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OBSERVABILITY ROBUSTNESS OF UNCERTAIN FUZZY-MODEL-BASED CONTROL SYSTEMS

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Abstract. *The problem considered in this study is the observability robustness of Takagi -Sugeno (TS) fuzzy-model-based control systems. Where a nominal TS-fuzzy-model-based control system is locally observable (i.e., where each fuzzy rule in the system has a full row rank for its observability matrix), a sufficient condition is proposed to preserve the assumed property when system uncertainties are considered. The proposed sufficient condition preserves the assumed property by indicating the explicit relationships of bounds on system uncertainties. A robustly global observability condition is also presented for uncertain TS-fuzzy-model-based control systems. Finally, the proposed sufficient conditions are applied in the example of a nonlinear mass-spring-damper mechanical system with system uncertainties.*

Keywords: Fuzzy system models, Fuzzy control, Robust observability, Takagi-Sugeno (TS) fuzzy model, System uncertainties

1. Introduction.

The fuzzy-model-based representation proposed by Takagi and Sugeno [1], known as the TS fuzzy model, has proven effective in many nonlinear control systems ([2-8] and references therein). The robust controllability of the uncertain TS-fuzzy-model-based control systems has also been studied by Chen et al. [9]. On the other hand, most applications of TS fuzzy control systems presented in the literature, however, assume that states are available for controller implementation, which may not be true in practice. Therefore, some researchers have proposed that the nominal TS-fuzzy-model-based control systems should be assumed to be locally observable (i.e., each fuzzy rule for a nominal TS-fuzzy-model-based control system should be assumed to have a full row rank for its observability matrix) when designing observer-based fuzzy parallel-distributed-compensation controllers (see, e.g., [10-16] and references therein).

In practice, however, obtaining accurate values may be difficult, if not impossible, for some system parameters due to inaccurate measurements or due to inaccessible or variable system parameters and sensor and actuator positions. These system uncertainties may negate the observability property of the TS-fuzzy-model-based control systems. The literature includes many studies of observability problems in various fuzzy systems [17-25]. A recent comprehensive literature review shows that the issue of robustly local and global observability has not been studied in uncertain TS-fuzzy-model-based control systems. Specifically, although the observability problem has been studied in various fuzzy systems, the problem of robustly local and global observability has not been considered in TS-fuzzy-model-based control systems.

This study presents a novel approach for measuring robustly local and global observability in TS-fuzzy-model-based control systems with system uncertainties. Under the assumption that the nominal TS-fuzzy-model-based control systems are locally observable, a sufficient condition is proposed to preserve the assumed property when system uncertainties are introduced. The presented approach uses a simple algebraic derivation, and the proposed sufficient condition explicitly indicates how the relationships among bounds on system uncertainties preserve the assumed property. A robustly global observability condition is also presented for uncertain TS-fuzzy-model-based control systems. An application of the proposed sufficient conditions is demonstrated in the example of a nonlinear mass-spring-damper mechanical system with both elemental parameter uncertainties and displacement-sensor position variations.

This paper is organized as follows. Section 2 presents an analysis of robust observability in uncertain TS-fuzzy-model-based control systems and the sufficient criteria for both robustly local and robustly global observability. Section 3 gives an illustrative example to demonstrate the applicability of the proposed sufficient criteria. Finally, Section 4 concludes the study.

2. Robust Observability Analysis.

When applying the sector nonlinearity approach to fuzzy model construction, both the fuzzy set of the premise part and the linear dynamic model with system uncertainties of the consequent part in the exact TS fuzzy control model with system uncertainties can be derived from a given nonlinear control model with system uncertainties [2]. The TS-fuzzy-model-based control system with system uncertainties for the nonlinear control system with system uncertainties can be obtained in the following form:

$\tilde{R}^i$: IF z_1 is M_{i1} and $\ldots$ and z_g is M_{ig},

$$\text{THEN } \dot{x}(t) = (A_i + \Delta A_i)\, x(t) + (B_i + \Delta B_i)\, u(t) \tag{1}$$

$$\text{and} \quad y(t) = (C_i + \Delta C_i)\, x(t), \tag{2}$$

with the initial state vector $x(0)$, where $\tilde{R}^i$ $(i = 1, 2, \ldots, N)$ denotes the i-th implication; N is the number of fuzzy rules; $x(t) = [x_1(t)\ ,\ x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ denotes the n-dimensional state vector; $y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^{\mathrm{T}}$ denotes the m-dimensional output vector; $u(t) = [u_1(t), u_2(t), \ldots, u_p(t)]^{\mathrm{T}}$ denotes the p-dimensional input vector; $z_i\,(i = 1, 2, \ldots, g)$ are the premise variables; A_i, B_i and C_i $(i = 1, 2, \ldots, N)$ are the consequent constant matrices $n \times n$, $n \times p$ and $m \times n$, respectively; ΔA_i, ΔB_i and ΔC_i $(i = 1, 2, \ldots, N)$ are uncertain matrices in the system matrices A_i, the input matrices B_i and the output matrices C_i, respectively, of the consequent part of the i-th rule due to inaccurate measurements, inaccessibility of system parameters, output-sensor measurement variations, or parameter variations, and M_{ij} $(i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, g)$ are the fuzzy sets.

