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EXPONENTIAL STABILITY OF SINGULARLY PERTURBED DISCRETE SYSTEMS WITH TIME-DELAY

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Abstract. *In this paper, the exponential stability of singularly perturbed discrete systems (SPDSs) with time-delay is investigated via the Lyapunov's direct method. In the previous results about the SPDS with time-delay, the asymptotic stability is obtained using the frequency-domain approach. However, we propose a composite Lyapunov function to show that the SPDS with time-delay is exponentially stable with the decay rate γ. In terms of the LMI, the sufficient condition for the exponential stability of the linear SPDS is presented. Moreover, based on the linear SPDS result, the exponential stability of the nonlinear SPDS with time-delay is also considered. Finally, numerical examples are given to validate the proposed results.*

Keywords: Singularly perturbed system, Discrete-time system, Time-delay

1. **Introduction.** There exist small parameters such as small time constants, masses, capacitances which increase the order of the system. In this respect, the singular perturbation method is generally effective to reduce the analysis complexity of the high-order systems, which is separated into two subsystems (slow subsystem and fast subsystem). In addition, there are many results about singularly perturbed systems ([1-3] and references therein). Recently, it becomes more feasible that the discrete-time approach is used for the system analysis [4-6], as the controller is usually implemented by the low-cost digital processors. The singularly perturbed systems are also extensively studied not only in the continuous-time but in the discrete-time with regard to the stability analysis and the controller design [7-11].

It is well known that the time-delay is commonly encountered in the practical systems, which degrades the system performance and leads to instability. Therefore, it is of great importance to study the stability for the system with time-delay. In the linear singularly perturbed discrete system with time-delay, the sufficient conditions of the asymptotic stability results are presented in [12, 13] using the frequency-domain approach. Furthermore, by means of the critical stability criteria, the stability problem for the linear singularly perturbed discrete system with multiple time-delay is considered. Under the D-stability conditions of the slow subsystem and the fast subsystem, the D-stability analysis of the singularly perturbed discrete system with time-delay is examined in [15, 16]. However, the results also do not present the exponential stability of the singulary perturbed system with time-delay. The D-stabilization controllers for the singularly perturbed discrete system with time-delay are designed by the composite feedback [17]. For the nonlinear singularly perturbed discrete system, there are just a few results without time-delay. Like the linear singularly perturbed discrete system, the stabilization controller for a class of the nonlinear singularly perturbed discrete system is designed based on the slow subsystem and the fast subsystem [18, 19]. In [20], the time-scale decomposition method is proposed to analyze the exponential stability of the nonlinear singularly perturbed discrete system. To the best of the authors' knowledge, the singularly perturbed discrete system with time-delay has not been fully investigated. Specifically, there are no results for the exponential stability of the singularly perturbed discrete system with time-delay.

In this paper, we propose the exponential stability analysis for the linear and the nonlinear singularly perturbed discrete system with time-delay. First, we deal with the linear singularly perturbed discrete system with time-delay. Compared with the previous results of the asymptotic stability, the exponential stability of the linear singularly perturbed discrete system with time-delay is presented using a composite Lyapunov function and the definition of the exponential stability. Then, the nonlinear singularly perturbed discrete system with time-delay is also considered. Based on the stability conditions of the slow subsystem and the fast subsystem, we analyze the exponential stability of the nonlinear singularly perturbed discrete system with time-delay. Finally, numerical examples are given to illustrate the results.

2. **Linear System.**

Consider the following linear singularly perturbed system with time-delay:

$$\begin{aligned} x(k+1) &= A_{11}x(k) + A_{12}z(k) + D_{11}x(k-d) + D_{12}z(k-d) \\ z(k+1) &= \epsilon\left\{A_{21}x(k) + A_{22}z(k) + D_{21}x(k-d) + D_{22}z(k-d)\right\} \end{aligned} \tag{1}$$

where $x \in \Re^n$ and $z \in \Re^m$ are state vectors, A_{ij} and D_{ij} for $i, j = 1, 2$ are real constant matrices, and ϵ is a small positive parameter.

