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ROBUST $\mathcal{H}_\infty$ FAULT DETECTION FOR UNCERTAIN LDTV SYSTEMS USING KREIN SPACE APPROACH

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Abstract. *This paper deals with the problem of robust $\mathcal{H}_\infty$ fault detection for a class of linear discrete time-varying systems with norm bounded model uncertainty. A generalized unknown input is introduced to represent the model uncertainty and, based on this, an observer-based fault detection filter (FDF) with accommodation of unknown input and fault is proposed. Then the problem of robust fault detection is formulated in a framework of finite horizon $\mathcal{H}_\infty$ filtering and the design of robust $\mathcal{H}_\infty$-FDF is converted into a minimum problem of indefinite quadratic form. A sufficient and necessary condition for the minimum is derived by using a Krein space approach and a solution to the $\mathcal{H}_\infty$-FDF is obtained by computation of Riccati recursions. A numerical example is given to illustrate the effectiveness of the proposed method.*

Keywords: Linear discrete time-varying system, Model uncertainty, $\mathcal{H}_\infty$ fault detection, Krein space, Riccati recursion

1. **Introduction.** During the past three decades, model-based fault detection (FD) has received much attention and significant progress has been achieved; see, for example, [1, 2, 3, 5, 7, 8, 11, 13, 14, 17, 18, 20] and references therein. For linear systems subject to $\mathcal{L}_2$ norm bounded unknown input, there are two typical $\mathcal{H}_\infty$ approaches to robust FD. One scheme is to use $\mathcal{H}_\infty$ norm as a measure of robustness of residual to unknown input, while the $\mathcal{H}_\infty$ norm or $\mathcal{H}_-$ index is used as a measure of sensitivity of residual to fault. Then the design of observer-based FDF can be formulated into an optimization problem in a framework of $\mathcal{H}_-/\mathcal{H}_\infty$ or $\mathcal{H}_\infty/\mathcal{H}_\infty$ maximization and a trade-off between the sensitivity and the robustness can be achieved; see, e.g., the frequency domain co-inner-outer factorization solution in [6] and the linear matrix inequality (LMI) based solution in [21]. Another scheme focuses on robust FD in the sense of $\mathcal{H}_\infty$ filtering; see, e.g., the LMI based solutions in [4, 12, 19, 24]. In [10], a comparison of the two typical FD schemes was given and it was revealed that both of them allowed the achievement of the same level performance.

In contrast to the numerous existing robust FD results for linear time invariant (LTI) systems, only a few studies have been devoted to linear time-varying (LTV) systems. In [23], the problem of robust FD for linear discrete time periodic systems was dealt with by extending the result in [6]. In [15, 16, 22], the result of [6] was extended to LTV systems. It has been demonstrated in [15, 16] that the robust FD problem under different performance indices, such as the $\mathcal{H}_-/\mathcal{H}_\infty$, $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_\infty/\mathcal{H}_\infty$ sensitivity/robustness ratio, can be solved by a unified optimal solution. In [25], an observer-based FDF with residual feedback was proposed for linear discrete time-varying (LDTV) systems and, through building a relationship with Krein space projection, a solution to the $\mathcal{H}_\infty$-FDF was obtained by computation of Riccati recursions. In [26], a unified solution to the robust FD problem of finite horizon $\mathcal{H}_\infty/\mathcal{H}_\infty$ or $\mathcal{H}_-/\mathcal{H}_\infty$ maximization was developed for LDTV systems. However, it should be pointed out that model uncertainty and unknown input are inevitable in practice. The approaches in [15, 16, 22, 26] are not applicable to uncertain LDTV systems, because the solution of the $\mathcal{H}_-/\mathcal{H}_\infty$ or $\mathcal{H}_\infty/\mathcal{H}_\infty$ based optimization problem will not exist anymore when model uncertainty appears. The result in [25] did not consider the influence of model uncertainty which may cause performance degradation of the $\mathcal{H}_\infty$-FDF for uncertain LDTV systems. To the authors' best knowledge, the problem of robust FD is still open and challenging when model uncertainty and unknown input are taken into account simultaneously, which motivates the present study.

In this paper, we focus our study on observer-based robust FD in the framework of finite horizon $\mathcal{H}_\infty$ filtering for LDTV systems subject to l_2 norm bounded unknown input and norm bounded model uncertainty. A generalized unknown input will be first introduced to represent the influence of the model uncertainty. Then an improved observer-based FDF will be developed of which the estimation and accommodation of model uncertainty, unknown input and fault are considered. Inspired by [25], the problem of robust FD will be solved by using a Krein space approach.

Notation. Elements in Krein space will be denoted by boldface letters and elements in the Euclidean space of complex numbers will be denoted by normal letters. The superscripts '-1' and 'T' stand for the inverse and transpose of a matrix, respectively. $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. I is the identity matrix with appropriate dimensions. $\langle \cdot, \cdot \rangle$ denotes the inner product in Krein space. $\mathrm{diag}(\cdots)$ denotes a block-diagonal matrix. $\theta(k) \in l_2[0, N]$ means $\sum_{k=0}^{N} \theta^T(k)\theta(k) < \infty$. $\mathcal{L}\{\{\theta_i\}_{i=1}^N\}$ denotes a linear space spanned by sequence $\theta_1, \cdots, \theta_N$. For a random variable α in Krein space, $\alpha \perp \mathcal{L}\{\{\theta_i\}_{i=1}^N\}$ means that α is orthogonal with $\mathcal{L}\{\{\theta_i\}_{i=1}^N\}$. $\mathrm{Proj}\{\alpha|\theta_1, \cdots, \theta_N\}$ stands for the orthogonal projection of α onto linear space $\mathcal{L}\{\{\theta_i\}_{i=1}^N\}$.

2. Problem Formulation.

Consider the following LDTV systems

$$\begin{cases} x(k+1) = (A(k) + \Delta A(k))x(k) + (B(k) + \Delta B(k))u(k) + B_d(k)d(k) + B_f(k)f(k) \\ y(k) = (C(k) + \Delta C(k))x(k) + (D(k) + \Delta D(k))u(k) + D_f(k)f(k) + v(k) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^m$, $u(k) \in \mathbb{R}^p$, $d(k) \in \mathbb{R}^q$, $v(k) \in \mathbb{R}^m$ and $f(k) \in \mathbb{R}^l$ are the state, measurement output, control input, unknown input, measurement noise and fault, respectively; $u(k)$, $d(k)$, $v(k)$, $f(k) \in l_2[0, N]$; x_0 denotes the initial state; $A(k)$, $B(k)$, $B_d(k)$, $B_f(k)$, $C(k)$, $D(k)$ and $D_f(k)$ are known matrices with appropriate dimensions; $\Delta A(k)$, $\Delta B(k)$, $\Delta C(k)$ and $\Delta D(k)$ are the uncertain model matrices described by

$$\begin{bmatrix} \Delta A(k) & \Delta B(k) \\ \Delta C(k) & \Delta D(k) \end{bmatrix} = \begin{bmatrix} E_1(k) \\ E_2(k) \end{bmatrix} \Sigma(k) \begin{bmatrix} F_1(k) & F_2(k) \end{bmatrix} \tag{2}$$

where $E_1(k)$, $E_2(k)$, $F_1(k)$ and $F_2(k)$ are known matrices with appropriate dimensions, $\Sigma(k) \in \mathbb{R}^{l_1 \times l_2}$ is unknown and satisfies

$$\Sigma^T(k)\Sigma(k) \leq I, \quad \forall k \in [0, N] \tag{3}$$

For the purpose of residual generation, we first consider the following observer-based FDF

$$\begin{cases} \hat{x}(k+1) = A(k)\hat{x}(k) + B(k)u(k) + H(k)(y(k) - C(k)\hat{x}(k) - D(k)u(k)) \\ r_f(k) = V(k)(y(k) - C(k)\hat{x}(k) - D(k)u(k)) \\ \hat{x}(0) = 0 \end{cases} \tag{4}$$

where $\hat{x}(k)$, $r_f(k)$ denote state estimation and residual, respectively.

