \\ S_{h12} \end{bmatrix} Y_{h12}^T = \begin{bmatrix} Y_h - X_h \\ Y_h - S_h \end{bmatrix} \bar{Y}_h, \quad \begin{bmatrix} X_{v12} \\ S_{v12} \end{bmatrix} Y_{v12}^T = \begin{bmatrix} Y_v - X_v \\ Y_v - S_v \end{bmatrix} \bar{Y}_v \quad (68)$$

2. *Compute* X_{h12}, X_{v12}, Y_{h12}, Y_{v12}, S_{h12} *and* S_{v12} *using the Singular Value Decomposition (SVD) in (68).*
3. *Construct a positive definite matrix* $X_{12} > 0$ *and* $Y_{12} > 0$ *of the form (55).*
4. *Then, by (52)-(54), compute the filter parameters* A_f, B_f *and* C_f.

4. Numerical Example.

This section presents an example that illustrates the effectiveness of the proposed results. For this, consider an uncertain 2-D continuous system (Σ) with the following parameters:

$$A = \left[\begin{array}{cc|cc} -1 & 0.2 & -0.5 & 0.3 \\ 0.6 & -2 & 0.2 & 0 \\ \hline -0.3 & 0.7 & -2.3 & 0.6 \\ -0.1 & 0.2 & 0.1 & -1.8 \end{array}\right], \quad A_d = \left[\begin{array}{cc|cc} -0.4 & 0.1 & -0.2 & 0.2 \\ 0.3 & -0.4 & 0.8 & 0 \\ \hline -0.2 & 0.3 & -0.9 & 0.3 \\ -0.4 & 0.8 & 0.1 & -0.8 \end{array}\right],$$

$$C=\left[\begin{array}{c}0.6\\0.1\\\hline -0.8\\0.5\end{array}\right]^T,\ C_1=\left[\begin{array}{cc}0.8&0.5\\-0.9&-0.3\\\hline 0.2&0\\-0.1&0.5\end{array}\right]^T,\ C_{1d}=\left[\begin{array}{cc}0.2&0.1\\-0.8&-0.6\\\hline 0.4&0.3\\-0.2&0.1\end{array}\right]^T,\ B=\left[\begin{array}{c}0.2\\-0.5\\\hline -0.8\\0.3\end{array}\right],$$

$$D_1=\left[\begin{array}{c}0.1\\0.3\end{array}\right],\ N_d=\left[\begin{array}{cc|cc}0.6&0.3&-0.4&0.8\end{array}\right],\ M_2=\left[\begin{array}{c}0.3\\0.1\end{array}\right],\ M_1=\left[\begin{array}{c}0.1\\0.2\\\hline 0.2\\-0.1\end{array}\right],$$

$$N_1=\left[\begin{array}{cc|cc}0.2&0.1&-0.2&0.3\end{array}\right],\ D=0.5,\ N_2=0.2.$$

To the best of our knowledge, there is no previous result dealing with the H_∞ filtering problem for 2D continuous systems with delays, so a direct comparison is not possible. The purpose of this example is to find matrices A_f, B_f and C_f of the filter Σ_f such that the system Σ_e is asymptotically stable and satisfies a prescribed H_∞ performance level γ, which is assumed to be 0.6 in this example.

Then, we resort to Matlab to solve the LMIs in (49) and (50); the solution is the following:

$$X=diag\left(\left[\begin{array}{cc}2.7096&0.9646\\0.9646&2.5763\end{array}\right],\left[\begin{array}{cc}2.5271&0.6081\\0.6081&2.0549\end{array}\right]\right)$$

$$Y=diag\left(\left[\begin{array}{cc}1.2111&0.0629\\0.0629&1.7058\end{array}\right],\left[\begin{array}{cc}1.3419&0.5932\\0.5932&1.7888\end{array}\right]\right)$$

$$Z=\left[\begin{array}{cccc}2.5031&0.2804&-0.8957&-0.8750\\0.2498&2.2700&0.9368&-1.7046\\-1.6468&-0.1829&7.4111&-1.8855\\-0.2420&-0.8398&-0.9728&1.9107\end{array}\right]$$

$$\Psi=\left[\begin{array}{cccc}-0.1255&-3.0194&-3.2943&1.9751\\-1.8908&3.6880&8.7438&-2.2558\end{array}\right]^T$$

$$\Theta=\left[\begin{array}{cccc}0.8064&-0.6856&-1.5351&0.5141\end{array}\right],\ \epsilon=0.2587.$$