Assumption 2.1. *There exists ϵ^* such that for all $\epsilon < \epsilon^*$, A_{11} and ϵA_0 are stable in the discrete-time sense where $A_0 = A_{22} - A_{21}A_{11}^{-1}A_{12}$.*

Definition 2.1. *The null solution of the discrete delay system is said to be exponentially stable if there exist constants $C > 0$ and $0 \le \gamma < 1$ such that*

$$\|x(k)\| \le C\gamma^k \|x_0\|_\infty \tag{2}$$

where $\|x_0\|_\infty = \max_{-d\le\theta\le 0} \|x_0(\theta)\|$ [4].

From Assumption 2.1, there exist $P_{11} > 0$ and $P_0 > 0$ that satisfy the following equations:

$$A_{11}{}^T P_{11} A_{11} - P_{11} = -Q_{11} \tag{3}$$

$$\epsilon A_0{}^T P_0 \epsilon A_0 - P_0 = -Q_0 \tag{4}$$

where Q_{11} and Q_0 are positive-definite matrices. For the exponential stability of the singularly perturbed system with time-delay (1), we introduce the following Lyapunov function.

$$\begin{aligned} \nu = {} & \eta^k y^T(k)P_{11}y(k) + \eta^k z^T(k)P_0 z(k) \\ & + \sum_{i=k-d}^{k-1} \left\{\eta^{i+1}y^T(i)P_1 y(i) + \eta^{i+1}z^T(i)P_2 z(i)\right\} \end{aligned} \tag{5}$$

where $1 < \eta < \infty$, $y(k) = x(k) + A_{11}^{-1}A_{12}z(k)$, P_1 and P_2 are arbitrary positive-definite matrices. The time difference of the Lyapunov function (5) is

$$\Delta\nu = \eta^{k+1}y^T(k+1)P_{11}y(k+1) + \eta^{k+1}z^T(k+1)P_0 z(k+1) - \eta^k y^T(k)P_{11}y(k)$$

$$
\begin{aligned}
& -\eta^k z^T(k) P_0 z(k) + \sum_{i=k-d+1}^{k} \left\{\eta^{i+1} y^T(i) P_1 y(i) + \eta^{i+1} z^T(i) P_2 z(i)\right\} \\
& - \sum_{i=k-d}^{k-1} \left\{\eta^{i+1} y^T(i) P_1 y(i) + \eta^{i+1} z^T(i) P_2 z(i)\right\} \\
\leq\; & \eta^{k+1}[A_{11} y(k) + D_{11} y(k-d) + D_0 z(k-d) \\
& + O(\epsilon)\{y(k) + z(k) + y(k-d) + z(k-d)\}]^T P_{11} \\
& \times [A_{11} y(k) + D_{11} y(k-d) + D_0 z(k-d) \\
& + O(\epsilon)\{y(k) + z(k) + y(k-d) + z(k-d)\}] \\
& + \eta^{k+1}[\epsilon A_0 z(k) + O(\epsilon)\{y(k) + y(k-d) + z(k-d)\}]^T P_0 \\
& \times [\epsilon A_0 z(k) + O(\epsilon)\{y(k) + y(k-d) + z(k-d)\}] \\
& - \eta^k y^T(k) P_{11} y(k) - \eta^k z^T(k) P_0 z(k) + \eta^{k+1} y^T(k) P_1 y(k) + \eta^{k+1} z^T(k) P_2 z(k) \\
& - \eta^{k-d+1} y^T(k-d) P_1 y(k-d) + \eta^{k-d+1} z^T(k-d) P_2 z(k-d) \\
=\; & -\eta^{k+1} Y^T [Q + O(\epsilon)] Y
\end{aligned} \tag{6}
$$

where $Y = \begin{bmatrix} y^T(k) & z^T(k) & y^T(k-d) & z^T(k-d) \end{bmatrix}^T$ and

$$
Q = \begin{bmatrix} \tilde{Q}_{11} & 0 & -A_{11}{}^T P_{11} D_{11} & -A_{11}{}^T P_{11} D_0 \\ 0 & \tilde{Q}_0 & 0 & 0 \\ -D_{11}{}^T P_{11} A_{11} & 0 & \eta^{-d} P_1 - D_{11}{}^T P_{11} D_{11} & -D_{11}{}^T P_{11} D_0 \\ -D_0{}^T P_{11} A_{11} & 0 & -D_0{}^T P_{11} D_{11} & \eta^{-d} P_2 - D_0{}^T P_{11} D_0 \end{bmatrix} \tag{7}
$$

with $\tilde{Q}_{11} = Q_{11} - (1-\eta^{-1}) P_{11} - P_1$, $\tilde{Q}_0 = Q_0 - (1-\eta^{-1}) P_0 - P_2$ and $D_0 = D_{12} - D_{11} A_{11}^{-1} A_{12}$.