Many interesting problems arise from a few uncertainties entering into many entries of the system, input and output matrices [26-29]. The proposed approach presents the uncertain matrices ΔA_i, ΔB_i and ΔC_i in the forms

$$\Delta A_i = \sum_{k=1}^{\bar{m}} \varepsilon_{ik} A_{ik}, \quad \Delta B_i = \sum_{k=1}^{\bar{m}} \varepsilon_{ik} B_{ik}, \text{ and } \Delta C_i = \sum_{k=1}^{\bar{m}} \varepsilon_{ik} C_{ik}, \tag{3}$$

respectively, where ε_{ik} ($i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, \bar{m}$) are the elemental parametric uncertainties and where A_{ik}, B_{ik} and C_{ik} are the given $n \times n$, $n \times p$ and $m \times n$ constant matrices, respectively, which are prescribed a priori to denote information that is linearly dependent on the elemental parametric uncertainties ε_{ik}, in which $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, \bar{m}$.

2.1. **Robustly local observability.** For the uncertain TS-fuzzy-model-based control system in (1) and (2), this subsection assumes that each fuzzy-rule-nominal model $\dot{x}(t) = A_i x(t) + B_i u(t)$ and $y(t) = C_i x(t)$ denoted by $\{A_i, C_i\}$ is observable (i.e., each fuzzy-rule-nominal model $\{A_i, C_i\}$ has a full row rank for its observability matrix). Due to inevitable uncertainties, each fuzzy-rule-nominal model $\{A_i, C_i\}$ is perturbed into the fuzzy-rule-uncertain model $\{A_i + \Delta A_i, C_i + \Delta C_i\}$. The considered problem is determining the condition under which each fuzzy-rule-uncertain model $\{A_i + \Delta A_i, C_i + \Delta C_i\}$ for the TS-fuzzy-model-based control system in (1) and (2) remains observable. Before investigating the uncertain TS-fuzzy-model-based control system in (1) and (2) in terms of the robustness of observability, several definition and lemmas must be introduced.

Definition 2.1. [3]: *The TS-fuzzy-model-based control system is locally observable if each fuzzy-rule model $\{A_i + \Delta A_i, C_i + \Delta C_i\}$ $(i = 1, 2, \ldots, N)$ is observable.*

Lemma 2.1. *The system model $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t)$ is observable if and only if the $n^2 \times n(n+m-1)$ matrix*

$$Q = \begin{bmatrix} I_n & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & 0 & C^{\mathrm{T}} \\ -A^{\mathrm{T}} & I_n & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & C^{\mathrm{T}} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & I_n & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & -A^{\mathrm{T}} & C^{\mathrm{T}} & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix} \tag{4}$$

has rank n^2, where $A \in R^{n\times n}$, $C \in R^{m\times n}$ and I_n denotes the $n \times n$ identity matrix.

Proof: In the above matrix Q of (4), add the product of A^{T} and the first (block) row to the second row. Then add the product of A^{T} and the second row to the third row, and so on. The resulting matrix is

$$\left[\begin{array}{cccccc|ccccc} I_n & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & 0 & C^{\mathrm{T}} \\ 0 & I_n & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & I_n & 0 & \bullet & \bullet & \bullet & \bullet & \left(A^{n-2}\right)^{\mathrm{T}} C^{\mathrm{T}} \\ \hline 0 & 0 & \bullet & \bullet & \bullet & 0 & C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \bullet & \bullet & \bullet & \left(A^{n-1}\right)^{\mathrm{T}} C^{\mathrm{T}} \end{array}\right]. \tag{5}$$

The observability matrix $\begin{bmatrix} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \bullet & \bullet & \bullet & \left(A^{n-1}\right)^{\mathrm{T}} C^{\mathrm{T}} \end{bmatrix}$ is of rank n if and only if the matrix in (5) has rank n^2 (i.e., the matrix in (4) has rank n^2) [30]. Therefore, we have the stated result.

Lemma 2.2. [31]: *The matrix measures of matrices $\bar{W}$ and $\bar{V}$, namely $\mu(\bar{W})$ and $\mu(\bar{V})$, respectively, are well defined for any norm and have the following properties:*

(i) $\mu(\pm I) = \pm 1$, *for the identity matrix* I;
(ii) $-\|\bar{W}\| \le -\mu(-\bar{W}) \le Re(\lambda(\bar{W})) \le \mu(\bar{W}) \le \|\bar{W}\|$, *for any norm* $\|\bullet\|$ *and any matrix* $\bar{W} \in C^{n\times n}$;
(iii) $\mu(\bar{W} + \bar{V}) \le \mu(\bar{W}) + \mu(\bar{V})$, *for any two matrices* $\bar{W}, \bar{V} \in C^{n\times n}$;
(iv) $\mu(\gamma\bar{W}) = \gamma\mu(\bar{W})$, *for any matrix* $\bar{W} \in C^{n\times n}$ *and any non-negative real number* γ;

where $\lambda(\bar{W})$ *denotes any eigenvalue of* $\bar{W}$, *and* $Re(\lambda(\bar{W}))$ *denotes the real part of* $\lambda(\bar{W})$.

Lemma 2.3. *For any* $\gamma < 0$ *and any matrix* $\bar{W} \in C^{n\times n}$, $\mu(\gamma\bar{W}) = -\gamma\mu(-\bar{W})$.

Proof: This lemma can be immediately obtained from property (iv) in Lemma 2.2.

