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PRODUCTION THROUGHPUT EVALUATION USING THE VASICEK MODEL

Kenji Shirai[1] and Yoshinori Amano[2]

[1]Faculty of Information Culture
Niigata University of International and Information Studies
3-1-1, Mizukino, Nishi-ku, Niigata 950-2292, Japan
shirai@nuis.ac.jp

[2]Kyohnan Elecs Co., Ltd.
8-48-2, Fukakusanishiura-cho, Fushimi-ku, Kyoto 612-0029, Japan
y_amano@kyohnan-elecs.co.jp

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Abstract. *The Vasicek model, which is used in mathematical finance, was adopted to evaluate the production throughput of a production flow system. The production throughput was assumed to behave as an average regression. The production consisting of asynchronous and synchronous processes was evaluated theoretically using the average regression. Three patterns, which combined asynchronous and synchronous methods involving nine workers and six stages each, were also tested. Both experiment and calculations gave almost the same production throughput data, thereby validating the mathematical model proposed in this study.*

Keywords: Production throughput, Vasicek model, Average regression, Stochastic differential equation of log-normal type

1. **Introduction.** Several studies have addressed the problem of productivity improvement in industrial production processes [1, 2]. Moreover, various theories have been applied to improve and reform production processes and increase productivity. In [3], an analysis that uses the queuing model and applies a log-normal distribution to model a system in the steel industry is described.

Several studies have reported approaches to shorten lead times [4, 5]. From the time of product ordering, the lead time is dependent on the work required to prepare the system for production.

We have reported that an analysis of the rate-of-return deviation for a certain equipment manufacturer over the past ten years displays "power-law distribution characteristics". Because the power-law distribution reveals the existence of a phase transition phenomenon, we expect that the rate-of-return deviation and the production system are correlated in a manner that is mediated by the power-law distribution [6]. By performing a data analysis, the relation between the rate-of-return deviation and production throughput has been clarified to some extent. The "fluctuation model of rate-of-return deviation" is self-similar and shows a fractal nature [7, 18]. Also, this power-law distribution characteristic has a "fluctuating" nature during phase transition. For example, occurrence of fluctuation is found at where the phase transition occurs at the point. Then, we have reported on the self-similarity of these fluctuations and noted the f^{-1} and f^{-2} fluctuations [8]. We have also verified self-similarity in the system through experiments on the supply chain system, and have used the supply chain system to produce control equipment. In total, nine workers were involved, and the production process was composed of six stages.

To compare the forms of production, we roughly conducted four patterns of asynchronous and synchronous methods. In this report, we propose that it is possible to increase manufacturing profits by adopting a management strategy that purposefully leads to a state of excessive production or excessive order entries. This management strategy is ideal on the basis of analysis of the cost rate of the production process.

Although the traditional approach to avoiding bottlenecks in production processes is to use the theory of constraints [9], we have reported that the synchronization method is superior for shortening throughput in production processes. This method requires synchronization between processes [10].

In our previous study [11], we constructed a state in which the production density of each process corresponded to the physical propagation of heat [18]. Using this approach, we showed that a diffusion equation dominates the production process. In other words, when minimizing the potential of the production field (stochastic field), the equation, which is defined by the production density function $S_i(x,t)$ and boundary conditions, is described by the use of diffusion equation with advection to move in transportation speed ρ. The boundary conditions describe a closed system in the production field. The adiabatic state in thermodynamics represents the same state [11].

With respect to the production flow system, generally, low volumes of a wide variety of products are produced through several stages in the production process. This method is good for producing specific control equipment such as semiconductor manufacturing equipment in our experience. We have reported many research findings in this area. The production flow process has nonlinear characteristics [12]. Moreover, we have made it clear that the manufacture of products proceeds in multiple stages from the beginning of production. Such volatility is encountered in every stage of manufacturing, and delays in the production line propagate this volatility to the successive steps. A delay in the production process is equivalent to a "fluctuation" in physical phenomena [13].

To achieve the production system goals, we propose the use of a mathematical model that focuses on the selection process and adaptation mechanism of the production lead time [14]. We model the throughput time of the production demand/production system in the production stage by using a stochastic differential equation of the log-normal type, which is derived from its dynamic behavior. Using this model and risk-neutral integral, we define and compute the evaluation equation for the compatibility condition of the production lead time. Furthermore, we apply the synchronization process and show that the throughput of the production process is reduced [14, 15].