Theorem 2.1. *Consider the singularly perturbed discrete system with time-delay (1). Suppose that Assumption 2.1 is satisfied. Then, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the singularly perturbed discrete system with time-delay is exponentially stable if there exist η, P_1, and P_2 for the given delay d such that Q is positive-definite.*

Proof: From Assumption 2.1, we set a Lyapunov function (5) with the time-difference as shown in (6). If the matrix Q is positive-definite, there exists ϵ^* such that for all $\epsilon < \epsilon^*$, the time-difference of the Lyapunov function (6) is negative-definite, that is, $\Delta\nu \leq 0$. Then, we have $\nu(k) \leq \nu(0)$. Now, we obtain the lower norm bound of $\nu(k)$ as

$$
\nu(k) \geq \eta^k Z^T(t) P Z(t) \geq \eta^k \lambda_{\min}(P) \|Z(t)\|^2 \tag{8}
$$

where $Z(t) = \begin{bmatrix} y^T(t) & z^T(t) \end{bmatrix}^T$, $P = diag(P_{11}, P_0)$, and $diag\{A_1, \cdots, A_n\}$ means a block diagonal matrix with A_i on the diagonal. Furthermore, we have the upper norm bound of $\nu(0)$ as

$$
\begin{aligned}
\nu(0) &\leq \lambda_{\max}(P) \|Z(0)\|^2 + \lambda_{\max}(\bar{P}) \max_{-d \leq \theta \leq 0} \|Z(\theta)\|^2 \sum_{i=-d}^{-1} \eta^{i+1} \\
&\leq \left\{\lambda_{\max}(P) + \lambda_{\max}(\bar{P}) \eta^{-d+1} \frac{\eta^d - 1}{\eta - 1}\right\} \max_{-d \leq \theta \leq 0} \|Z(\theta)\|^2
\end{aligned} \tag{9}
$$

where $\bar{P} = diag(P_1, P_2)$. Thus, from (8) and (9) with $\nu(k) \leq \nu(0)$, we obtain

$$
\|Z(t)\| \leq \sqrt{\frac{1}{\lambda_{\min}(P)} \left\{\lambda_{\max}(P) + \lambda_{\max}(\bar{P}) \eta^{-d+1} \frac{\eta^d - 1}{\eta - 1}\right\}} \gamma^k \max_{-d \leq \theta \leq 0} \|Z(\theta)\| \tag{10}
$$

where $0 < \gamma = \sqrt{1/\eta} < 1$. Therefore, the singularly perturbed discrete system with time-delay is exponentially stable from the definition of the exponential stability.

Remark 2.1. *Consider the following singularly perturbed discrete system with multiple time delays:*

$$
\begin{aligned}
x(k+1) &= A_{11}x(k) + A_{12}z(k) + \sum_{j=1}^{l} D_{1j}x(k-d_j) + D_{2j}z(k-d_j) \\
z(k+1) &= \epsilon A_{21}x(k) + \epsilon A_{22}z(k) + \epsilon \sum_{j=1}^{l} D_{3j}x(k-d_j) + D_{4j}z(k-d_j)
\end{aligned} \tag{11}
$$

Suppose that Assumption 2.1 is satisfied. Then, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the singularly perturbed discrete system with multiple time delays is exponentially stable if there exist η, $\bar{P}_i$, and $\tilde{P}_i$ for the given delay d_i with $i = 1, \cdots, l$ such that the following matrix $\tilde{Q}$ is positive-definite.

$$
\tilde{Q} = diag\left\{\tilde{Q}_1, \tilde{Q}_2, \bar{P}_d, \tilde{P}_d\right\} - E^T {A_{11}}^T P_{11} D - D^T P_{11} A_{11} E - D^T P_{11} D \tag{12}
$$