Let $e(k) = x(k) - \hat{x}(k)$. Applying to (1) and (4) yields

$$\begin{cases} e(k+1) = (A(k) - H(k)C(k))e(k) + B_d(k)d(k) - H(k)v(k) \\ \qquad\qquad + (B_f(k) - H(k)D_f(k))f(k) + (\Delta A(k) - H(k)\Delta C(k))x(k) \\ \qquad\qquad + (\Delta B(k) - H(k)\Delta D(k)))u(k) \\ r_f(k) = V(k)(C(k)e(k) + D_f(k)f(k) + v(k) + \Delta C(k)x(k) + \Delta D(k)u(k)) \\ x(0) = x_0, \quad e(0) = x_0 \end{cases} \tag{5}$$

Substituting (2) into (5) and introducing

$$\varphi(k) = \Sigma(k)(F_1(k)x(k) + F_2(k)u(k))$$

we then have

$$\begin{cases} e(k+1) = (A(k) - H(k)C(k))e(k) + B_d(k)d(k) - H(k)v(k) \\ \qquad\qquad + (B_f(k) - H(k)D_f(k))f(k) + (E_1(k) - H(k)E_2(k))\varphi(k) \\ r_f(k) = V(k)(C(k)e(k) + D_f(k)f(k) + v(k) + E_2(k)\varphi(k)) \end{cases} \tag{6}$$

A typical way is to handle $\varphi(k)$ as an unknown input and formulate the finite horizon robust FD problem in order to find $H(k)$ and $V(k)$ such that

$$\frac{\sum_{k=0}^{N-1} \|r_f(k) - f(k)\|^2}{x_0^T P_0^{-1} x_0 + \sum_{k=0}^{N} \|w(k)\|^2} < \gamma^2 \tag{7}$$

where γ is a given positive scalar, $w(k) = \begin{bmatrix} d^T(k) & f^T(k) & \varphi^T(k) & v^T(k) \end{bmatrix}^T$, and P_0 is a given positive definite weighting matrix. There is no doubt that this may lead to large conservatism.

In order to improve the performance of the residual generator, we propose a structure of FDF with the estimation and accommodation of $f(k)$, $d(k)$ and $\varphi(k)$. For this purpose, the following modified FDF is established

$$\begin{cases} \hat{x}(k+1|k) = A(k)\hat{x}(k) + B(k)u(k) + H_1(k)\varepsilon(k) \\ \hat{x}(k+1) = \hat{x}(k+1|k) + H_2(k)\varepsilon(k) + H_3(k)r(k) \\ \varepsilon(k) = y(k) - C(k)\hat{x}(k) - D(k)u(k) \\ \varepsilon(k+1|k) = y(k+1) - C(k+1)\hat{x}(k+1|k) - D(k+1)u(k+1) \\ r(k) = V_1(k)\varepsilon(k) + V_2(k+1)\varepsilon(k+1|k) \\ \hat{x}(0) = 0 \end{cases} \tag{8}$$

where $\hat{x}(k+1|k)$, $\hat{x}(k)$ denote the one-step state prediction and state estimation, respectively; $H_i(k)$ $(i = 1, 2, 3)$, $V_1(k)$ and $V_2(k+1)$ are parameter matrices to be designed; $r(i) = \begin{bmatrix} r_f^T(i) & r_\varphi^T(i) & r_d^T(i) \end{bmatrix}^T$; $r_\varphi(k)$ and $r_d(k)$ stand for the estimation of $\varphi(k)$ and $d(k)$, respectively.

Now, the problem of finite horizon robust $\mathcal{H}_\infty$ FD can be re-formulated in order to find $H_i(k)$ $(i = 1, 2, 3)$, $V_1(k)$ and $V_2(k+1)$ such that (7) is satisfied.

Remark 2.1. *It is worth pointing out that the uncertain matrices $\Delta A(k)$, $\Delta B(k)$, $\Delta C(k)$ and $\Delta D(k)$ in (1) may be norm bounded type, polytopic type or stochastic ones. In this paper, only the norm bounded type described by (2) and (3) is dealt with. Moreover, the way of handling model uncertainty as unknown input is well known in robust FD for LTI systems, but it is a novel idea to consider the estimation and accommodation of $f(k)$, $d(k)$, $\varphi(k)$ in the residual generator.*

Remark 2.2. *Different from the scheme of finite horizon $\mathcal{H}_\infty/\mathcal{H}_\infty$ and $\mathcal{H}_-/\mathcal{H}_\infty$ optimization in* [26]*, this paper deals with the problem of robust FD in the framework of $\mathcal{H}_\infty$ filtering* [25]*, which can be regarded as a finite horizon time-varying version of* [4]*. If $H_3(k) = 0$, $V_2(k+1) = 0$, $N \to \infty$ and system (1) is LTI, then (8) with (7) leads to the $\mathcal{H}_\infty$ filtering formulation of FD in* [4]*, which can be considered as a special case of this paper. If the accommodation of model uncertainty and unknown input is not taken into account, then (8) becomes the observer-based FDF in* [25]*. The key feature of the proposed robust FD scheme is the simultaneous estimation and accommodation of the model uncertainty, unknown input and fault. Compared with the $\mathcal{H}_\infty - FDF$ in* [25]*, the robust FDF (8) can achieve better FD performance when model uncertainty appears.*

Remark 2.3. *It has been shown in* [9, 25] *that the $\mathcal{H}_\infty$ filtering problem can be formulated as calculating the minimum of a certain quadratic form and, through establishing a relationship with the orthogonal projection and innovation analysis in Krein space, a necessary and sufficient existence condition for the minimum can be derived. A solution to the robust $\mathcal{H}_\infty$-FDF can be obtained by computation of Riccati recursions. Inspired by these, the similar techniques with* [25] *will be employed to solve the formulated robust FD problem.*

3. Main Results.

3.1. The $\mathcal{H}_\infty$-FDF problem and indefinite quadratic form.

We start with the introduction of the following indefinite quadratic form

$$
\begin{aligned}
J_N = {} & x_0^T P_0^{-1} x_0 + \sum_{k=0}^{N} \|w(k)\|^2 - \gamma^{-2} \sum_{k=0}^{N-1} \|r_f(k) - f(k)\|^2 \\
& -\rho^2 \sum_{k=0}^{N-1} (\|r_\varphi(k) - \varphi(k)\|^2 + \|r_d(k) - d(k)\|^2)
\end{aligned} \tag{9}
$$

where $\rho > 0$ is a sufficient small scalar. Referring to the robust $\mathcal{H}_\infty$-FDF problem, we note that (7) is satisfied for all nonzero x_0 and $w(k)$ if $J_N > 0$. Let

$$
v_f(i) = r_f(i) - f(i), \quad v_\varphi(i) = r_\varphi(i) - \varphi(i), \quad v_d(i) = r_d(i) - d(i) \tag{10}
$$

$$
v_s(i) = \left\{ \begin{array}{l} \begin{bmatrix} v(i) \\ v_{f_s}(i) \end{bmatrix}, \quad i \le k-1 \\ v(k), \quad i = k \end{array} \right., \quad v_{f_s}(i) = \begin{bmatrix} v_f(i) \\ v_\varphi(i) \\ v_d(i) \end{bmatrix}, \quad f_s(i) = \begin{bmatrix} f(i) \\ \varphi(i) \\ d(i) \end{bmatrix} \tag{11}
$$

$$
v_{sN} = \begin{bmatrix} v_s^T(0) & v_s^T(1) & \cdots & v_s^T(N) \end{bmatrix}^T \tag{12}
$$

$$
f_{sN} = \begin{bmatrix} f_s^T(0) & f_s^T(1) & \cdots & f_s^T(N) \end{bmatrix}^T \tag{13}
$$

$$
Q_{v_s N} = \mathrm{diag}(Q_{v_s}(0), Q_{v_s}(1), \cdots, Q_{v_s}(N)) \tag{14}
$$

$$
Q_{v_s}(k) = \left\{ \begin{array}{l} \mathrm{diag}(I, -\gamma^2 I, -\rho^{-2} I), \quad k \le N-1 \\ I, \quad k = N \end{array} \right. \tag{15}
$$