In accordance with this result, we show that Kalman filter theory, conventionally used in state estimation problems in control theory, can be applied under an incomplete information state. In addition, by applying a theory of ongoing assessment in real option, the conditions that determine throughput rate are clarified and confirmed by numerical value calculations [15].

In this study, the Vasicek financial model was used for the first time as a tool in the throughput evaluation of a production flow system. Our previous studies showed that synchronization generally increased the throughput [10]. In an actual production setting, the throughput exhibits an average regression behavior. The average regression obeys a normal logarithm-type stochastic partial differential equation and can be evaluated at the termination time. Three tests, which consisted of an asynchronous method (Testrun1), a synchronous method involving pre- and post-processing (Testrun2), and a synchronous method (Testrun3), were performed to validate this theory. The number of devices produced during each test run and the resulting production throughput were compared. The same production volumes were obtained, validating our study. To the best of our knowledge, a work analysis of this type using the Vasicek model has not been undertaken by previous studies.

2. **Applying an Average Regression Model to a Production Flow System.** We utilize the average regression model from financial engineering for a production flow system, because the Vasicek model is appropriate to workers in the production process and provides productivity evaluation at the end time of a process.

The average regression model is as follows (See Appendix A).

$$dS(t) = a(t)[W(t) - S(t)]dt + \sigma S(t)dB(t) \tag{1}$$

where $a(t) \equiv constant$, $W(t) \equiv constant$ for simplicity. We obtain the following, which is expressed in another equation.

$$dS(t) = a(t)(W - S(t))dt + \sigma(S(t), t)dB(t) \tag{2}$$

where, noise is assumed to be not state-dependent. Thus, we obtain as follows.

$$dS(t) = a(t)(W - S(t))dt + \sigma S(t)dB(t) \tag{3}$$

where Equation (3) is assumed to represent the production throughput model.

3. **Production Flow Process.** Figure 1 depicts a manufacturing process that is termed as a production flow process. This manufacturing process is employed in the production of control equipment. In this example, the production flow process consists of six stages. In each step S1-S6 of the manufacturing process, materials are being produced.

The direction of the arrows represents the direction of the production flow. In this process, production materials are supplied through the inlet and the end-product is shipped from the outlet.

Figure 2 depicts the queuing situation that occurs between stages by irregular work. This queue greatly affects the throughput performance.

Next, based on Equation (3), we consider the model that represents the value of the product in the flow production system.