where $diag\{A_1, \cdots, A_n\}$ means a block diagonal matrix with A_i on the diagonal, $\tilde{Q}_1 = Q_{11} - (1-\eta^{-1})P_{11} - \sum_{j=1}^{l} \bar{P}_j$, $\tilde{Q}_2 = Q_0 - (1-\eta^{-1})P_0 - \sum_{j=1}^{l} \tilde{P}_j$, $\bar{P}_d = diag\left\{\eta^{-d_1}\bar{P}_1, \cdots, \eta^{-d_l}\bar{P}_l\right\}$, $\tilde{P}_d = diag\left\{\eta^{-d_1}\tilde{P}_1, \cdots, \eta^{-d_l}\tilde{P}_l\right\}$, $E = \left[\begin{array}{cccc} I_{n\times n} & 0_{n\times m} & 0_{n\times nl} & 0_{n\times ml}\end{array}\right]$, $I_{n\times n}$ is an $n\times n$ identity matrix, $0_{n\times m}$ is an $n\times m$ zero matrix, $D = \left[\begin{array}{cccc} 0_{n\times n} & 0_{n\times m} & D_1 & D_2\end{array}\right]$, $D_1 = \left[\begin{array}{ccc} D_{11} & \cdots & D_{1l}\end{array}\right]$, $D_2 = \left[\begin{array}{ccc} D_{01} & \cdots & D_{0l}\end{array}\right]$, and $D_{0i} = D_{2i} - D_{1i}{A_{11}}^{-1}A_{12}$ for $i = 1, \cdots, l$. The proof of this remark is obtained from that of Theorem 2.1 with the following Lyapunov function:

$$
\begin{aligned}
\nu &= \eta^k y^T(k) P_{11} y(k) + \eta^k z^T(k) P_0 z(k) \\
&\quad + \sum_{j=1}^{l} \sum_{i=k-d_j}^{k-1} \left\{\eta^{k+1} y^T(i) \bar{P}_j y(i) + \eta^{k+1} z^T(i) \tilde{P}_j z(i)\right\}
\end{aligned} \tag{13}
$$

3. Nonlinear System.

Consider the following nonlinear singularly perturbed discrete system with time-delay:

$$
\begin{aligned}
x(k+1) &= f(x(k), z(k)) + f_d(x(k-d), z(k-d)) \\
z(k+1) &= \epsilon g(x(k), z(k)) + \epsilon g_d(x(k-d), z(k-d))
\end{aligned} \tag{14}
$$

where $x \in D_x \subset \Re^n$ and $z \in D_z \subset \Re^m$ are state vectors, $f : \Re^n \times \Re^m \to \Re^n$ and $g : \Re^n \times \Re^m \to \Re^m$ are assumed to be continuously differentiable functions with $f(0,0) = g(0,0) = 0$, $f_d(0,0) = g_d(0,0) = 0$, and ϵ is a small positive parameter. Moreover, f_d and g_d are bounded as follows:

$$
\|f_d(x(k), z(k))\| \le k_{fd}\{\|x(k)\| + \|z(k)\|\} \tag{15}
$$

$$
\|g_d(x(k), z(k))\| \le k_{gd}\{\|x(k)\| + \|z(k)\|\} \tag{16}
$$

First, the nominal nonlinear singularly perturbed discrete system of the time-delay system (14) is

$$
\begin{aligned}
x(k+1) &= f(x(k), z(k)) \\
z(k+1) &= \epsilon g(x(k), z(k))
\end{aligned} \tag{17}
$$

Assumption 3.1. *The followings are satisfied for all $(x, z) \in D_x \times D_z$ in the nominal system (17).*

- *The equation $0 = f(x(k), z(k))$ has an isolated root $x(k) = h_f(z(k))$ such that $\|h_f(z)\| \leq k_h \|z\|$.*
- *The origin of the slow system $y(k+1) = f(y(k) + h_f(z(k)), z(k))$ is exponentially stable, uniformly in $z(k)$ where $y(k) = x(k) - h_f(z(k))$.*
- *The origin of the fast system $z(k+1) = \epsilon g(h_f(z(k)), z(k))$ is exponentially stable.*