To construct the desired filter, using the steps discussed in Remark 3.8, we have that

$$X_{h12}=\left[\begin{array}{cc}-1.0386&0.2170\\-0.6840&-0.3199\end{array}\right],\ Y_{h12}=\left[\begin{array}{cc}1.1315&-0.1704\\0.5405&0.3567\end{array}\right],\tag{69}$$

$$X_{v12}=\left[\begin{array}{cc}-1.0156&-0.0458\\0.1084&-0.3647\end{array}\right],\ Y_{v12}=\left[\begin{array}{cc}1.0092&0.1243\\-0.3447&0.3638\end{array}\right].\tag{70}$$

It can be verified that the matrices X_{h12}, X_{v12}, Y_{h12} and Y_{v12} chosen in (69) are nonsingular and satisfy (56). Then, from Theorem 3.2, the corresponding filter parameters are

$$A_f=\left[\begin{array}{cc|cc}-1.2581&0.0636&-0.4664&1.8546\\0.9566&-9.1271&-3.9566&3.2678\\\hline 0.9321&-1.4224&-7.2548&2.6707\\1.0795&2.0964&2.5319&-6.3226\end{array}\right],\ B_f=\left[\begin{array}{cc}1.4466&-0.4067\\6.3460&-10.6605\\\hline 3.4417&-8.7709\\-4.3932&3.5793\end{array}\right],$$

$$C_f=\left[\begin{array}{cc|cc}0.3482&-1.7253&-1.5584&0.6711\end{array}\right].$$

We note here that Theorem 3.2 is valid for any time varying perturbation matrix $F(t_1,t_2)$ that verifies (5). For simulation only, we fix $F=0.8$: the corresponding responses $\tilde{\xi}_1^h(t_1,t_2)$, $\tilde{\xi}_2^h(t_1,t_2)$ of the error system are shown in Figures 1 and 2, respectively, whereas Figure 3 gives the response of the error $\tilde{z}(t_1,t_2)$. The frequency response of the error