Lemma 2.4. *Let* $\bar{N} \in C^{n\times n}$. *If* $\mu(-\bar{N}) < 1$, *then* $\det(I + \bar{N}) \ne 0$.

Proof: Since $\mu(-\bar{N}) < 1$, property (ii) in Lemma 2.2 gets $Re(\lambda(\bar{N})) \ge -\mu(-\bar{N}) > -1$. This implies that $\lambda(\bar{N}) \ne -1$. Therefore, we have the stated result.

According to Lemma 2.1, for the uncertain TS-fuzzy-model-based control system in (1) and (2), each fuzzy-rule-uncertain model $\{A_i + \Delta A_i, C_i + \Delta C_i\}$ in (1) and (2) is observable if and only if the $n^2 \times n(n + m - 1)$ matrix

$$\tilde{Q}_i = Q_i + \sum_{k=1}^{\bar{m}} \varepsilon_{ik} E_{ik} \tag{6}$$

has a full row rank n^2, where

$$Q_i = \begin{bmatrix} I_n & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & 0 & C_i^{\mathrm{T}} \\ -A_i^{\mathrm{T}} & I_n & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & C_i^{\mathrm{T}} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & I_n & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & -A_i^{\mathrm{T}} & C_i^{\mathrm{T}} & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix} \tag{7}$$

and

$$E_{ik} = \begin{bmatrix} 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & 0 & C_{ik}^{\mathrm{T}} \\ -A_{ik}^{\mathrm{T}} & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & C_{ik}^{\mathrm{T}} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & -A_{ik}^{\mathrm{T}} & C_{ik}^{\mathrm{T}} & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix}. \tag{8}$$

Let the singular value decomposition of Q_i be

$$Q_i = U_i \begin{bmatrix} S_i & 0_{n^2\times n(m-1)} \end{bmatrix} V_i^H, \tag{9}$$

where $U_i \in R^{n^2\times n^2}$ and $V_i \in R^{n(n+m-1)\times n(n+m-1)}$ are the unitary matrices, V_i^H denotes the complex-conjugate transpose of matrix V_i, $S_i = \mathrm{diag}[\sigma_{i1}, \ldots, \sigma_{in^2}]$, and $\sigma_{i1} \ge \sigma_{i2} \ge \cdots \ge \sigma_{in^2} > 0$ are the singular values of Q_i.

The sufficient criterion presented next ensures that the uncertain TS-fuzzy-model-based control system in (1) and (2) is robustly locally observable.

Theorem 2.1. *Suppose that each fuzzy-rule-nominal model $\{A_i, C_i\}$ is observable. The uncertain TS-fuzzy-model-based control system in (1) and (2) is robustly locally observable if the following conditions simultaneously hold true:*

$$\sum_{k=1}^{\bar{m}} \varepsilon_{ik}\phi_{ik} < 1, \tag{10}$$

where $i = 1, 2, \ldots, N$;

$$\phi_{ik} = \begin{cases} \mu\left(-S_i^{-1}U_i^H E_{ik} V_i [I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \text{for } \varepsilon_{ik} \geq 0; \\ -\mu\left(S_i^{-1}U_i^H E_{ik} V_i [I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \text{for } \varepsilon_{ik} < 0; \end{cases}$$

the matrices E_{ik}, S_i, U_i and V_i $(i = 1, 2, \ldots, N)$ are defined by (8) and (9), respectively, and I_{n^2} denotes the $n^2 \times n^2$ identity matrix.

Proof: Since each fuzzy-rule-nominal model $\{A_i, C_i\}$ $(i = 1, 2, \ldots, N)$ is observable, matrix Q_i in (7) has a full row rank (i.e., $\text{rank}(Q_i) = n^2$) according to Lemma 2.1. We know that

$$\text{rank}(Q_i) = \text{rank}\left(S_i^{-1}U_i^H Q_i V_i\right). \tag{11}$$

Thus, instead of $\text{rank}(\tilde{Q}_i)$, we can discuss the rank of

$$\begin{bmatrix} I_{n^2} & 0_{n^2\times n(m-1)} \end{bmatrix} + \sum_{k=1}^{\bar{m}} \varepsilon_{ik} R_{ik}, \tag{12}$$

where $R_{ik} = S_i^{-1}U_i^H E_{ik} V_i$, for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, \bar{m}$. Since a matrix has rank of at least n^2 if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in (12) to have rank n^2 is the nonsingularity

$$G_i = I_{n^2} + \sum_{k=1}^{\bar{m}} \varepsilon_{ik} \bar{R}_{ik}, \tag{13}$$

where $\bar{R}_{ik} = S_i^{-1}U_i^H E_{ik} V_i [I_{n^2}, 0_{n^2\times n(p-1)}]^{\mathrm{T}}$, for $i = 1, 2, \ldots, N$.