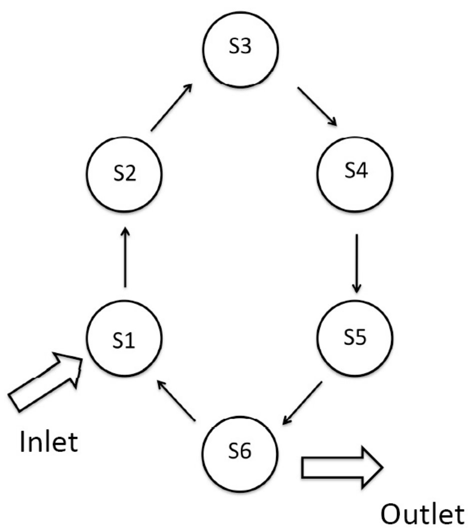

FIGURE 1. Production flow process

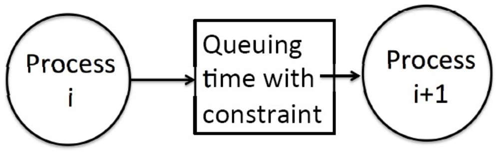

Figure 2. Queuing time with constraint

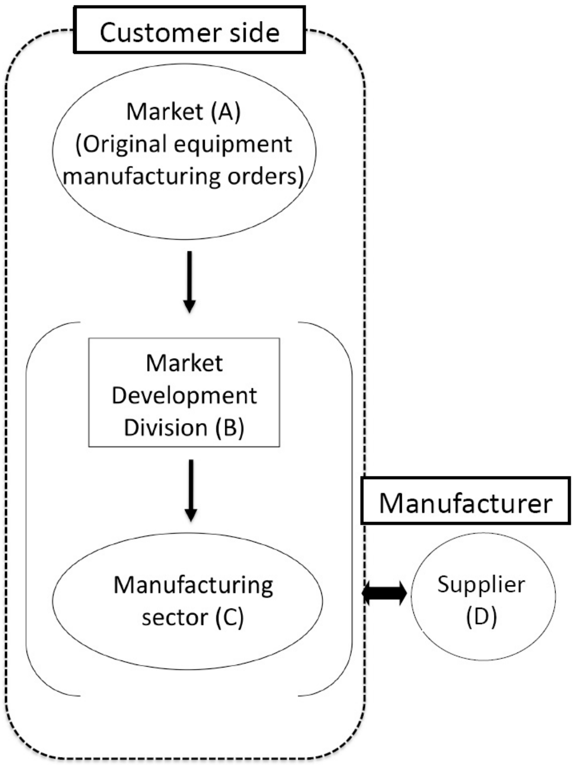

Figure 3. Business structure of company of research target

4. **Production Systems in the Manufacturing Equipment Industry.** The production methods used in manufacturing equipment are briefly covered in this paper. More information is provided in our study [6].

This system is considered to be a "Make-to-order system with version control", which enables manufacturing after orders are received from clients, resulting in "volatility" according to its delivery date and lead time. In addition, there is volatility in the lead time, depending on the content of the make-to-order products (production equipment).

In Figure 3(A), the "Customer side" refers to an ordering company and "Supplier (D)" means the target company in this paper. The product manufacturer, which is the source of the ordered manufacturing equipment presents an order that takes into account the market price. In Figure 3(B), the market development department at the customer's factory receives the order through the sale contract based on the predetermined strategy.

5. **Production Process Model.** It is often represented by a log-normal distribution [3]. The sales figures for the probability density function of the rate of return shows the log-normal distribution in Figure 4. Because small-to-midsize firms often do not have enough working capital, to sustain company operations, they are forced to raise working capital from financial institutions. It is non-linear in the case such as the products of

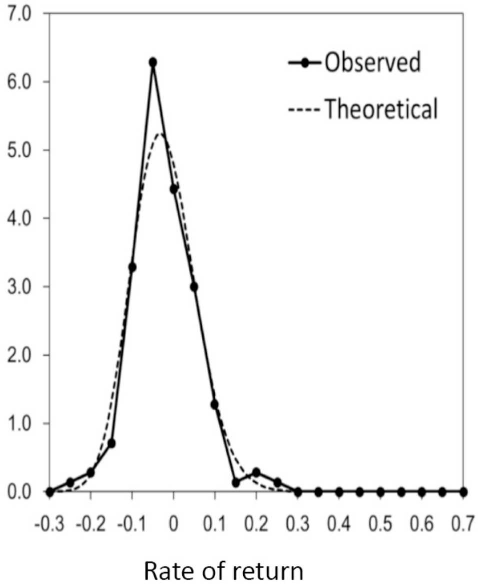

FIGURE 4. Probability density function of rate-of-return deviation: actual data (solid line) and data based on theoretical formula (dotted line)

different product specifications with fluctuations in demand or multi-kind small lot. We will report about this separately.

Thus, if the rate of return follows a log-normal distribution, we can assume that the cash flow will also follow the same log-normal distribution. Therefore, a cash flow model is defined as follows [20].

Definition 5.1. *Definition of a cash flow model*

$$\frac{dQ(t)}{Q(t)} = \mu dt + \sigma dW^Q(t) \tag{4}$$

where $Q(t)$ is an expected money amount of production for each month. The left-hand side is a monthly rate of return, and a rate of return varies with expected value μ. Further, σ represents a volatility, and $W^Q(t)$ standard Brownian motion.

From the data of monthly rate of return observed, its probability density function was calculated (Figure 4). As a result, it was found that the probability density function conforms to log-normal distribution (Figure 4, Theoretical).