From the above assumption, there exist Lyapunov functions, $V(y(k), z(k))$ and $W(z(k))$, for the slow and fast subsystems of the nominal system (17) as follows [20]:

$$\begin{aligned}
&b_1 \|y(k)\| \leq V(y(k), z(k)) \leq b_2 \|y(k)\| \\
&V\left[f\left\{y(k) + h_f(k), z(k)\right\}, z(k+1)\right] - V(y(k), z(k)) \leq -b_3 \|y(k)\| \\
&V(y(k), z(k)) - V(y^{'}(k), z(k)) \leq b_4 \left\|y(k) - y^{'}(k)\right\| \\
&c_1 \|z(k)\| \leq W(z(k)) \leq c_2 \|z(k)\| \\
&W\left[\epsilon g\left\{h_f(z(k)), z(k)\right\}\right] - W(z(k)) \leq -c_3 \|z(k)\| \\
&W(z(k)) - W(z^{'}(k)) \leq c_4 \left\|z(k) - z^{'}(k)\right\|
\end{aligned}$$

where b_i and c_i are positive constants with $i = 1, \cdots, 4$. From the above Lyapunov functions, we set a Lyapunov function for the nonlinear singularly perturbed discrete system with time-delay (14) as follows:

$$\nu = \eta^k V(y(k), z(k)) + \eta^k W(z(k)) + \sum_{i=k-d}^{k-1} \left\{\eta^i p_1 \|y(i)\| + \eta^i p_2 \|z(i)\|\right\} \tag{18}$$

where p_1 and p_2 are positive constants, and $1 < \eta < \infty$. Now, we have the time-difference of the Lyapunov function along the nonlinear singularly perturbed discrete system with time-delay (14) as

$$\begin{aligned}
\Delta\nu = {} & \eta^k[\eta V\{f(x(k), z(k)) + f_d(x(k-d), z(k-d)) \\
& -h_f(\epsilon g(x(k), z(k)) + \epsilon g_d(x(k-d), z(k-d))), z(k+1)\} \\
& -V(y(k), z(k)) + \eta W\{\epsilon g(x(k), z(k)) + \epsilon g_d(x(k-d), z(k-d))\} \\
& -W(z(k)) + p_1 \|y(k)\| + p_2 \|z(k)\| - \eta^{-d} p_1 \|y(k-d)\| - \eta^{-d} p_2 \|z(k-d)\|] \\
\leq {} & \eta^k[\eta b_4 \|f_d(x(k-d), z(k-d)) - h_f(\epsilon g(x(k), z(k)) + \epsilon g_d(x(k-d), z(k-d)))\| \\
& +V\{f(y(k) + h_f(z(k)), z(k)), z(k+1)\} - V\{y(k), z(k)\} \\
& +(\eta - 1)V\{f(y(k) + h_f(z(k)), z(k)), z(k+1)\} \\
& +\eta c_4 \|\epsilon g(x(k), z(k)) - \epsilon g(h_f(z(k)), z(k)) + \epsilon g_d(x(k-d), z(k-d))\| \\
& +W\{\epsilon g(h_f(z(k)), z(k))\} - W\{z(k)\} + (\eta - 1)W\{\epsilon g(h_f(z(k)), z(k))\} \\
& +p_1 \|y(k)\| + p_2 \|z(k)\| - \eta^{-d} p_1 \|y(k-d)\| - \eta^{-d} p_2 \|z(k-d)\|]
\end{aligned} \tag{19}$$

From a continuously differentiable functions f and g with Assumption 3.1, there exist positive constants k_f, k_{gx} and k_{gz} such that

$$\|f(y(k) + h_f(z(k)), z(k))\| \leq k_f \|y(k)\| \tag{20}$$

$$\|g(x(k), z(k)) - g(x'(k), z'(k))\| \leq k_{gx} \|x(k) - x'(k)\| + k_{gz} \|z(k) - z'(k)\| \tag{21}$$

Using the above inequalities and the norm bounds of the time-delay terms (15) and (16), we rewrite the time-difference of Lyapunov function as

$$\begin{aligned}\Delta\nu \le\ & \eta^k[\eta b_4\{k_{fd}\left\|y(k-d)\right\| + k_{fd}k_h\left\|z(k-d)\right\| + k_{fd}\left\|z(k-d)\right\| \\ & +O(\epsilon)\{\|y(k)\| + \|z(k)\| + \|y(k-d)\| + \|z(k-d)\|\}\} \\ & -b_3\left\|y(k)\right\| + (\eta-1)b_2k_f\left\|y(k)\right\| \\ & +\eta c_4\{O(\epsilon)(\|y(k)\| + \|z(k)\| + \|y(k-d)\| + \|z(k-d)\|)\} \\ & -c_3\left\|z(k)\right\| + p_1\left\|y(k)\right\| + p_2\left\|z(k)\right\| \\ & -\eta^{-d}p_1\left\|y(k-d)\right\| - \eta^{-d}p_2\left\|z(k-d)\right\|] \\ =\ & \eta^k[-(b_3-(\eta-1)b_2k_f-p_1-O(\epsilon))\left\|y(k)\right\| \\ & -(c_3-p_2-O(\epsilon))\left\|z(k)\right\| \\ & -(\eta^{-d}p_1-\eta b_4k_{fd}-O(\epsilon))\left\|y(k-d)\right\| \\ & -(\eta^{-d}p_2-\eta b_4k_{fd}k_h-\eta b_4k_{fd}-O(\epsilon))\left\|z(k-d)\right\|]\end{aligned} \tag{22}$$