According to Lemmas 2.2 and 2.3 and (10),

$$\begin{aligned} \mu\left(-\sum_{k=1}^{\bar{m}} \varepsilon_{ik}\bar{R}_{ik}\right) &= \mu\left(-\sum_{k=1}^{\bar{m}} \varepsilon_{ik}\left(S_i^{-1}U_i^H E_{ik} V_i \left[I_{n^2}, 0_{n^2\times n(p-1)}\right]^{\mathrm{T}}\right)\right) \\ &\leq \sum_{k=1}^{\bar{m}} \mu\left(-\varepsilon_{ik}\left(S_i^{-1}U_i^H E_{ik} V_i \left[I_{n^2}, 0_{n^2\times n(p-1)}\right]^{\mathrm{T}}\right)\right) \\ &= \sum_{k=1}^{\bar{m}} \varepsilon_{ik}\phi_{ik} \\ &< 1. \end{aligned} \tag{14}$$

Thus, from Lemma 2.4, we have

$$\det(G_i) = \det\left(I_{n^2} + \sum_{k=1}^{\bar{m}} \varepsilon_{ik}\bar{R}_{ik}\right) \neq 0. \tag{15}$$

Hence, matrix G_i in (13) is nonsingular. That is, matrix $\tilde{Q}_i$ in (6) has a full row rank of n^2. According to the above results and Lemma 2.1, the local observability of the uncertain TS-fuzzy-model-based control system in (1) and (2) is ensured. Thus, the proof is completed.

Additionally, this study considers the following uncertainty forms for the parametric uncertainty matrices ΔA_i, ΔB_i and ΔC_i [2]:

$$\Delta A_i = M_{A_i}\Delta N_{A_i}, \quad \Delta B_i = M_{B_i}\Delta N_{B_i} \text{ and } \Delta C_i = M_{C_i}\Delta N_{C_i}, \tag{16}$$

for $i = 1, 2, \ldots, N$, where M_{A_i}, M_{B_i}, M_{C_i}, N_{A_i}, N_{B_i} and N_{C_i} are known constant real matrices with appropriate dimensions, and Δ is an unknown matrix function where

$$\Delta \in \Omega := \{\Delta \,|\|\Delta\| \leq 1, \text{ the elements of } \Delta \text{ are Lebesgue measurable}\}.$$

Following the same procedures for the proof given in Theorem 2.1 gets the following corollary for ensuring that the uncertain TS-fuzzy-model-based control system in (1) and (2) with the uncertainty forms in (16) is robustly locally observable.

Corollary 2.1. *Suppose that each fuzzy-rule-nominal model $\{A_i, C_i\}$ is observable. The uncertain TS-fuzzy-model-based control system in (1) and (2) with the uncertainty forms in (16) is robustly locally observable if the following conditions simultaneously hold true:*

$$\alpha_i \beta_{1i} \beta_{2i} < 1, \tag{17}$$

where $\alpha_i = \|F_i\|$, $\beta_{1i} = \left\|S_i^{-1}U_i^H\right\|$, $\beta_{2i} = \left\|V_i[I_{n^2}, 0_{n_1^2\times n(m-1)}]^T\right\|$, *and*

$$F_i = \begin{bmatrix} 0 & 0 \bullet\bullet\bullet & 0 & 0 & \bullet\bullet\bullet & 0 & (M_{C_i}\Delta N_{C_i})^T \\ -(M_{A_i}\Delta N_{A_i})^T & 0 \bullet\bullet\bullet & 0 & 0 & \bullet\bullet\bullet & (M_{C_i}\Delta N_{C_i})^T & 0 \\ \bullet & \bullet\bullet\bullet\bullet & \bullet & \bullet & \bullet\bullet\bullet & \bullet & \bullet \\ \bullet & \bullet\bullet\bullet\bullet & \bullet & \bullet & \bullet\bullet\bullet & \bullet & \bullet \\ \bullet & \bullet\bullet\bullet\bullet & \bullet & \bullet & \bullet\bullet\bullet & \bullet & \bullet \\ 0 & 0 \bullet\bullet\bullet & 0 & 0 & \bullet\bullet\bullet & 0 & 0 \\ 0 & 0 \bullet\bullet\bullet & -(M_{A_i}\Delta N_{A_i})^T & (M_{C_i}\Delta N_{C_i})^T & \bullet\bullet\bullet & 0 & 0 \end{bmatrix}, \tag{18}$$

for $i = 1, 2, \ldots, N$; matrices S_i, U_i and V_i $(i = 1, 2, \ldots, N)$ are defined in (9), and I_{n^2} denotes the $n^2 \times n^2$ identity matrix.

2.2. **Robustly global observability.** The resulting TS-fuzzy-model-based control system with parametric uncertainties inferred from (1) and (2) is represented as

$$\dot{x}(t) = \sum_{i=1}^{N} h_i(z)\left((A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t)\right), \tag{19a}$$

and

$$y(t) = \sum_{i=1}^{N} h_i(z)\left((C_i + \Delta C_i)x(t)\right), \tag{19b}$$

where $z = [z_1, z_2, \ldots, z_g]^{\mathrm{T}}$ denotes the g-dimensional premise vector, $h_i(z) = w_i(z) \Big/ \sum\limits_{i=1}^{N} w_i(z)$, $w_i(z) = \prod\limits_{j=1}^{g} M_{ij}(z_j)$, and $M_{ij}(z_j)$ are the membership grades of z_j in the fuzzy sets M_{ij} $(i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, g)$. Thus, $h_i(z) \geq 0$ and $\sum\limits_{i=1}^{N} h_i(z) = 1$.