Theoretical curve was calculated using EasyFit software (http://www.mathwave.com/), and as a result of Kolmogorov and Smirnov test, the observed values conformed to a log-normal type probability density function. Because, in the goodness-of-fit test of Kolmogorov-Smirnov, a null hypothesis that it is "log-normal" was not rejected with rejection rate 0.2, this data conforms to "log-normal" distribution. *P-value* was 0.588. The parameters of a theoretical curve were: $\mu_p = -0.134$ (average), $\sigma_p = 0.0873$ (standard deviation), $\gamma_p = -0.900$. The theoretical curve is given by the following formula.

$$f(x) = \frac{1}{\sqrt{2\pi}(x-\gamma_p)\sigma_p} \exp\Big\{-\frac{1}{2}\Big(\frac{(\ln x - \gamma_p) - \mu_p}{\sigma_p}\Big)^2\Big\} \tag{5}$$

We assumed manufacturing process follows a log-normal probability distribution. In fact, we found to be the log-normal probability distribution by analyzing the rate of return on monthly data of manufacturing operations (1999/1 to 2008/12) over the past 10 years (Figure 4 reference). We think a rate of return is proportional to the manufacturing process lead time. The throughput model is derived as follows [10].

$$dC(t) = \mu C(t)dt + \sigma C(t)dW(t) \tag{6}$$

where $C(t)$ is the only time related function same as $S(t)$.

The problem is to determine the product price $C(t)$ as $t = T$ for evaluation of production value, where $C(T) = 1$. However, T represents sufficiently a long time. In the production flow system, as the number of the product to be produced in each process is the maximum quantity of the stage number N, the maximum throughput is equivalent to $N/N = 1$. Appendix B gives the results of Testrun1-Testrun3 in the production flow system [10].

We obtain from Equation (3) as follows.

$$\frac{\partial C(t)}{\partial t} + a(W - S(t))\frac{\partial C(t)}{\partial S(t)} + \frac{1}{2}\sigma^2\frac{\partial^2 C(t)}{\partial S^2(t)} - S(t)C(t) = 0 \tag{7}$$

where $C(T) = 1$.

Equation (7) in the Vasicek model is well known, in that coupon bonds are satisfied by the partial differential equation. $S(t)$ represents the production throughput, and it is derived as follows.

$$S(t) = \exp(-at)\Big\{S(0) + W(\exp(at) - 1) + \sigma\int_0^t \exp(au)dB(u)\Big\} \tag{8}$$

We apply the above theory to the production flow system.

The model is derived as follows.

$$\frac{d\epsilon(t)}{\epsilon(t)} = \Big[1 - \frac{D}{2M}\Big]dt + \sigma dB(t) \tag{9}$$

We deform Equation (9) as follows.

$$dS(t) = a(W - S(t))dt + \sigma S(t)dB(t) \tag{10}$$

where $W = 1 - \frac{D}{2M}$.

The throughput model is affected by worker ability. This model holds the assumption of average regression model. Thus, the production value is affected by the throughput function.

For the production deviation that gives this function, we represent the productivity throughput deviation with respect to the standard deviation. Therefore, under the assumptions of the above mentioned model, the partial differential equation for $C(t)$, which provides product value as described above, has a meaning.

Therefore, the production value $C(t)$ is described by the fluctuation of $S(t)$ as follows.

$$C(t) = \exp\{\alpha(t) - \beta(t)S(t)\} \tag{11}$$

From the results of mathematical finance, we obtain as follows.

$$C(t) = \exp\Big\{-\beta S(t) + \Big(W - \frac{\sigma^2}{2a^2}\Big)\Big(\beta - (T - t)\Big) - \frac{\sigma^2}{4a}\beta^2\Big\} \tag{12}$$

$$\beta = \frac{1}{a}\Big(1 - \exp\{-a(T - t)\}\Big) \tag{13}$$

$$dS(t) = a(W - S(t))dt + \sigma dB(t) \tag{14}$$

where $C(T) = 1$.

Then $t = 0$, the value $C(T)$ at termination time T is as follows.

$$C(T) = \exp\Big\{\frac{1}{a}\Big(W - \frac{\sigma^2}{2a^2} - S_0\Big)\hat{\beta} - \Big(W - \frac{\sigma^2}{2a^2}\Big)T - \frac{\sigma^2}{4a^3}\hat{\beta}^2\Big\} \tag{15}$$

$$\hat{\beta} = 1 - \exp(-aT) \tag{16}$$

where S_0 is an initial value.