Then, J_N can be rewritten as the following indefinite quadratic form

$$
J_N = \begin{bmatrix} x_0 \\ f_{sN} \\ v_{sN} \end{bmatrix}^T \begin{bmatrix} P_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{v_s N} \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ f_{sN} \\ v_{sN} \end{bmatrix} \tag{16}
$$

For the sake of notation simplicity, we further denote

$$
y_s(i) = \left\{ \begin{array}{l} y_a(i), \quad i \le k-1 \\ y(k), \quad i = k \end{array} \right., \quad y_a(i) = \begin{bmatrix} y(i) \\ r(i) \end{bmatrix}
$$

$$B_{sf}(i) = \left[\begin{array}{ccc} B_f(i) & E_1(i) & B_d(i) \end{array}\right], \quad D_{fd}(i) = \left[\begin{array}{ccc} D_f(i) & E_2(i) & 0 \end{array}\right]$$

$$C_s(i) = \left\{\begin{array}{ll} \bar{C}(i), & i \le k-1 \\ C(k), & i = k \end{array}\right., \quad \bar{C}(i) = \left[\begin{array}{c} C(i) \\ 0 \end{array}\right]$$

$$D_s(i) = \left\{\begin{array}{ll} \bar{D}(i), & i \le k-1 \\ D(k), & i = k \end{array}\right., \quad \bar{D}(i) = \left[\begin{array}{c} D(i) \\ 0 \end{array}\right]$$

$$D_{sf}(i) = \left\{\begin{array}{ll} \bar{D}_{fd}(i), & i \le k-1 \\ C(k), & i = k \end{array}\right., \quad \bar{D}_{fd}(i) = \left[\begin{array}{c} D_{fd}(i) \\ I \end{array}\right]$$

$$H_s(i) = \left\{\begin{array}{ll} \bar{H}(i), & i \le k-1 \\ H_1(k), & i = k \end{array}\right., \quad \bar{H}(i) = \left[\begin{array}{cc} H_1(i) & H_2(i) \end{array}\right]$$

for $k = 0, 1, \cdots, N$. Rewrite (1) as

$$\begin{cases} x(i+1) = A(i)x(i) + B_{sf}(i)f_s(i) \\ y_s(i) = C_s(i)x(i) + D_s(i)u(i) + D_{sf}(i)f_s(i) + v_s(i) \\ x(0) = x_0 \end{cases} \tag{17}$$

It is evident that (7) is satisfied if J_N subject to (8) and (17) has a minimum over $\{x_0, f_{sN}\}$ and its value at the minimum is positive. Thus, the design of robust $\mathcal{H}_\infty$-FDF can be converted into the minimum problem of indefinite quadratic form J_N.

Remark 3.1. *The ρ in (9) is a weighting positive scalar. Increasing ρ means weighting the influence of the estimation error of model uncertainty and unknown input more strongly. To guarantee the feasibility of* $\min J_N > 0$*, the allowed maximal ρ and minimal γ should be considered. From the viewpoint of $\mathcal{H}_\infty$ fault filtering, the ρ is usually set as a sufficient small positive scalar.*

3.2. **Existence conditions for the minimum of J_N.** In this subsection, we will introduce a stochastic Krein space system model and build a relationship between the minimum problem of J_N with Krein space orthogonal projection. Similar to [25], a sufficient and necessary condition for the minimum will be given in terms of a certain Krein space Gramian matrix (essentially a Krein space variance matrix).

Consider the following Krein space stochastic system

$$\begin{cases} \mathbf{x}(i+1) = A(i)\mathbf{x}(i) + B(i)\mathbf{u}(i) + B_{sf}(i)\mathbf{f}_s(i) \\ \mathbf{y}_s(i) = C_s(i)\mathbf{x}(i) + D_s(i)\mathbf{u}(i) + D_{sf}(i)\mathbf{f}_s(i) + \mathbf{v}_s(i) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{18}$$

where $\mathbf{x}(i)$, $\mathbf{f}_s(i)$, $\mathbf{u}(i)$ and $\mathbf{v}_s(i)$ are Krein space vectors and

$$\mathbf{y}_s(i) = \left\{\begin{array}{ll} \mathbf{y}_a(i), & i \le k-1 \\ \mathbf{y}(k), & i = k \end{array}\right., \quad \mathbf{y}_a(i) = \left[\begin{array}{c} \mathbf{y}(i) \\ \mathbf{r}(i) \end{array}\right] \tag{19}$$

$$\mathbf{v}_s(i) = \left\{\begin{array}{ll} \left[\begin{array}{c} \mathbf{v}(i) \\ \mathbf{v}_{f_s}(i) \end{array}\right], & i \le k-1 \\ \mathbf{v}(k), & i = k \end{array}\right., \quad \mathbf{r}(i) = \mathbf{f}_s(i) + \mathbf{v}_{f_s}(i) \tag{20}$$

$$\left\langle \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{f}_s(i) \\ \mathbf{v}_s(i) \end{array}\right], \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{f}_s(j) \\ \mathbf{v}_s(j) \end{array}\right] \right\rangle = \mathrm{diag}(P_0, \;\; I\delta_{ij}, \;\; Q_{vs}(i)\delta_{ij}) \tag{21}$$

The following Lemma 3.1 presents an existing condition for the minimum of J_N.

Lemma 3.1. [25] *The J_N subject to (17) has a minimum over $\{x_0, f_{sN}\}$ if and only if $R_{\mathbf{y}_s}(k) = \langle \mathbf{y}_s(k), \mathbf{y}_s(k)\rangle$ has the same inertia with $Q_{vs}(k)$ and, if this is the case, we have*

$$\min J_N = \sum_{k=0}^{N} y_s^T(k) R_{\mathbf{y}_s}^{-1}(k) y_s(k) \tag{22}$$

We should mention that it is difficult to compare the inertia of $R_{\mathbf{y}_s}(k)$ with $Q_{vs}(k)$. As shown later, innovation re-organization provides the block diagonal factorization of a matrix equivalent to $R_{\mathbf{y}_s}(k)$, and thereby allows us to do it easily.

Define innovations

$$\tilde{\mathbf{y}}_a(i,1) = \mathbf{y}_a(i) - \bar{C}(i)\hat{\mathbf{x}}(i,1) - \bar{D}(i)\mathbf{u}(i), \quad R_{\tilde{\mathbf{y}}_a}(i,1) = \langle \tilde{\mathbf{y}}_a(i), \tilde{\mathbf{y}}_a(i)\rangle \tag{23}$$
$$\tilde{\mathbf{y}}(i,2) = \mathbf{y}(i) - C(i)\hat{\mathbf{x}}(i,2) - D(i)\mathbf{u}(i), \quad R_{\tilde{\mathbf{y}}}(i,2) = \langle \tilde{\mathbf{y}}(i,2), \tilde{\mathbf{y}}(i,2)\rangle \tag{24}$$

where $\hat{\mathbf{x}}(0,1) = 0$, $\hat{\mathbf{x}}(0,2) = 0$; $\hat{\mathbf{x}}(i,1)$ and $\hat{\mathbf{x}}(i,2)$ denote the orthogonal projections of $\mathbf{x}(i)$ onto linear space spanned by $\{\tilde{\mathbf{y}}_a(j,1)\}_{j=0}^{i-1}$ and $\{\{\tilde{\mathbf{y}}_a(j,1)\}_{j=0}^{i-2}, \tilde{\mathbf{y}}(i-1,2)\}$, respectively.