Theorem 3.1. *Consider the nonlinear singularly perturbed discrete system with time-delay (14). Suppose that Assumption 3.1 is satisfied. Then, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the nonlinear singularly perturbed discrete system with time-delay is exponentially stable if there exists a positive constant η with $1 < \eta < \infty$ for the given time-delay d such that the following inequalities are satisfied:*

$$b_3 - (\eta-1)b_2k_f > \eta^{d+1}b_4k_{fd} \tag{23}$$

$$c_3 > \eta^{d+1}b_4k_{fd}(1+k_h) \tag{24}$$

Proof: From Assumption 3.1, we set a Lyapunov function (18) with the time-difference as shown in (22). If the inequalities (23) and (24) are satisfied, then we obtain positive constants p_1 and p_2 such that the following inequalities hold:

$$b_3 - (\eta-1)b_2k_f > p_1 \tag{25}$$

$$c_3 > p_2 \tag{26}$$

$$\eta^{-d}p_1 > \eta b_4k_{fd} \tag{27}$$

$$\eta^{-d}p_2 > \eta b_4k_{fd}k_h + \eta b_4k_{fd} \tag{28}$$

From the above inequalities, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the time-difference of Lyapunov function (22) is $\Delta\nu \le 0$. Then, we obtain $\nu(k) \le \nu(0)$. Now, we have the lower norm bound of $\nu(k)$ as

$$\nu(k) \ge \eta^k\{b_1\left\|y(k)\right\| + c_1\left\|z(k)\right\|\} \tag{29}$$

Moreover, we have the upper norm bound of $\nu(0)$ as

$$\begin{aligned}\nu(0) =\ & V(y(0),z(0)) + W(z(0)) + \sum_{i=-d}^{-1}\left\{\eta^ip_1\left\|y(i)\right\| + \eta^ip_2\left\|z(i)\right\|\right\} \\ \le\ & b_2\left\|y(k)\right\| + c_2\left\|z(k)\right\| + \sum_{i=-d}^{-1}\left\{\max_{-d\le\theta\le 0}\eta^ip_1\left\|y(\theta)\right\| + \eta^ip_2\left\|z(\theta)\right\|\right\} \\ \le\ & b_2\max_{-d\le\theta\le 0}\left\|y(\theta)\right\| + c_2\max_{-d\le\theta\le 0}\left\|z(\theta)\right\| + \bar{\eta}\max_{-d\le\theta\le 0}\{\left\|y(\theta)\right\| + p_2\left\|z(\theta)\right\|\} \\ =\ & (b_2+\bar{\eta}p_1)\|y\|_\infty + (c_2+\bar{\eta}p_2)\|z\|_\infty\end{aligned} \tag{30}$$

where $\bar{\eta} = (1-\eta^{-d})/(\eta-1)$. From (29) and (30) with $\nu(k) \le \nu(0)$, we obtain

$$\{b_1\left\|y(k)\right\| + c_1\left\|z(k)\right\|\} \le (b_2+\bar{\eta}p_1)\gamma^k\|y\|_\infty + (c_2+\bar{\eta}p_2)\gamma^k\|z\|_\infty \tag{31}$$

where $0 < \gamma = 1/\eta < 1$. Thus, from the definition of the exponential stability, the nonlinear singularly perturbed discrete system with time-delay is exponentially stable.