The production throughput is as follows.

$$y(t,T) = -\frac{1}{T-t}\ln C(t)$$
$$= \Big(W - \frac{\sigma^2}{2a^2}\Big) - \frac{1}{T-t}\Big\{\Big(W - \frac{\sigma^2}{2a^2} - S(t)\Big)\beta - \frac{\sigma^2}{4a}\beta^2\Big\} \tag{17}$$

where the relationship between $C(T)$ and $y(t,T)$ is as follows.

$$C(t) = \exp\Big\{-(T-t)y(t,T)\Big\} \tag{18}$$

The equipment production for the price $C(t)$ is carried out until the termination time T (cycle termination time) under the production flow system. However, if T is sufficiently long, $C(t)$ does not decrease ($S(T) = 1.0$).

6. **Numerical Example of Production Throughput.** In a numerical example of a production throughput in Figure 5, the risk evaluation value was calculated using Equation (16). Therefore, the production progress was determined at the time of completion. In Figure 6, the evaluation value was estimated using 1− Equation (16). In other words, the production value corresponds to the production progress at the completion time. In Figure 6, the thin and thick lines ultimately overlapped, indicating that the production was completed after a long time.

We present an evaluation of the production process model as follows. Our basic idea is under average regression.

Herein, we describe about $D/2M$. M and D are the actual data. M represents the elements of $[6 \times 9]$ (six workers and nine stages in the process). D is $\Delta X > K$ (now $K \geq 4$), which represents the number of elements for target throughput (WS in actual data). Then, D/M represents the error rate in Testrun1-Testrun3. $D/2M$ represents the average of D/M under normal distribution. Therefore, $\mu = 1 - (D/2M)$.

$$\frac{dy(t)}{y(t)} = \Big[1 - \frac{D}{2M}\Big]dt + \sigma dW(t) \tag{19}$$

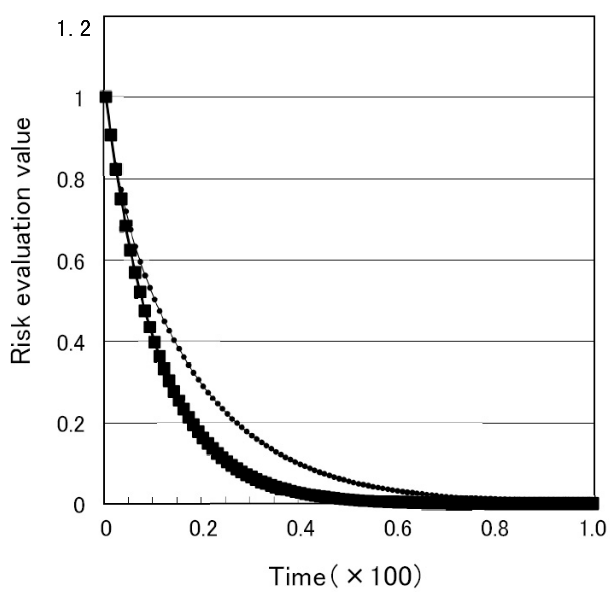

FIGURE 5. Risk evaluation value

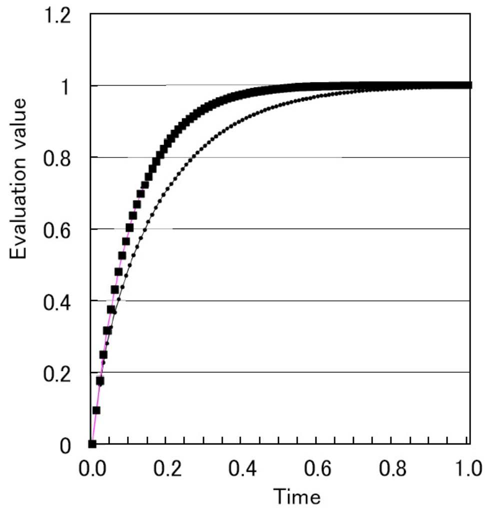

FIGURE 6. Throughput using the average regression model

TABLE 1. Parameters of Figure 5 and Figure 6

Fig	Figure 5(Thin-line)	Figure 5(Thick-line)	Figure 6(Thick-line)	Figure 6(Thin-line)
a	0.4	0.4	0.6	0.4
W	0.08	0.09	0.095	0.076
σ	0.09	0.01	0.01	0.1
S_0	0.1	0.1	0.1	0.1

where $\sigma \cong P/M$ represents deviation.