According to Lemma 2.1, the resulting uncertain TS-fuzzy-model-based control system in (19) is robustly globally observable if and only if the $n^2 \times n(n+m-1)$ matrix

$$\begin{aligned} \tilde{Q} &= \sum_{i=1}^{N} h_i(z)Q_i + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} h_i(z)\varepsilon_{ik}E_{ik} \\ &= \sum_{i=1}^{N} h_i(z)\left(\bar{Q} + D_i\right) + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} h_i(z)\varepsilon_{ik}E_{ik} \end{aligned}$$

$$= \bar{Q} + \sum_{i=1}^{N} h_i(z) D_i + \sum_{i=1}^{N} \sum_{k=1}^{\bar{m}} h_i(z) \varepsilon_{ik} E_{ik} \tag{20}$$

has full row rank n^2, where $\bar{Q}$ is any given $n^2 \times n(n+m-1)$ constant matrix having a full row rank, Q_i and E_{ik} are as given in (7) and (8), respectively, and $D_i = Q_i - \bar{Q}$.

Let the singular value decomposition of $\bar{Q}$ be

$$\bar{Q} = \bar{U} \begin{bmatrix} \bar{S} & 0_{n^2 \times n(m-1)} \end{bmatrix} \bar{V}^H, \tag{21}$$

where $\bar{U} \in R^{n^2 \times n^2}$ and $\bar{V} \in R^{n(n+m-1) \times n(n+m-1)}$ are the unitary matrices, $\bar{V}^H$ denotes the complex-conjugate transpose of matrix $\bar{V}$, $\bar{S} = \mathrm{diag}[\bar{\sigma}_1, \ldots, \bar{\sigma}_{n^2}]$, and $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_{n^2} > 0$ are the singular values of $\bar{Q}$. A sufficient criterion is presented for ensuring that the resulting uncertain TS-fuzzy-model-based control system in (19) is robustly globally observable.

Theorem 2.2. *The resulting uncertain TS-fuzzy-model-based control system in (19) is robustly globally observable if the following condition holds true:*

$$\sum_{i=1}^{N} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{N} \sum_{k=1}^{\bar{m}} \varepsilon_{ik} \bar{\phi}_{ik} < 1, \tag{22}$$

where $\bar{\Lambda}_i = \bar{S}^{-1} \bar{U}^H D_i \bar{V} [I_{n^2}, 0_{n^2 \times n(m-1)}]^T$; $\bar{\Omega}_{ik} = \bar{S}^{-1} \bar{U}^H E_{ik} \bar{V} [I_{n^2}, 0_{n^2 \times n(m-1)}]^T$;

$$\bar{\phi}_{ik} = \begin{cases} \mu(-\bar{\Omega}_{ik}), & \text{for } \varepsilon_{ik} \geq 0; \\ -\mu(\bar{\Omega}_{ik}), & \text{for } \varepsilon_{ik} < 0; \end{cases}$$

matrices E_{ik}, D_i, $\bar{S}$, $\bar{U}$ *and* $\bar{V}$ *are as defined in (8), (20) and (21); and* I_{n^2} *denotes the* $n^2 \times n^2$ *identity matrix.*

Proof: We know that

$$\mathrm{rank}(\bar{Q}) = \mathrm{rank}(\bar{S}^{-1} \bar{U}^H \bar{Q} \bar{V}). \tag{23}$$

Thus, instead of $\mathrm{rank}(\tilde{Q})$, we can discuss the rank of

$$\begin{bmatrix} I_{n^2} & 0_{n^2 \times n(m-1)} \end{bmatrix} + \sum_{i=1}^{N} h_i(z) \Lambda_i + \sum_{i=1}^{N} \sum_{k=1}^{\bar{m}} h_i(z) \varepsilon_{ik} \Omega_{ik}, \tag{24}$$

in which $\Lambda_i = \bar{S}^{-1} \bar{U}^H D_i \bar{V}$ and $\Omega_{ik} = \bar{S}^{-1} \bar{U}^H E_{ik} \bar{V}$. Since a matrix has at least rank n^2 if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient criterion for the matrix in (24) to have rank n^2 is the nonsingularity

$$G = I_{n^2} + \sum_{i=1}^{N} h_i(z) \bar{\Lambda}_i + \sum_{i=1}^{N} \sum_{k=1}^{\bar{m}} h_i(z) \varepsilon_{ik} \bar{\Omega}_{ik}, \tag{25}$$

where $\bar{\Lambda}_i = \bar{S}^{-1} \bar{U}^H D_i \bar{V} [I_{n^2}, 0_{n^2 \times n(m-1)}]^{\mathrm{T}}$ and $\bar{\Omega}_{ik} = \bar{S}^{-1} \bar{U}^H E_{ik} \bar{V} [I_{n^2}, 0_{n^2 \times n(m-1)}]^{\mathrm{T}}$.