Furthermore, let

$$\tilde{\mathbf{r}}(k) = \mathbf{r}(k) - \hat{\mathbf{r}}(k|k+1), \quad R_{\tilde{\mathbf{r}}}(k) = \langle \tilde{\mathbf{r}}(k), \tilde{\mathbf{r}}(k)\rangle \tag{25}$$
$$\tilde{\mathbf{y}}_r(k) = \begin{bmatrix} \tilde{\mathbf{y}}^T(k,2) & \tilde{\mathbf{r}}^T(k) \end{bmatrix}^T, \quad R_{\tilde{\mathbf{y}}_r}(k) = \langle \tilde{\mathbf{y}}_r(k), \tilde{\mathbf{y}}_r(k)\rangle \tag{26}$$
$$\tilde{\mathbf{y}}_{rs}(k) = \begin{cases} \tilde{\mathbf{y}}_r(k), & k \le N-1 \\ \tilde{\mathbf{y}}(N,2), & k = N \end{cases}, \quad R_{\tilde{\mathbf{y}}_{rs}}(k) = \langle \tilde{\mathbf{y}}_{rs}(k), \tilde{\mathbf{y}}_{rs}(k)\rangle \tag{27}$$

where $\hat{\mathbf{r}}(k|k+1)$ is the projection of $\mathbf{r}(k)$ onto $\mathcal{L}\{\{\tilde{\mathbf{y}}_a(i,1)\}_{i=0}^{k-1}; \tilde{\mathbf{y}}(k,2), \tilde{\mathbf{y}}(k+1,2)\}$. By using projection formulas, we have

$$\begin{aligned}
\mathbf{y}(0) &= \tilde{\mathbf{y}}(0,2) \\
\mathbf{r}(0) &= \tilde{\mathbf{r}}(0) + \hat{\mathbf{r}}(0|1) = \tilde{\mathbf{r}}(0) + \mathrm{Proj}\{\mathbf{r}(0)|\tilde{\mathbf{y}}(0,2), \tilde{\mathbf{y}}(1,2)\} \\
&= \tilde{\mathbf{r}}(0) + \langle \mathbf{r}(0), \tilde{\mathbf{y}}(0,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(0,2)\tilde{\mathbf{y}}(0,2) + \langle \mathbf{r}(0), \tilde{\mathbf{y}}(1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(1,2)\tilde{\mathbf{y}}(1,2) \\
\mathbf{y}(1) &= \tilde{\mathbf{y}}(1,2) + \mathrm{Proj}\{\mathbf{y}(1)|\mathbf{y}(0)\} = \tilde{\mathbf{y}}(1,2) + \mathrm{Proj}\{\mathbf{y}(1)|\tilde{\mathbf{y}}(0,2)\} \\
&= \tilde{\mathbf{y}}(1,2) + \langle \mathbf{y}(1), \tilde{\mathbf{y}}(0,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(0,2)\tilde{\mathbf{y}}(0,2) \\
\mathbf{r}(1) &= \tilde{\mathbf{r}}(1) + \hat{\mathbf{r}}(1|2) = \tilde{\mathbf{r}}(1) + \mathrm{Proj}\{\mathbf{r}(1)|\tilde{\mathbf{y}}_a(0,1); \tilde{\mathbf{y}}(1,2), \tilde{\mathbf{y}}(2,2)\} \\
&= \tilde{\mathbf{r}}(1) + \langle \mathbf{r}(1), \tilde{\mathbf{y}}_a(0,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(0,1)\tilde{\mathbf{y}}_a(0,1) + \langle \mathbf{r}(1), \tilde{\mathbf{y}}(1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(1,2)\tilde{\mathbf{y}}(1,2) \\
&\quad + \langle \mathbf{r}(1), \tilde{\mathbf{y}}(2,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(2,2)\tilde{\mathbf{y}}(2,2) \\
&\;\;\vdots \\
\mathbf{y}(i-1) &= \tilde{\mathbf{y}}(i-1,2) + \hat{\mathbf{y}}(i-1,2) \\
&= \tilde{\mathbf{y}}(i-1,2) + \mathrm{Proj}\{\mathbf{y}(i-1)|\tilde{\mathbf{y}}_a(0,1), \tilde{\mathbf{y}}_a(1,1), \cdots, \tilde{\mathbf{y}}_a(i-3,1); \tilde{\mathbf{y}}(i-2,2)\} \\
&= \tilde{\mathbf{y}}(i-1,2) + \sum_{j=0}^{i-3} \langle \mathbf{y}(i-1), \tilde{\mathbf{y}}_a(j,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(j,1)\tilde{\mathbf{y}}_a(j,1) \\
&\quad + \langle \mathbf{y}(i-1), \tilde{\mathbf{y}}(i-2,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i-2,2)\tilde{\mathbf{y}}(i-2,2) \\
\mathbf{r}(i-1) &= \tilde{\mathbf{r}}(i-1) + \hat{\mathbf{r}}(i-1|i) \\
&= \tilde{\mathbf{r}}(i-1) + \mathrm{Proj}\{\mathbf{r}(i-1)|\tilde{\mathbf{y}}_a(0,1), \tilde{\mathbf{y}}_a(1,1), \cdots, \tilde{\mathbf{y}}_a(i-2,1); \tilde{\mathbf{y}}(i-1,2), \tilde{\mathbf{y}}(i,2)\} \\
&= \tilde{\mathbf{r}}(i-1) + \sum_{j=0}^{i-2} \langle \mathbf{r}(i-1), \tilde{\mathbf{y}}_a(j)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(j,1)\tilde{\mathbf{y}}_a(j,1) \\
&\quad + \langle \mathbf{r}(i-1), \tilde{\mathbf{y}}(i-1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i-1,2)\tilde{\mathbf{y}}(i-1,2)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \langle \mathbf{r}(i-1), \tilde{\mathbf{y}}(i,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i,2)\tilde{\mathbf{y}}(i,2) \\
\mathbf{y}(i) &= \tilde{\mathbf{y}}(i,2) + \hat{\mathbf{y}}(i,2) \\
&= \tilde{\mathbf{y}}(i,2) + \mathrm{Proj}\{\mathbf{y}(i)|\tilde{\mathbf{y}}_a(0,1), \tilde{\mathbf{y}}_a(1,1), \cdots, \tilde{\mathbf{y}}_a(i-2,1); \tilde{\mathbf{y}}(i-1,2)\} \\
&= \tilde{\mathbf{y}}(i,2) + \sum_{j=0}^{i-2} \langle \mathbf{y}(i), \tilde{\mathbf{y}}_a(j,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(j,1)\tilde{\mathbf{y}}_a(j,1) \\
&\quad + \langle \mathbf{y}(i), \tilde{\mathbf{y}}(i-1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i-1,2)\tilde{\mathbf{y}}(i-1,2)
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{y}_{sk} &= \begin{bmatrix} \mathbf{y}_s^T(0) & \mathbf{y}_s^T(1) & \cdots & \mathbf{y}_s^T(k) \end{bmatrix}^T, \quad R_{\mathbf{y}_{sk}} = \mathrm{diag}(R_{\mathbf{y}_s}(0), R_{\mathbf{y}_s}(1), \cdots, R_{\mathbf{y}_s}(k)) \\
\tilde{\mathbf{y}}_{sk} &= \begin{bmatrix} \tilde{\mathbf{y}}_{rs}^T(0) & \tilde{\mathbf{y}}_{rs}^T(1) & \cdots & \tilde{\mathbf{y}}_{rs}^T(k) \end{bmatrix}^T, \quad R_{\tilde{\mathbf{y}}_{sk}} = \mathrm{diag}(R_{\tilde{\mathbf{y}}_{rs}}(0), R_{\tilde{\mathbf{y}}_{rs}}(1), \cdots, R_{\tilde{\mathbf{y}}_{rs}}(k))
\end{aligned}
$$