Remark 3.1. *Consider the following nonlinear singularly perturbed discrete system with multiple time delays:*

$$\begin{aligned} x(k+1) &= f(x(k), z(k)) + \sum_{j=1}^{l} f_{dj}(x(k-d_j), z(k-d_j)) \\ z(k+1) &= \epsilon g(x(k), z(k)) + \epsilon \sum_{j=1}^{l} g_{dj}(x(k-d_j), z(k-d_j)) \end{aligned} \tag{32}$$

Suppose that Assumption 3.1 is satisfied. Then, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the nonlinear singularly perturbed discrete system with multiple time delays is exponentially stable if there exists a positive constant η with $1 < \eta < \infty$ for the given time delays d_j such that the following inequalities are satisfied:

$$b_3 - (\eta - 1)b_2 k_f > \sum_{j=1}^{l} \eta^{d_j+1} b_4 k_{fdj} \tag{33}$$

$$c_3 > \sum_{j=1}^{l} \eta^{d_j+1} b_4 k_{fdj}(1 + k_h) \tag{34}$$

where

$$\|f_{dj}(x(k-d_j), z(k-d_j))\| \leq k_{fdj} \{\|x(k-d_j)\| + \|z(k-d_j)\|\} \tag{35}$$

$$\|g_{dj}(x(k-d_j), z(k-d_j))\| \leq k_{gdj} \{\|x(k-d_j)\| + \|z(k-d_j)\|\} \tag{36}$$

for $j = 1, \cdots, l$. The proof of this remark is obtained from that of Theorem 3.1 with the following Lyapunov function:

$$\nu = \eta^k V(y(k), z(k)) + \eta^k W(z(k)) + \sum_{j=1}^{l} \sum_{i=k-d_j}^{k-1} \left\{\eta^i p_{1j} \|y(i)\| + \eta^i p_{2j} \|z(i)\|\right\} \tag{37}$$

4. **Illustrative Examples.**

Example 4.1. *Consider the following linear singularly perturbed discrete system with time-delay:*

$$\begin{aligned} x(k+1) &= 0.1x(k) + 0.3z(k) + 0.3x(k-d) + 0.1z(k-d) \\ z(k+1) &= \epsilon \{0.4x(k) + 0.7z(k) + 0.4x(k-d) + 0.2z(k-d)\} \end{aligned} \tag{38}$$

where ϵ is a sufficiently small positive constant, x and z are state vectors. In the above system, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, Assumption 2.1 is satisfied with $A_0 = -0.5$. Then, from (3) and (4), we set $P_{11} = 3.0$ and $P_0 = 3.5$ so that Q_{11} and Q_0 are positive-definite. Now, we select η according to the given time-delay d with $P_1 = 1.2$ and $P_2 = 3$ such that the matrix Q is positive-definite as shown in Table 1. Hence, from Theorem 2.1, there exists ϵ^* such that for all $\epsilon < \epsilon^*$, the singularly perturbed discrete system with time-delay is exponentially stable as shown in Figure 1.