According to Lemmas 2.2 and 2.3 and (22),

$$\begin{aligned}
&\mu\left(-\sum_{i=1}^{N} h_i(z)\bar{\Lambda}_i - \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} h_i(z)\varepsilon_{ik}\bar{\Omega}_{ik}\right)\\
&\le \sum_{i=1}^{N} h_i(z)\mu(-\bar{\Lambda}_i) + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} h_i(z)\mu(-\varepsilon_{ik}\bar{\Omega}_{ik})\\
&\le \sum_{i=1}^{N} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} \mu(-\varepsilon_{ik}\bar{\Omega}_{ik})\\
&= \sum_{i=1}^{N} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} \varepsilon_{ik}\bar{\phi}_{ik}\\
&< 1.
\end{aligned} \tag{26}$$

Thus, Lemma 2.4 gets

$$\det(G) = \det\left(I_{n^2} + \sum_{i=1}^{N} h_i(z)\bar{\Lambda}_i + \sum_{i=1}^{N}\sum_{k=1}^{\bar{m}} h_i(z)\varepsilon_{ik}\bar{\Omega}_{ik}\right) \neq 0. \tag{27}$$

Therefore, matrix G in (25) is nonsingular. That is, matrix $\tilde{Q}$ in (20) has the full row rank n^2. Therefore, the above results and Lemma 2.1 ensure the global observability of the resulting uncertain TS-fuzzy-model-based control system in (19). Thus, the proof is completed.

On the other hand, following the same proof procedures given in Theorem 2.2 gets the following corollary for ensuring that the uncertain TS-fuzzy-model-based control system in (1) and (2) with the uncertainty forms in (16) is robustly globally observable.

Corollary 2.2. *The resulting uncertain TS-fuzzy-model-based control system in (19) with the uncertainty forms in (16) is robustly globally observable if the following condition holds true:*

$$\sum_{i=1}^{N} \mu\left(-\bar{\Lambda}_i\right) + \sum_{i=1}^{N} \alpha_i \bar{\beta}_1 \bar{\beta}_2 < 1, \tag{28}$$

where $\bar{\Lambda}_i = \bar{S}^{-1}\bar{U}^H D_i \bar{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T$*;* $\alpha_i = \|F_i\|$*;* $\bar{\beta}_1 = \left\|\bar{S}^{-1}\bar{U}^H\right\|$*;* $\bar{\beta}_2 = \left\|\bar{V}[I_{n^2}, 0_{n_1^2\times n(m-1)}]^T\right\|$*; matrices* F_i*,* D_i*,* $\bar{S}$*,* $\bar{U}$ *and* $\bar{V}$ *are defined by (18), (20) and (21);* I_{n^2} *denotes the* $n^2 \times n^2$ *identity matrix.*

Remark 2.1. *The proposed sufficient conditions (10) and (22) give the explicit relationship of the bounds on* ε_{ik} *for preserving observability. Additionally, the bounds on* ε_{ik}*, which are obtained by using the proposed sufficient conditions, are not necessarily symmetric with respect to the origin of the parameter space regarding* ε_{ik}*, in which* $i = 1, 2, \ldots, N$*, and* $k = 1, 2, \ldots, \bar{m}$*.*

Remark 2.2. *The proposed robustly global observability conditions in (22) and (28) are very conservative. Therefore, approaches to deriving the relaxed robustly global observability conditions and to using the evolutionary algorithm to choose a suitable matrix* $\bar{Q}$ *[32-36] for reducing the conservatism of the proposed conditions are worthy of further study.*

Remark 2.3. *Recent works in the literature have tended to focus on the controller design problem of uncertain TS-fuzzy-model-based control systems. Although the literature* *on robust control theory agrees that LMI obtains good results, it cannot solve the rank preservation problem considered here. The LMI is also difficult to use for determining the robustly local and global observability of uncertain TS-fuzzy-model-based control systems with uncertainties. Researchers have studied observability problems in various fuzzy systems (e.g., [17-25]). Until now, however, no studies have discussed the robustly local and global observability of uncertain TS-fuzzy-model-based control systems. Additionally, although the observability problem has been studied in various fuzzy systems, the problem of robustly local and global observability in uncertain TS-fuzzy-model-based control systems has not been reported. Therefore, we present a new algebraic approach for studying the robustly local and global observability of uncertain TS-fuzzy-model-based control systems. Since the recent literature shows no comparable approaches, a performance comparison is not possible. Therefore, only an illustrative example is given to demonstrate the applicability of the sufficient criteria proposed here.*

3. Illustrative Example.

Consider the nonlinear mass-spring-damper mechanical system in Lee et al. [38]. The dynamic equation for the nonlinear mass-spring-damper mechanical system with elemental parametric uncertainties is

$$\bar{M}\ddot{y}(t) + \bar{g}(y(t), \dot{y}(t)) + \bar{f}(y(t)) = \varphi(\dot{y}(t))u(t), \tag{29}$$

where $\bar{M}$ is the mass, $u(t)$ is the force, $y(t)$ is the displacement, $\dot{y}(t)$ is the velocity, $\bar{g}(y(t), \dot{y}(t))$ is the nonlinear or uncertain term with respect to the damper, $\bar{f}(y(t))$ is the nonlinear or uncertain term with respect to the spring, and $\varphi(\dot{y}(t))$ is the nonlinear or uncertain term with respect to the input term. Here, the assumptions are that $y(t) \in [-1.5 \quad 1.5]$, $\dot{y}(t) \in [-1.5 \quad 1.5]$, $\bar{g}(y(t), \dot{y}(t)) = (1 + c_1)\dot{y}(t)$, $\bar{f}(y(t)) = (0.01 + c_2)y(t) + 0.1y^3(t)$, and $\varphi(\dot{y}(t)) = 1 + 0.13\dot{y}^3(t)$, where c_k $(k = 1, 2)$ are parametric uncertainties. The parameters in this example are set as $\bar{M} = 1.0$ kg, $-0.1 \le c_1 \le 0.11$, and $-0.2 \le c_2 \le 0.18$.