It is easy to find

$$
\mathbf{y}_{sk} = \Phi_k \tilde{\mathbf{y}}_{sk} \tag{28}
$$

where

$$
\Phi_k = \Psi_k \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \phi_k(2,1) & \phi_k(2,2) & 0 & \cdots & 0 \\ \phi_k(3,1) & \phi_k(3,2) & \phi_k(3,3) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \phi_k(k+1,1) & \phi_k(k+1,2) & \phi_k(k+1,3) & \cdots & \phi_k(k+1,k+1) \end{bmatrix} \Psi_k^T
$$

$$
\begin{aligned}
\Psi_k &= \mathrm{diag}(I, \psi, \cdots, \psi), \quad \psi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\
\phi_k(i,i) &= \begin{bmatrix} I & 0 \\ \langle \mathbf{r}(i-2), \tilde{\mathbf{y}}(i-1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i-1,2) & I \end{bmatrix}, \quad i = 2, 3, \cdots, k+1 \\
\phi_k(i,1) &= \begin{bmatrix} \langle \mathbf{y}(i-1), \tilde{\mathbf{y}}(0,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(0,2) \\ \langle \mathbf{r}(i-2), \tilde{\mathbf{y}}(0,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(0,2) \end{bmatrix}, \quad i = 2, 3, \cdots, k+1 \\
\phi_k(i,j) &= \begin{bmatrix} \langle \mathbf{y}(i-1), \tilde{\mathbf{y}}(j-1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(j-1,2) & \langle \mathbf{y}(j-1,2), \tilde{\mathbf{r}}(j-2)\rangle R_{\tilde{\mathbf{r}}}^{-1}(j-2) \\ \langle \mathbf{r}(i-2), \tilde{\mathbf{y}}(j-1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(j-1,2) & \langle \mathbf{r}(i-2), \tilde{\mathbf{r}}(j-2)\rangle R_{\tilde{\mathbf{r}}}^{-1}(j-2) \end{bmatrix} \\
&\quad i = 3, 4, \cdots, k+1, \quad j = 2, 3, \cdots, i-1
\end{aligned}
$$

Thus, we have

$$
R_{\mathbf{y}_{sk}} = \Phi_k R_{\tilde{\mathbf{y}}_{sk}} \Phi_k^T \tag{29}
$$

which implies that $R_{\mathbf{y}_{sk}}$ is equivalent to $R_{\tilde{\mathbf{y}}_{sk}}$. Therefore, $R_{\mathbf{y}_s}(k)$ and $R_{\tilde{\mathbf{y}}_{rs}}(k)$ have the same inertia.

On the other hand, it is obvious from the definition of $\tilde{\mathbf{r}}(k)$ and the orthogonality that

$$
\tilde{\mathbf{r}}(k) \perp \mathcal{L}\{\{\tilde{\mathbf{y}}_a(i,1)\}_{i=0}^{k-1}; \tilde{\mathbf{y}}(k,2), \mathbf{y}(k+1,2)\}
$$

Hence,

$$
R_{\tilde{\mathbf{y}}_{rs}}(k) = \begin{bmatrix} \langle \tilde{\mathbf{y}}(k,2), \tilde{\mathbf{y}}(k,2)\rangle & \langle \tilde{\mathbf{y}}(k,2), \tilde{\mathbf{r}}(k)\rangle \\ \langle \tilde{\mathbf{r}}(k), \tilde{\mathbf{y}}(k,2)\rangle & \langle \tilde{\mathbf{r}}(k), \tilde{\mathbf{r}}(k)\rangle \end{bmatrix} = \mathrm{diag}(R_{\tilde{\mathbf{y}}}(k,2), \ R_{\tilde{\mathbf{r}}}(k)) \tag{30}
$$

for $k \le N-1$, $R_{\tilde{\mathbf{y}}_{rs}}(N,2) = R_{\tilde{\mathbf{y}}}(N,2)$. This means that $R_{\tilde{\mathbf{y}}_{rs}}(k)$ is a block diagonal matrix.

According to Lemma 3.1, J_N has a minimum if and only if $R_{\mathbf{y}_s}(k)$ and $Q_{vs}(k)$ have the same inertia. In view of (15) and (30), the inertia of $R_{\tilde{\mathbf{y}}_{rs}}(k)$ and $Q_{vs}(k)$ coincide if and only if $R_{\tilde{\mathbf{y}}}(k,2) > 0$ and $R_{\tilde{\mathbf{r}}}(k) < 0$. This gives the following Theorem 3.1.

Theorem 3.1. *The J_N subject to (8) and (17) has a minimum over $\{x_0, f_{sN}\}$ if and only if $R_{\tilde{\mathbf{y}}}(k,2) > 0$ and $R_{\tilde{\mathbf{r}}}(k) < 0$.*

Remark 3.2. *Theorem 3.1 provides a sufficient and necessary condition for the existence of the minimum of J_N. In the next subsection, the calculation of $R_{\tilde{\mathbf{y}}}(k,2)$ and $R_{\tilde{\mathbf{r}}}(k)$ will be given by using an orthogonal projection in Krein space.*

3.3. **The recursive calculation of $R_{\tilde{\mathbf{y}}}(k,2)$ and $R_{\tilde{\mathbf{r}}}(k)$.** Inspecting (20) and (21), it is easy to see that

$$\langle \mathbf{r}(i), \mathbf{r}(j)\rangle = 0, \quad \langle \mathbf{r}(i), \mathbf{y}(j)\rangle = 0, \quad \forall i > j$$

Thus, the projection of $\mathbf{r}(i)$ onto $\mathcal{L}\{\tilde{\mathbf{y}}_a(j,1)\}_{j=0}^{i-1}$ can be calculated from

$$\hat{\mathbf{r}}(i,1) = \mathrm{Proj}\left(\mathbf{r}(i)|\mathcal{L}\{\tilde{\mathbf{y}}_a(j)\}_{j=0}^{i-1}\right) = 0$$

Let $\mathbf{e}_1(i) = \mathbf{x}(i) - \hat{\mathbf{x}}(i,1)$. It follows from (18), (19) and (23) that

$$\tilde{\mathbf{y}}_a(i,1) = \bar{C}(i)\mathbf{e}_1(i) + \mathbf{v}_s(i) + \bar{D}_{fd}(i)\mathbf{f}_s(i) \tag{31}$$
$$R_{\tilde{\mathbf{y}}_a}(i,1) = \bar{C}(i)P_1(i)\bar{C}^T(i) + \bar{D}_{fd}(i)\bar{D}_{fd}^T(i) + Q_{vs}(i) \tag{32}$$

Then, the projection of $\mathbf{x}(i)$ onto $\mathcal{L}\{\tilde{\mathbf{y}}_a(j,1)\}_{j=0}^{i-1}$ can be calculated by

$$\begin{cases} \hat{\mathbf{x}}(i+1,1) = \mathrm{Proj}(\mathbf{x}(i+1)|\mathcal{L}\{\tilde{\mathbf{y}}_a(j)\}_{j=0}^{i}) \\ \qquad = \mathrm{Proj}(\mathbf{x}(i+1)|\mathcal{L}\{\tilde{\mathbf{y}}_a(j)\}_{j=0}^{i-1}) + \mathrm{Proj}(\mathbf{x}(i+1)|\mathcal{L}\{\tilde{\mathbf{y}}_a(i)\}) \\ \qquad = A(i)\sum_{j=0}^{i-1} \langle \mathbf{x}(i), \tilde{\mathbf{y}}_a(j,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(j,1)\tilde{\mathbf{y}}_a(j,1) + B(i)\mathbf{u}(i) \\ \qquad\quad + \langle \mathbf{x}(i+1), \tilde{\mathbf{y}}_a(i,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(i,1)\tilde{\mathbf{y}}_a(i,1) \\ \qquad = A(i)\hat{\mathbf{x}}(i,1) + B(i)\mathbf{u}(i) + K_1(i)\tilde{\mathbf{y}}_a(i,1) \\ \hat{\mathbf{x}}(0,1) = 0 \end{cases} \tag{33}$$

where

$$K_1(i) = (A(i)P_1(i)\bar{C}^T(i) + B_{sf}(i)\bar{D}_{fd}^T(i))R_{\tilde{\mathbf{y}}_a}^{-1}(i,1) \tag{34}$$
$$P_1(i+1) = A(i)P_1(i)A^T(i) + B_{sf}(i)B_{sf}^T(i) - K_1(i)R_{\tilde{\mathbf{y}}_a}(i,1)K_1^T(i) \tag{35}$$
$$P_1(0) = P_0 \tag{36}$$

Likewise, define $\mathbf{e}_2(i) = \mathbf{x}(i) - \hat{\mathbf{x}}(i,2)$. Applying to (18) and (24), we get

$$\tilde{\mathbf{y}}(i,2) = C(i)\mathbf{e}_2(i) + \mathbf{v}(i) + D_{fd}(i)\mathbf{f}_s(i) \tag{37}$$
$$R_{\tilde{\mathbf{y}}}(i,2) = C(i)P_2(i)C^T(i) + D_{fd}(i)D_{fd}^T(i) + I \tag{38}$$