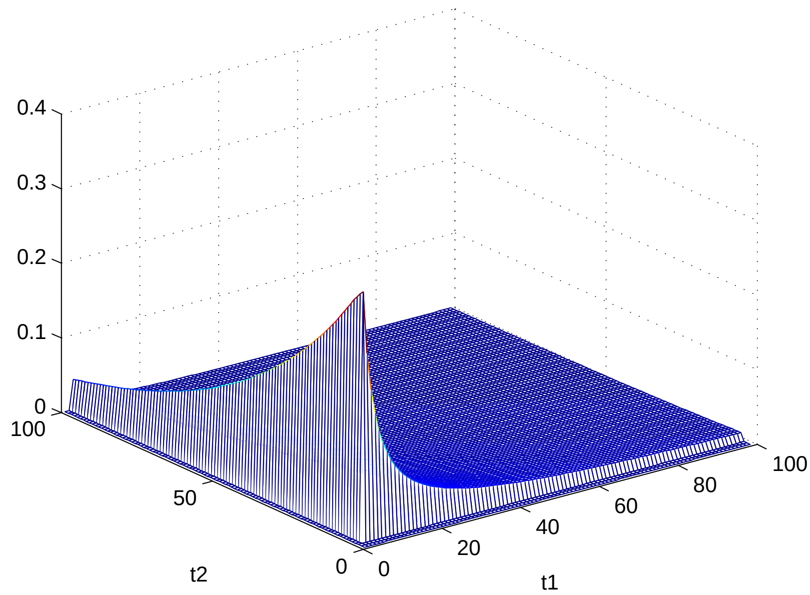

Figure 1. Response of $\tilde{\xi}_1^h(t_1, t_2)$

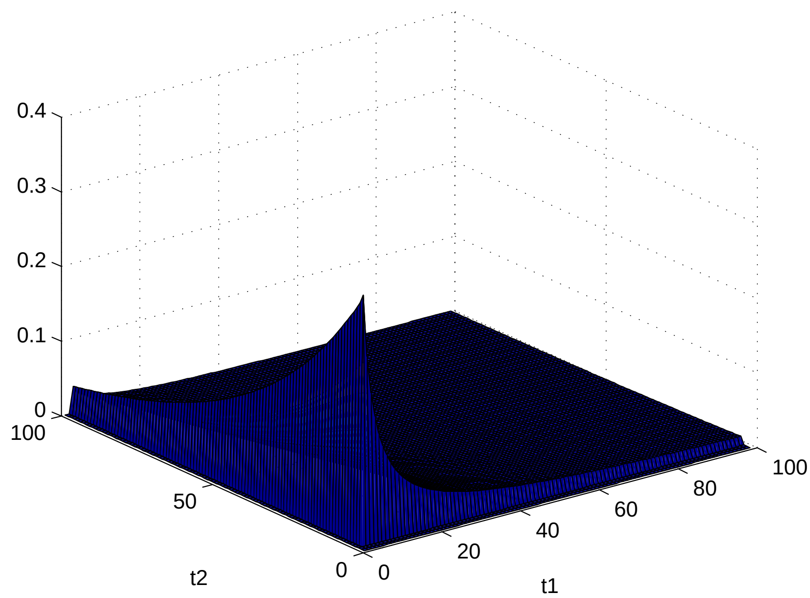

Figure 2. Response of $\tilde{\xi}_2^h(t_1, t_2)$

system is given in Figure 4; the achieved H_∞ norm is 0.4838, which compares well with the proposed value $\gamma = 0.6$. In summary, the simulation results show the effectiveness of the designed filter.

5. **Conclusions.** This paper has proposed a methodology to design H_∞ filters, for 2-D continuous systems described by a Roesser state-space model with state delays and norm-bounded parameter uncertainties in the state and measurement equations. More