Therefore, by using the sector nonlinearity approach for fuzzy model construction [2], the uncertain nonlinear system in (29) can be represented by the following exact TS fuzzy model with system uncertainties, in which the output uncertain matrices ΔC_i $(i = 1, 2, 3, 4)$ are also considered due to variations in the displacement-sensor position:

$\tilde{R}^1$: IF z_1 is M_{11} and z_2 is M_{12},

$$\text{THEN} \quad \dot{x}(t) = (A_1 + \Delta A_1)\, x(t) + B_1 u(t), \tag{30a}$$

$$\text{and} \qquad y(t) = (C_1 + \Delta C_1)\, x(t); \tag{30b}$$

$\tilde{R}^2$: IF z_1 is M_{21} and z_2 is M_{22},

$$\text{THEN} \quad \dot{x}(t) = (A_2 + \Delta A_2)\, x(t) + B_2 u(t), \tag{31a}$$

$$\text{and} \qquad y(t) = (C_2 + \Delta C_2)\, x(t); \tag{31b}$$

$\tilde{R}^3$: IF z_1 is M_{31} and z_2 is M_{32},

$$\text{THEN} \quad \dot{x}(t) = (A_3 + \Delta A_3)\, x(t) + B_3 u(t), \tag{32a}$$

$$\text{and} \qquad y(t) = (C_3 + \Delta C_3)\, x(t); \tag{32b}$$

$\tilde{R}^4$: IF z_1 is M_{41} and z_2 is M_{42},

$$\text{THEN} \quad \dot{x}(t) = (A_4 + \Delta A_4)\, x(t) + B_4 u(t), \tag{33a}$$

$$\text{and} \qquad y(t) = (C_4 + \Delta C_4)\, x(t); \tag{33b}$$

where $z_1 = y^2(t)$, $z_2 = \dot{y}^3(t)$, $x(t) = [y(t) \quad \dot{y}(t)]^{\mathrm{T}}$, $x(0) = [-1 \quad -1]^{\mathrm{T}}$, $\Delta A_i = \sum_{k=1}^{3} \varepsilon_{ik} A_{ik}$, $\Delta C_i = \sum_{k=1}^{3} \varepsilon_{ik} C_{ik}$, $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix}$, $B_1 = B_3 = \begin{bmatrix} 0 \\ 1.43875 \end{bmatrix}$, $B_2 = B_4 = \begin{bmatrix} 0 \\ 0.56125 \end{bmatrix}$, $C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $A_{i3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_{11} = A_{21} = A_{31} = A_{41} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{12} = A_{22} = A_{32} = A_{42} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C_{13} = C_{23} = C_{33} = C_{43} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_{i1} = C_{i2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $M_{11} = M_{21} = 1 - \frac{z_1}{2.25}$, $M_{31} = M_{41} = \frac{z_1}{2.25}$, $M_{12} = M_{32} = 0.5 + \frac{z_2}{6.75}$, $M_{22} = M_{42} = 0.5 - \frac{z_2}{6.75}$, $\varepsilon_{i1} \in [-0.1 \quad 0.11]$, $\varepsilon_{i2} \in [-0.2 \quad 0.18]$, and $\varepsilon_{i3} \in [-0.01 \quad 0.9]$, where $i = 1, 2, 3, 4$, and where uncertainties ε_{i3} result from variations in the displacement-sensor measurement.

Applying the proposed conditions in (10) for the robustly local observability then gets

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.16778 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \end{aligned} \tag{34a}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.17778 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \end{aligned} \tag{34b}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.18778 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \end{aligned} \tag{34c}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.17778 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \end{aligned} \tag{34d}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.16071 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \end{aligned} \tag{34e}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.17071 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \end{aligned} \tag{34f}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.18071 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \end{aligned} \tag{34g}$$

$$\begin{aligned} &\sum_{k=1}^{3} \varepsilon_{1k} \phi_{1k} \le 0.17071 < 1, \\ &\text{for } \varepsilon_{11} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \ \varepsilon_{12} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix}, \text{ and } \varepsilon_{13} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \end{aligned} \tag{34h}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.17778 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{35a}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.17778 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{35b}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.18778 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{35c}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.17778 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{35d}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.16071 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{35e}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.17071 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{35f}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.18071 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{35g}$$

$$\sum_{k=1}^{3} \varepsilon_{2k}\phi_{2k} \leq 0.17071 < 1,$$
$$\text{for } \varepsilon_{21} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{22} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{23} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{35h}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \leq 0.17638 < 1,$$
$$\text{for } \varepsilon_{31} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{36a}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \leq 0.18638 < 1,$$
$$\text{for } \varepsilon_{31} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{36b}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}(t)\phi_{3k} \leq 0.18774 < 1,$$
$$\text{for } \varepsilon_{31} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{36c}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \le 0.17774 < 1, \quad \text{for } \varepsilon_{31} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{36d}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \le 0.16931 < 1, \quad \text{for } \varepsilon_{31} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{36e}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \le 0.17931 < 1, \quad \text{for } \varepsilon_{31} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{36f}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \le 0.18067 < 1, \quad \text{for } \varepsilon_{31} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{36g}$$