Let $\hat{\mathbf{x}}(i,2) = \hat{\mathbf{x}}(i,1)$. Then the projection of $\mathbf{x}(i+1)$ onto $\mathcal{L}\{\{\tilde{\mathbf{y}}_a(j,1)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i,2)\}$, i.e., $\hat{\mathbf{x}}(i+1,2)$, can be given by

$$\begin{aligned} \hat{\mathbf{x}}(i+1,2) &= \sum_{j=0}^{i-1} \langle \mathbf{x}(i+1), \tilde{\mathbf{y}}_a(j,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(j,1)\tilde{\mathbf{y}}_a(j,1) + B(i)\mathbf{u}(i) \\ &\quad + \langle \mathbf{x}(i+1), \tilde{\mathbf{y}}(i,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(i,2)\tilde{\mathbf{y}}(i,2) \\ &= A(i)\hat{\mathbf{x}}(i,2) + B(i)\mathbf{u}(i) + K_2(i)\tilde{\mathbf{y}}(i,2) \end{aligned} \tag{39}$$

where

$$\begin{aligned} K_2(i) &= (A(i)P_2(i)C^T(i) + B_{sf}(i)D_{fd}^T(i))R_{\tilde{\mathbf{y}}}^{-1}(i,2) &(40)\\ P_2(i) &= A(i-1)P_2(i-1)A^T(i-1) + B_{sf}(i-1)B_{sf}^T(i-1) \\ &\quad -K_2(i-1)R_{\tilde{\mathbf{y}}}(i-1,2)K_2^T(i-1) &(41)\\ P_2(i-1) &= P_1(i-1) &(42)\end{aligned}$$

Furthermore, we obtain the projection of $\mathbf{r}(k)$ onto $\mathcal{L}\{\{\tilde{\mathbf{y}}_a(i)\}_{i=0}^{k-1}; \tilde{\mathbf{y}}(k,2), \tilde{\mathbf{y}}(k+1,2)\}$ as follows:

$$\begin{aligned} \hat{\mathbf{r}}(k|k+1) &= \sum_{i=0}^{k-1} \langle \mathbf{r}(k), \tilde{\mathbf{y}}_a(i,1)\rangle R_{\tilde{\mathbf{y}}_a}^{-1}(i,1)\tilde{\mathbf{y}}_a(i,1) + \langle \mathbf{r}(k), \tilde{\mathbf{y}}(k,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(k,2)\tilde{\mathbf{y}}(k,2) \\ &\quad + \langle \mathbf{r}(k), \tilde{\mathbf{y}}(k+1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(k+1,2)\tilde{\mathbf{y}}(k+1,2) \\ &= \langle \mathbf{r}(k), \tilde{\mathbf{y}}(k,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(k,2)\tilde{\mathbf{y}}(k,2) \\ &\quad + \langle \mathbf{r}(k), \tilde{\mathbf{y}}(k+1,2)\rangle R_{\tilde{\mathbf{y}}}^{-1}(k+1,2)\tilde{\mathbf{y}}(k+1,2) \\ &= (B_{sf}(k) - K_2(k)D_{fd}(k))^T C^T(k+1)R_{\tilde{\mathbf{y}}}^{-1}(k+1,2)\tilde{\mathbf{y}}(k+1,2) \\ &\quad + D_{fd}^T(k)R_{\tilde{\mathbf{y}}}^{-1}(k,2)\tilde{\mathbf{y}}(k,2) &(43)\end{aligned}$$

Hence,

$$\begin{aligned} R_{\tilde{\mathbf{r}}}(k) &= \langle \mathbf{r}(k) - \hat{\mathbf{r}}(k|k+1), \mathbf{r}(k) - \hat{\mathbf{r}}(k|k+1)\rangle \\ &= \langle \mathbf{r}(k), \mathbf{r}(k)\rangle - \langle \hat{\mathbf{r}}(k|k+1), \hat{\mathbf{r}}(k|k+1)\rangle \\ &= (1-\gamma^2)I - D_{fd}^T(k)R_{\tilde{\mathbf{y}}}^{-1}(k,2)D_{fd}(k) - (B_{sf}(k) - K_2(k)D_{fd}(k))^T \\ &\quad \times C^T(k+1)R_{\tilde{\mathbf{y}}}^{-1}(k+1,2)C(k+1)(B_{sf}(k) - K_2(k)D_{fd}(k)) &(44)\end{aligned}$$

Now the calculation of $R_{\tilde{\mathbf{y}}}(k,2)$ and $R_{\tilde{\mathbf{r}}}(k)$ can be summarized in **Algorithm 1**:

Step 1. Set $P_0 > 0$, $\gamma > 0$ and $\rho > 0$;

Step 2. Calculate $K_1(i)$ and $P_1(i)$ $(i = 0, 1, \cdots, k-1)$ using (34)-(36);

Step 3. Let $P_2(k-1) = P_1(k-1)$. Update $P_2(k)$ using (41) with $i = k$; calculate $K_2(k)$, $P_2(k+1)$ using (40) and (41) with $i = k, k+1$;

Step 4. Calculate $R_{\tilde{\mathbf{y}}}(k,2)$ and $R_{\tilde{\mathbf{y}}}(k+1,2)$ using (38) with $i = k, k+1$;

Step 5. Calculate $R_{\tilde{\mathbf{r}}}(k)$ using (44).

Remark 3.3. *So far, we have obtained the recursions for computing $R_{\tilde{\mathbf{y}}}(k,2)$, $R_{\tilde{\mathbf{r}}}(k)$ and the way for checking the existence condition of the minimum of J_N. In the next subsection, a recursive state estimation and residual generation will be derived by choosing $H_i(k)$ $(i = 1, 2, 3)$, $V_1(k)$ and $V_2(k+1)$ such that* $\min J_N > 0$.

3.4. **A solution to the $\boldsymbol{\mathcal{H}_\infty}$-FDF.** Suppose that $\hat{x}(i,1)$ $(i \le k-1)$, $\hat{x}(k,2)$ and $\hat{r}(k|k+1)$ are obtained from the Krein space projection formulas of $\hat{\mathbf{x}}(i,1)$ $(i \le k-1)$, $\hat{\mathbf{x}}(k,2)$ and $\hat{\mathbf{r}}(k|k+1)$, respectively. Let

$$\begin{aligned} \tilde{r}(k) &= r(k) - \hat{r}(k|k+1), \\ \tilde{y}(k,2) &= y(k) - C(k)\hat{x}(k,2) - D(k)u(k), \\ \tilde{y}_r(k) &= \begin{bmatrix} \tilde{y}(k,2) \\ \tilde{r}(k) \end{bmatrix}, \\ \tilde{y}_{rs}(k) &= \begin{cases} \tilde{y}_r(k), & k \le N-1 \\ \tilde{y}(N,2), & k = N \end{cases}\end{aligned}$$

$$\tilde{y}_{sk} = \left[\begin{array}{ccccc} \tilde{y}_{rs}^T(0) & \tilde{y}_{rs}^T(1) & \cdots & \tilde{y}_{rs}^T(k-1) & \tilde{y}^T(k,2) \end{array} \right]^T$$
$$y_{sk} = \left[\begin{array}{ccccc} y_s^T(0) & y_s^T(1) & \cdots & y_s^T(k-1) & y^T(k) \end{array} \right]^T$$

Similar to (28), it is easy to have $y_{sk} = \Phi_k \tilde{y}_{sk}$. Substituting into (22) and applying (29) yields

$$\begin{aligned} \min J_N &= y_{sN}^T R_{\mathbf{y}_{sN}}^{-1} y_{sN} = \tilde{y}_{sN}^T R_{\tilde{\mathbf{y}}_{sN}}^{-1} \tilde{y}_{sN} \\ &= \sum_{k=0}^{N} \tilde{y}^T(k) R_{\tilde{\mathbf{y}}}^{-1}(k,2) \tilde{y}(k) + \sum_{k=0}^{N-1} \tilde{r}^T(k) R_{\tilde{\mathbf{r}}}^{-1}(k) \tilde{r}(k) \end{aligned}$$