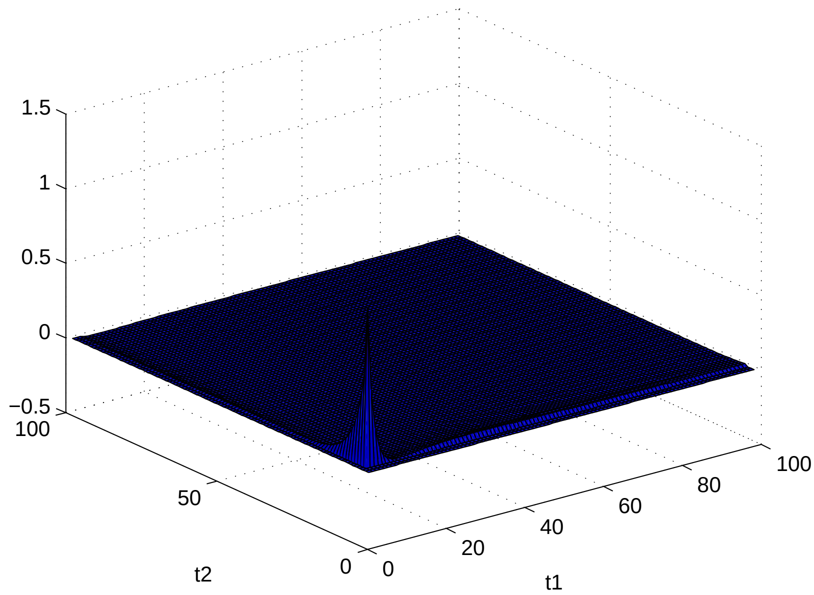

FIGURE 3. Filtering error response of $\tilde{z}(t_1, t_2)$

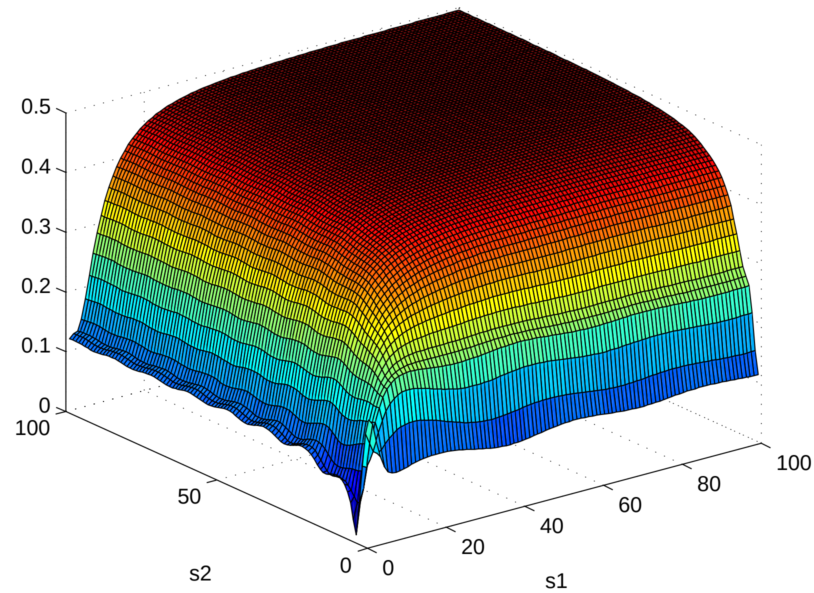

FIGURE 4. Frequency response of error system

precisely, an LMI approach for designing a 2-D continuous filter has been developed, to ensure asymptotic stability and H_∞ performance of the 2-D filter. In addition, an optimization problem for optimizing the H_∞ performance is given. A numerical example has shown the effectiveness of the proposed method.

Acknowledgment. This work was funded by AECID A/030426/10, AP/034911/11 and MiCInn DPI2010-21589-c05 projects.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Englewood Cliffs, Prentice-Hall, NJ, 1979.

[2] A. Elsayed and M. J. Grimble, A new approach to H_∞ design of optimal digital linear filters, *IMA J. Math. Contr. Inform.*, vol.6, pp.233-251, 1989.

[3] H. Xu, Y. Zou and S. Xu, Robust H_∞ control for a class of uncertain nonlinear two-dimensional systems, *International Journal Innovative Computing, Information and Control*, vol.1, no.2, pp.181-191, 2005.

[4] E. Gershon, D. J. N. Limebeer, U. Shaked and I. Yaesh, Robust H_∞ filtering of stationery continuous-times linear systems with stochastic uncertainties, *IEEE Trans. Automat. Control*, vol.46, pp.1788-1793, 2001.

[5] K. M. Nagpal and P. P. Khargonekov, Filtering and smoothing in an H_∞ setting, *IEEE Trans. Automatic Control*, vol.36, pp.152-166, 1991.

[6] P. G. Park and T. Keileth, H_∞ via convex optimization, *Int. J. Control*, vol.66, pp.15-22, 1997.

[7] S. H. Jiu and J. B. Park, Robust H_∞ filtering for polytopic uncertain systems via convex optimization, *Proc. of Inst. Elect. Eng. Control Theory Appl.*, vol.148, pp.55-59, 2001.

[8] S. Xu and P. Van Dooren, Robust H_∞ filtering for a class of nonlinear systems with state delay and parameter uncertainty, *Int. J. Control*, vol.75, pp.766-774, 2002.

[9] C. Du and L. Xie, *H_∞ Control and Filtering of Two-Dimensional Systems*, Springer Verlog, Heidelberg, Germany, 2002.

[10] N. E. Mastorakis and M. Swamy, A new method for computing the stability margin of two dimensional continuous systems, *IEEE Trans. Circuit and Systems I*, vol.49, pp.869-872, 2002.

[11] H. D. Tuan, P. Apkarian, T. Q. Nguyen and T. Narikiys, Robust mixed H_2/H_∞ filtering of 2-D systems, *IEEE Trans. Signal Processing*, vol.50, pp.1759-1771, 2002.

[12] L. Xie, C. Du, C. Zhang and Y. C. Soh, H_∞ deconvolution filtering of 2-D digital systems, *IEEE Trans. Signal Processing*, vol.50, pp.2319-2332, 2003.

[13] S. Xu, J. Lam, Y. Zou, Z. Lin and W. Paszke, Robust H_∞ filtering for uncertain 2-D continuous systems, *IEEE Trans. Signal Processing*, vol.53, pp.1731-1738, 2005.

[14] K. Galkowski, LMI based stability analysis for 2-D continuous systems, *Proc. of the 9th IEEE Int. Conf. Electron., Circuits Syst.*, Dubrovnik, Croatia, pp.923-926, 2002.

[15] A. Hmamed, M. Alfidi, A. Benzaouia and F. Tadeo, LMI conditions for robust stability of 2-D linear discrete-time systems, *Mathematical Problems in Engineering*, 2008.