$$\sum_{k=1}^{3} \varepsilon_{3k}\phi_{3k} \le 0.17067 < 1, \quad \text{for } \varepsilon_{31} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{32} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{33} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{36h}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.16778 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{37a}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.17778 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{37b}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.18778 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{37c}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.17778 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} 0 & 0.11 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} -0.01 & 0 \end{bmatrix}; \tag{37d}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.16071 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} 0 & 0.18 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{37e}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.17071 < 1, \quad \text{for } \varepsilon_{41} \in \begin{bmatrix} -0.1 & 0 \end{bmatrix},\ \varepsilon_{42} \in \begin{bmatrix} -0.2 & 0 \end{bmatrix},\ \text{and } \varepsilon_{43} \in \begin{bmatrix} 0 & 0.9 \end{bmatrix}; \tag{37f}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.18071 < 1, \quad \text{for } \varepsilon_{41} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{42} \in \left[-0.2 \quad 0 \right], \ \text{and } \varepsilon_{43} \in \left[-0.01 \quad 0 \right]; \tag{37g}$$

$$\sum_{k=1}^{3} \varepsilon_{4k}\phi_{4k} \le 0.17071 < 1, \quad \text{for } \varepsilon_{41} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{42} \in \left[0 \quad 0.18 \right], \ \text{and } \varepsilon_{43} \in \left[-0.01 \quad 0 \right]. \tag{37h}$$

The results for (34)-(37) confirm that the uncertain TS-fuzzy-model-based control system in (30)-(33) is robustly locally observable.

However, entering the presented robustly global observability criterion in (22) in the constant matrix $\bar{Q} = (Q_1 + Q_2 + Q_3 + Q_4)/4$, gets

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.90472 < 1, \quad \text{for } \varepsilon_{i1} \in \left[0 \quad 0.11 \right], \ \varepsilon_{i2} \in \left[0 \quad 0.18 \right], \ \text{and } \varepsilon_{i3} \in \left[0 \quad 0.9 \right]; \tag{38a}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.94472 < 1, \quad \text{for } \varepsilon_{i1} \in \left[0 \quad 0.11 \right], \ \varepsilon_{i2} \in \left[-0.2 \quad 0 \right], \ \text{and } \varepsilon_{i3} \in \left[0 \quad 0.9 \right]; \tag{38b}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.97609 < 1, \quad \text{for } \varepsilon_{i1} \in \left[0 \quad 0.11 \right], \ \varepsilon_{i2} \in \left[-0.2 \quad 0 \right], \ \text{and } \varepsilon_{i3} \in \left[-0.01 \quad 0 \right]; \tag{38c}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.93609 < 1, \quad \text{for } \varepsilon_{i1} \in \left[0 \quad 0.11 \right], \ \varepsilon_{i2} \in \left[0 \quad 0.18 \right], \ \text{and } \varepsilon_{i3} \in \left[-0.01 \quad 0 \right]; \tag{38d}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.87644 < 1, \quad \text{for } \varepsilon_{i1} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{i2} \in \left[0 \quad 0.18 \right], \ \text{and } \varepsilon_{i3} \in \left[0 \quad 0.9 \right]; \tag{38e}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.91644 < 1, \quad \text{for } \varepsilon_{i1} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{i2} \in \left[-0.2 \quad 0 \right], \ \text{and } \varepsilon_{i3} \in \left[0 \quad 0.9 \right]; \tag{38f}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.94780 < 1, \quad \text{for } \varepsilon_{i1} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{i2} \in \left[-0.2 \quad 0 \right], \ \text{and } \varepsilon_{i3} \in \left[-0.01 \quad 0 \right]; \tag{38g}$$

$$\sum_{i=1}^{4} \mu(-\bar{\Lambda}_i) + \sum_{i=1}^{4}\sum_{k=1}^{3} \varepsilon_{ik}\bar{\phi}_{ik} \le 0.90780 < 1, \quad \text{for } \varepsilon_{i1} \in \left[-0.1 \quad 0 \right], \ \varepsilon_{i2} \in \left[0 \quad 0.18 \right], \ \text{and } \varepsilon_{i3} \in \left[-0.01 \quad 0 \right]. \tag{38h}$$

The results for (38) confirm that the uncertain TS-fuzzy-model-based control system in (30)-(33) is robustly globally observable. However, although the robustly global observability criterion ensures that the uncertain TS-fuzzy-model-based control system is robustly and globally observable, the data obtained data in (34)-(38) show that the criterion for robustly global observability is more conservative than that for robustly local observability.

4. **Conclusions.** In this study of the robust observability problem for uncertain TS-fuzzy-model-based control systems, the problem of rank preservation for robust observability of uncertain TS-fuzzy-model-based control systems is converted to a nonsingularity analysis problem. Under the assumption that each fuzzy rule of a nominal TS-fuzzy-model-based control system has a full row rank for its observability matrix, a sufficient criterion was proposed for preserving the assumed property when the system uncertainties are included in the nominal TS-fuzzy-model-based control systems. The proposed sufficient criterion indicates the explicit relationships of bounds on system uncertainties that are needed to preserve the assumed property. The criterion for the robustly global observability of the TS-fuzzy-model-based control system is also presented. A nonlinear mass-spring-damper mechanical system with both elemental parameter uncertainties and displacement-sensor measurement variations is also given to illustrate the application of the proposed sufficient criteria.

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