TABLE 1. Selection of η according to d

Delay d	*1*	*5*	*10*	*20*	*50*
η	*1.1321*	*1.0258*	*1.0128*	*1.0064*	*1.0025*

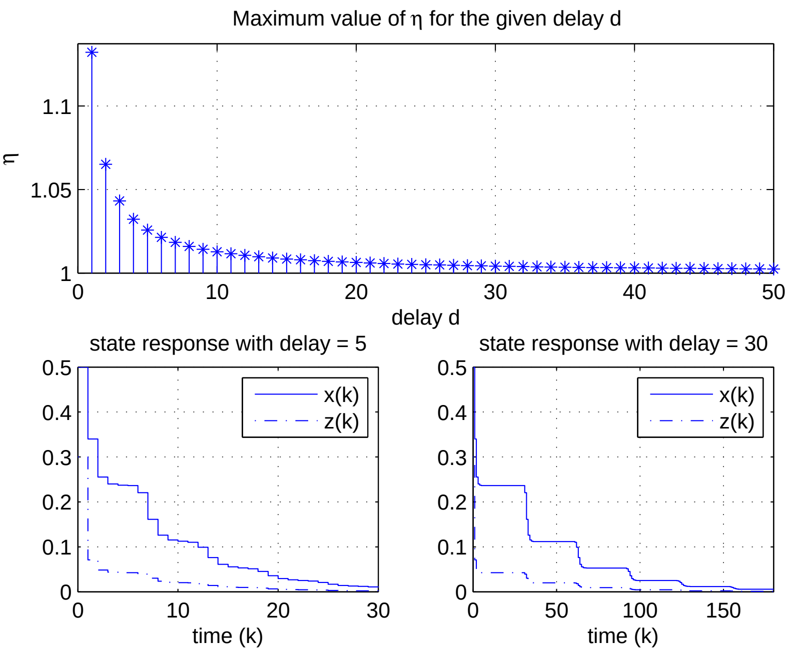

FIGURE 1. State response and maximum value of η with $\epsilon = 0.1$

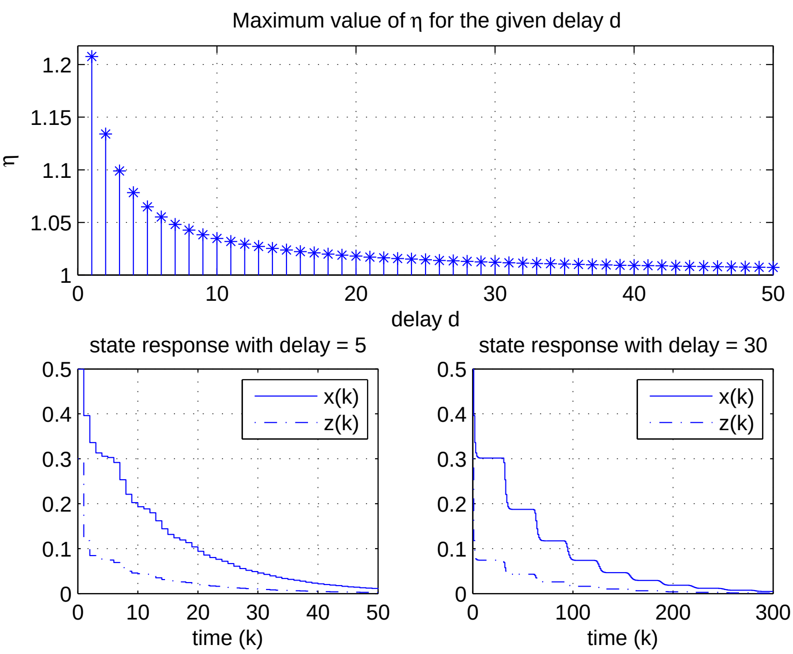

FIGURE 2. State response and maximum value of η with $\epsilon = 0.1$

Example 4.2. *Consider the following nonlinear singularly perturbed system discrete system with time-delay:*

$$\begin{aligned} x(k+1) &= 0.3\tan^{-1}(x(k)+z(k)) + 0.3\sin(x(k-d)) + 0.1z(k-d) \\ z(k+1) &= \epsilon\left\{0.5x^2(k) + (1-x(k))z(k) + 0.7x(k-d) + 0.5z(k-d)\right\} \end{aligned} \tag{39}$$

where ϵ is a sufficiently small positive constant, $x \in \{x : \|x\| < 0.5\}$ and $z \in \{z : \|z\| < 0.5\}$ are state vectors. From the nominal system of (39), Assumption 3.1 is satisfied. Using the isolated root $x(k) = -z(k)$, we obtain the slow subsystem $y(k+1) = 0.3\tan^{-1} y(k)$ where $y(k) = x(k) + z(k)$ and the fast subsystem $z(k+1) = \epsilon(0.1z^2(k) + (1 - z(k))z(k))$ with $V(y(k)) = \|y(k)\|$ and $W(z(k)) = \|z(k)\|$. Now, as shown in Table 2, we select η according to the given time-delay d such that the inequalities (23) and (24) are satisfied by using $b_3 = 0.7$, $b_2 = 1$, $k_f = 0.3$, $b_4 = 1$, $k_{fd} = 0.3$, $c_3 = 0.875$ and $k_h = 1$. Therefore, from Theorem 3.1, there exists ϵ^* such that for all $\epsilon < \epsilon^*$, the nonlinear singularly perturbed discrete system with time-delay is exponentially stable as shown in Figure 2.

TABLE 2. Selection of η according to d

Delay d	*1*	*5*	*10*	*20*	*50*
η	*1.2076*	*1.0649*	*1.0348*	*1.0181*	*1.0074*

5. Conclusions.

We investigate the exponential stability of the linear and the nonlinear singularly perturbed discrete systems with time-delay. First, the linear singularly perturbed discrete system with time-delay is considered to show the exponential stability. Next, based on the stability of the slow and fast subsystems of the nominal nonlinear singularly perturbed discrete systems, we propose a composite Lyapunov function. Then, we present the exponential stability of the nonlinear singularly perturbed discrete system with time-delay. Finally, some examples are given to illustrate the proposed results.

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