[16] A. Hmamed, M. Alfidi, A. Benzaouia and F. Tadeo, Robust stabilization under linear fractional parametric uncertainties of two-Dimensional system with Roesser models, *Int. J. of Sciences and Techniques of Automatic Control and Computer Engineering*, pp.336-348, 2007.

[17] A. Hmamed, F. Mesquine, F. Tadeo, M. Benhayoun and A. Benzaouia, Stabilization of 2D saturated systems by state feedback control, *Multidim. Syst. and Sign. Process.*, vol.21, no.3, pp.277-292, 2010.

[18] M. S. Pickarski, Algebraic characterization of matrices whose multivariable characteristic polynomial is Hermitian, *Proc. of Int. Symp. Operator Theory*, Lubbock, TX, USA, pp.126-126, 1977.

[19] E. Rogers, K. Galkowski and D. H. Owens, Delay differential control theory applied to differential linear repetitive processes, *Amer. Control Conf.*, Anchorage, AK, USA, 2002.

[20] J. Huang, G. Lu and X. Zou, Existence of traveling wave fronts of delayed lattice differential equations, *J. Math. Anal. Appl.*, vol.298, pp.538-558, 2004.

[21] B. G. Zhang and C. J. Tian, Oscillation criteria of a class of partial difference equations with delays, *Comput. Math. with Appl.*, vol.48, pp.291-303, 2004.

[22] C. W. Chen, J. S. H. Tsai and L. S. Shieh, Modeling and solution of two-dimensional input time-delay systems, *J. Franklin Inst.*, vol.337, pp.569-578, 2002.

[23] W. Paszke, J. Lam, K. Galkowski, S. Xu and Z. Lin, Robust stability and stabilization of 2-D discrete state-delayed systems, *Syst. Control Lett.*, vol.51, pp.277-291, 2004.

[24] A. Benzaouia, M. Benhayoun and F. Tadeo, State feedback stabilization of 2-D continuous systems with delays, *International Journal of Innovative Computing, Information and Control*, vol.7, no.2, pp.977-988, 2011.

[25] El-K. Boukas and N. F. Al-Muthairi, Delay-dependent stabilization of singular linear systems with delays, *International Journal of Innovative Computing, Information and Control*, vol.2, no.2, pp.283-291, 2006.

[26] B. Chen, J. Lam and S. Xu, Memory state feedback guaranteed cost control for neutral delay systems, *International Journal of Innovative Computing, Information and Control*, vol.2, no.2, pp.293-303, 2006.

[27] H. Gao and C. Wang, A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems, *IEEE Trans. Signal Process.*, vol.52, no.6, pp.1631-1640, 2004.
[28] H. Gao and C. Wang, Robust $L_2 - L_\infty$ filtering for uncertain systems with multiple time-varying state delays, *IEEE Trans. Circuits Syst. I*, vol.50, no.4, pp.594-599, 2003.
[29] C. H. Liu, M. Y. Hsiao and S. H. Tsai, Stabilization for a class of time-delay discrete bilinear system, *Information Technology Journal*, vol.10, no.2, pp.457-460, 2011.
[30] M. Y. Shieh, J. S. Chiou and C. M. Cheng, Delay independence stability analysis and switching law design for the switched time-delay systems, *Information Technology Journal*, vol.10, no.6, pp.1201-1207, 2011.
[31] Y. Zhang, A. G. Wu and G. R. Duan, Enhanced H_∞ filtering for continuous-time state-delayed systems, *International Journal of Automation and Computing*, vol.6, no.2, pp.159-163, 2009.
[32] W. Paszke, J. Lam, K. Galkowski, S. Xu and E. Rogers, H_∞ control of 2-D linear state-delayed systems, *The 4th IFAC Workshop Time-Delay Systems*, Rocquencourt, France, 2003.
[33] S. Xu, T. Chen and J. Lam, Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time delays, *IEEE Trans. Autom. Control*, vol.48, no.5, pp.900-907, 2003.
[34] El-K. Boukas and Z. K. Liu, *Deterministic and Stochastic Time Delay Systems*, 1st Edition, Montreal, Canada, 2002.
[35] E. I. Jury, Stability of multi-dimensional scalar and matrix polynomials, *Proc. of IEEE*, vol.66, no.9, pp.1018-1047, 1978.
[36] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, Linear matrix inequalities in system and control theory, *SIAM Studies in Applied Mathematics*, SIAM, Philadephia, Pennsylvania, 1994.