Choosing

$$H_1(k) = K_2(k), \quad \left[\begin{array}{cc} H_1(k) + H_2(k) & H_3(k) \end{array} \right] = K_1(k) \tag{45}$$
$$V_1(k) = D_{fd}^T(k) R_{\tilde{\mathbf{y}}}^{-1}(k,2) \tag{46}$$
$$V_2(k+1) = (B_{sf}(k) - K_2(k) D_{fd}(k))^T C^T(k+1) R_{\tilde{\mathbf{y}}}^{-1}(k+1,2) \tag{47}$$

where $K_1(k)$, $K_2(k)$ and $R_{\tilde{\mathbf{y}}}(i,2)$ $(i = k, k+1)$ are calculated by (34), (40) and (38), respectively; it is easy to verify that

$$\hat{x}(k) = \hat{x}(k,1), \quad \hat{x}(k+1|k) = \hat{x}(k+1,2) \tag{48}$$

In view of (8), (43) and (48), we have $r(k) = \hat{r}(k|k+1)$. Therefore,

$$\min J_N = \sum_{k=0}^{N} \tilde{y}^T(k,2) R_{\tilde{\mathbf{y}}}^{-1}(k,2) \tilde{y}(k,2) > 0 \tag{49}$$

From the above analysis, we thus have the following result.

Theorem 3.2. *Suppose that $K_1(i)$ $(i \le k-1)$, $K_2(k)$, $R_{\tilde{\mathbf{y}}}(i,2)$ $(i = k, k+1)$ and $R_{\tilde{\mathbf{r}}}(k)$ are respectively calculated by (34), (40), (38), (44). If $R_{\tilde{\mathbf{y}}}(i,2) > 0$, $R_{\tilde{\mathbf{r}}}(k) < 0$ are satisfied, then the FDF (8) with parameter matrices given by*

$$H_1(k) = K_2(k) \tag{50}$$
$$H_2(i) = K_1(i) \left[\begin{array}{c} I \\ 0 \end{array} \right] - K_2(i) \tag{51}$$
$$H_3(i) = K_1(i) \left[\begin{array}{c} 0 \\ I \end{array} \right], \quad i \le k-1 \tag{52}$$
$$V_1(k) = D_{fd}^T(k) R_{\tilde{\mathbf{y}}}^{-1}(k,2) \tag{53}$$
$$V_2(k+1) = (B_{sf}(k) - H_1(k) D_{fd}(k))^T C^T(k+1) R_{\tilde{\mathbf{y}}}^{-1}(k+1,2) \tag{54}$$

is a robust $\mathcal{H}_\infty$-FDF satisfying (7).

Having obtained a solution to the parameter matrices $H_i(k)$ $(i = 1,2,3)$, $V_1(k)$ and $V_2(k+1)$, one can get $r(k)$ according to (8). Now the calculation of residual $r_f(k)$ is summarized in **Algorithm 2**:

Step 1. Set $k = 0$, $\hat{x}(0) = 0$;
Step 2. Calculate $\varepsilon(k)$, $\hat{x}(k+1|k)$, $\varepsilon(k+1|k)$ and $r(k)$ in turn by (8);
Step 3. Calculate residual signal $r_f(k)$ by $r_f(k) = \left[\begin{array}{ccc} I & 0 & 0 \end{array} \right] r(k)$;
Step 4. Let $k = k+1$, go to Step 2, till $k = N$.

Remark 3.4. *As a summary of this section, we should like to say that the results of robust FD for uncertain LDTV systems are new, but the used techniques in Krein space are the same with* [9, 25]*. The major contribution of this paper is the consideration of* *the estimation and accommodation of model uncertainty, unknown input and fault in the residual generation.*

4. A Numerical Example.

Consider system (1) with

$$A(k) = \begin{bmatrix} 0.2e^{-\frac{k}{100}} & 0.6 & 0 \\ 0 & 0.5 & \sin(k) \\ 0 & 0 & 0.7 \end{bmatrix},$$

$$B_f(k) = \begin{bmatrix} 0.2 \\ 1.8 \\ 0.3 \end{bmatrix}, \quad B_d(k) = \begin{bmatrix} 1.3 \\ 0.5 \\ 0.6 \end{bmatrix}$$

$$F_1(k) = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}, \quad C(k) = \begin{bmatrix} -0.5 & 1.5 & 0 \end{bmatrix}, \quad D_f(k) = 2$$

$$E_1(k) = \begin{bmatrix} 1 & 0 & 0.5 \end{bmatrix}^T, \quad E_2(k) = 1, \quad u(k) = 0$$

In order to show the effectiveness of the proposed method, we first design a robust $\mathcal{H}_\infty$-FDF using Algorithm 1 and Theorem 3.2, i.e., the Case 1. Set $x(0) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^{\mathrm{T}}$, $P_0 = I$, $\gamma = 0.85$ and $\rho = 0.1$. Let $v(k) = 0.1\cos(k)$, $d(k)$ be uniformly distributed random numbers between -0.5 and 0.5. We calculate $r_f(k)$ using Algorithm 2 for the case of an impulse fault and sine wave fault, respectively. Figure 1 shows the fault and the corresponding residual $r_f(k)$ for $N = 100$.

Moreover, it is easy to know from (7) that, for any chosen x_0 and $w(k)$ such that $x_0^T P_0^{-1} x_0 + \sum_{k=0}^{N} \|w(k)\|^2 \neq 0$, the smaller $J_{vf}(N) = \Sigma_{k=0}^{N-1} ||r_f(k) - f(k)||^2$ is, the better is the $\mathcal{H}_\infty$-FDF performance. To show the influence of γ, the corresponding $J_{vf}(N)$ of unit impulse fault is calculated also for different γ, which is listed in the following Table 1.

In addition, the relationship between ρ and γ is also considered. The allowed minimal value of γ is greater than 0.8, while the allowed maximal value of ρ is less than 1. Table 2 shows the corresponding maximal ρ for some different values of γ.

TABLE 1. The $J_{vf}(N)$ for different γ

γ	0.85	0.95	1.05	1.15	1.25
$J_{vf}(N)$	3.3666	3.5257	3.6987	3.7813	3.8294

TABLE 2. The maximal ρ for different γ

γ	0.81	0.85	0.90	0.95	1.00
ρ	0.02	0.30	0.73	0.87	0.99

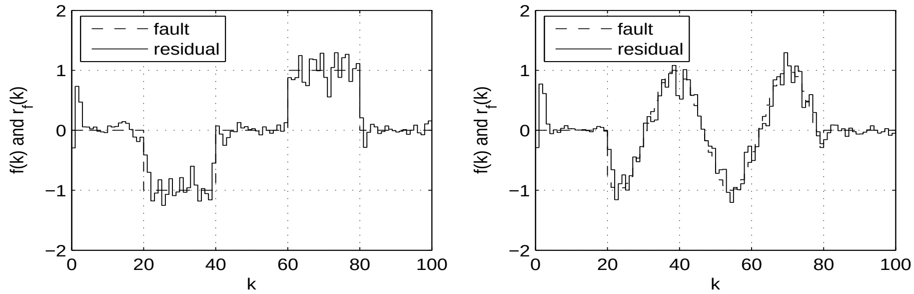

FIGURE 1. Case 1: the impulse fault (left), sine wave fault (right) and corresponding residual

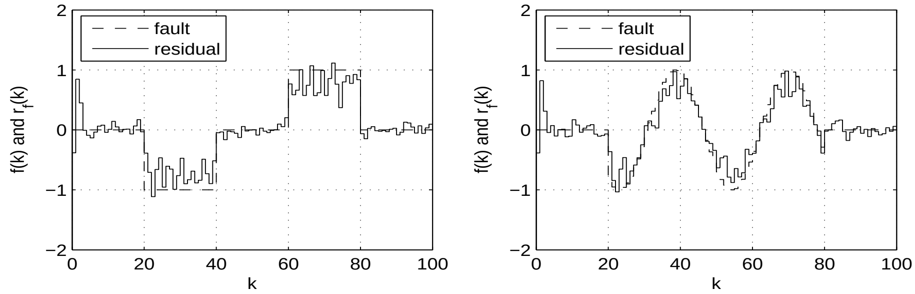

FIGURE 2. Case 2: the impulse fault (left), sine wave fault (right) and corresponding residual

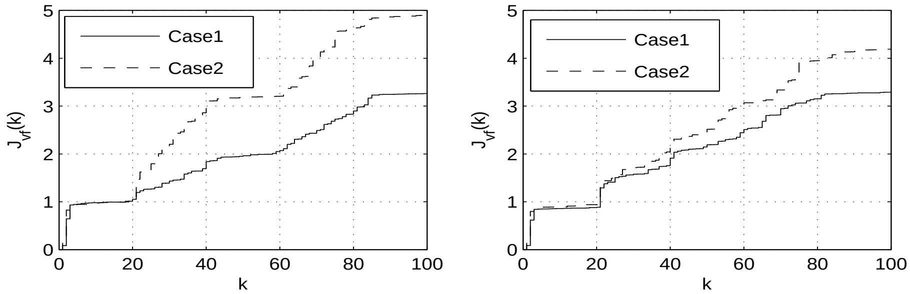

FIGURE 3. The two cases $J_{vf}(k)$ with the impulse fault (left) and the sine wave fault (right)

To further show the improvement of residual performance compared with [25], we consider $\varphi(k)$ as unknown input, set $\gamma = 0.85$ and design $\mathcal{H}_\infty$-FDF using the method of [25], i.e., Case 2. The impulse fault, sine wave fault and the corresponding $r_f(k)$ of Case 2 are shown in Figure 2, respectively. As a comparison with Case 1, the evolutions of the two cases $J_{vf}(k)$ are also presented in Figure 3. It can be seen that better performances are achieved by applying the new developed robust $\mathcal{H}_\infty$-FDF.

5. **Conclusion.** The problem of robust $\mathcal{H}_\infty$ FD has been investigated for LDTV systems subject to norm bounded model uncertainty. A modified observer-based FDF with the estimation and accommodation of unknown input, fault and model uncertainty has been proposed as a residual generator. It has been shown that the design of robust $\mathcal{H}_\infty$-FDF can be converted into a minimum problem of indefinite quadratic form and the minimum problem can be solved by applying orthogonal projection and innovation analysis in Krein space. A sufficient and necessary condition for the minimum has been derived and a solution to the robust $\mathcal{H}_\infty$-FDF has been given in terms of Riccati recursions. A numerical example has been given to illustrate the effectiveness of the proposed method.

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REFERENCES

[1] A. E. Ashari, R. Nikoukhah and S. L. Campbell, Auxiliary signal design for robust active fault detection of linear discrete-time systems, *Automatica*, vol.47, no.9, pp.1887-1895, 2011.

[2] S. Armeni, A. Casavola and E. Mosca, A robust fault detection and isolation filter design under sensitivity constraint: An LMI approach, *Int. J. Robust and Nonlinear Control*, vol.18, no.5, pp.1493-1506, 2009.

[3] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, 1999.

[4] J. Chen and R. J. Patton, Standard H_∞ filtering formulation of robust fault detection, *Proc. of SAFEPROCESS'2000*, Budapest, Hungary, pp.256-261, 2000.

[5] S. X. Ding, *Model-Based Fault Diagnosis Technique*, Spring, Berlin, 2008.

[6] S. X. Ding, T. Jeinsch, P. M. Frank and E. L. Ding, A unified approach to the optimization of fault detection systems, *Int. J. Adaptive Control and Signal Processing*, vol.14, no.7, pp.725-745, 2000.

[7] P. M. Frank and S. X. Ding, Survey of robust residual generation and evaluation methods in observer-based fault detection systems, *J. Process Control*, vol.7, no.6, pp.403-424, 1997.

[8] P. M. Frank, S. X. Ding and B. Koppen-Seliger, Current developments in the theory of FDI, *Proc. of SAFEPROCESS'2000*, Budapest, Hungary, pp.16-27, 2000.

[9] B. Hassibi, A. H. Sayed and T. Kailath, *Indefinite Quadratic Estimation and Control: A Unified Approach to H_2 and H_∞ Theories*, SIAM, Philadelphia, PA, 1999.

[10] D. Henry and A. Zolghadri, Norm-based design of robust FDI schemes for uncertain systems under feedback control: Comparison of two approaches, *Control Engineering Practice*, vol.14, no.9, pp.1081-1097, 2006.

[11] I. Hwang, S. Kim and Y. Kim, A survey of fault detection, isolation, and reconfiguration methods, *IEEE Trans. Control Systems Technology*, vol.18, no.3, pp.636-653, 2010.

[12] X. He, Z. Wang, Y. Ji and D. Zhou, Robust fault detection for networked systems with distributed sensors, *IEEE Trans. Aerospace and Electronic Systems*, vol.47, no.1, pp.166-177, 2011.

[13] B. Jiang, M. Staroswiecki and V. Cocquempot, Fault diagnosis based on adaptive observer for a class of non-linear systems with unknown parameters, *Int. J. Control*, vol.77, no.4, pp.415-426, 2004.

[14] B. Jiang, K. Zhang and P. Shi, Integrated fault estimation and accommodation design for discrete-time Takagi-Sugeno fuzzy systems with actuator faults, *IEEE Trans. Fuzzy Systems*, vol.19, no.2, pp.291-304, 2011.

[15] X. Li and K. Zhou, A time domain approach to robust fault detection of linear time-varying systems, *Automatica*, vol.45, no.1, pp.94-102, 2009.

[16] X. Li, *Fault Detection Filter Design for Linear Systems*, Ph.D. Thesis, Louisiana State University, USA, 2009.

[17] Z. Mao and B. Jiang, Fault identification and fault-tolerant control for a class of networked control systems, *International Journal of Innovative Computing, Information and Control*, vol.3, no.5, pp.1121-1130, 2007.

[18] R. S. Mangoubi and A. M. Edelmayer, Model based fault detection: The optimal past, the robust present and a few thoughts on the future, *Proc. of SAFEPROCESS'2000*, Budapest, Hungary, pp.64-75, 2000.

[19] E. Nobrega, M. Abdalla and K. Grigoriadis, Robust fault estimation of uncertain systems using an LMI-based approaches, *Int. J. Robust and Nonlinear Control*, vol.18, no.18, pp.1657-1680, 2008.

[20] S. K. Nguang, P. Shi and S. X. Ding, Delay-dependent fault estimation for uncertain time-delay nonlinear systems: An LMI approach, *Int. J. Robust and Nonlinear Control*, vol.16, no.18, pp.913-933, 2006.

[21] J. L. Wang, G.-H. Yang and J. Liu, An LMI approach to H_- index and mixed H_-/H_∞ fault detection observer design, *Automatica*, vol.43, no.9, pp.1656-1665, 2007.

[22] P. Zhang and S. X. Ding, Observer-based fault detection of linear time-varying systems, *Automatisierungstechniik*, vol.52, no.8, pp.370-376, 2004 (in German).

[23] P. Zhang, S. X. Ding, G. Wang and D. Zhou, Fault detection of linear discrete-time periodic systems, *IEEE Trans. Automatic Control*, vol.50, no.2, pp.239-244, 2005.

[24] M. Zhong, S. X. Ding, J. Lam and H. Wang, An LMI approach to design robust fault detection filter for uncertain LTI systems, *Automatica*, vol.39, no.3, pp.543-550, 2003.

[25] M. Zhong, D. Zhou and S. X. Ding, Optimal fault detection for linear discrete time-varying systems, *IEEE Trans. Automatic Control*, vol.55, no.7, pp.1689-1695, 2010.

[26] M. Zhong, S. X. Ding and E. L. Ding, Optimal fault detection for linear discrete time-varying systems, *Automatica*, vol.46, no.8, pp.1395-1340, 2010.