Assuming that the drift term (average) in Equation (19) is $\mu = (1 - D/2M)$, the throughput probability is derived from the average value to the deviation η as follows.

$$P(\ln y + \eta > \theta) = P\Big(y > \frac{\theta - (1 - D/2M)}{\sigma} - \eta\Big) \tag{20}$$

$$P(\ln y - \eta > \theta) = P\Big(y > \frac{\theta - (1 - D/2M)}{\sigma} + \eta\Big) \tag{21}$$

In regard to Equation (20) and Equation (21), see Figure 7. The probability $P(\ln y+\eta > \theta)$ and $P(\ln y - \eta > \theta)$ for the threshold θ are as follows.

$$P\Big(\ln y > \frac{\theta - (1 - D/2M)}{\sigma} - \eta\Big) = P\Big(y > \exp\Big(\frac{\theta - (1 - D/2M)}{\sigma} - \eta\Big)\Big) \tag{22}$$

$$P\Big(\ln y > \frac{\theta - (1 - D/2M)}{\sigma} + \eta\Big) = P\Big(y > \exp\Big(\frac{\theta - (1 - D/2M)}{\sigma} + \eta\Big)\Big) \tag{23}$$

The probability is obtained as follows.

$$P(y > \theta) = \Phi\Big(\exp\Big(\frac{\theta - (1 - D/2M)}{\sigma} + \eta\Big)\Big) - \Phi\Big(\exp\Big(\frac{\theta - (1 - D/2M)}{\sigma} - \eta\Big)\Big) \tag{24}$$

Figure 8 shows the time transition which is used in $M = 54$, $D = 25$, and $\sigma = 0.169$ for parameters of Equation (9). Figure 9 shows considerable variation in work capacity. Therefore, the throughput probability indicates a lower value and corresponds to Testrun1. Figure 10 shows slightly variation in the ability to work. Therefore, the throughput probability indicates a high value and corresponds to Testrun2. Figure 11 shows slightly variation in the ability to work same as Figure 10. Therefore, the throughput probability indicates a high value and corresponds to Testrun3.

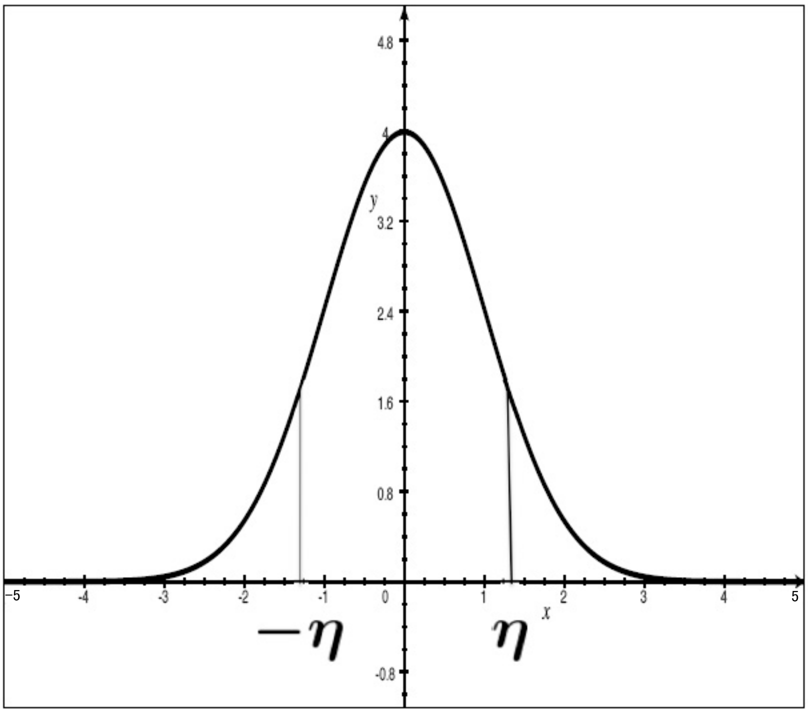

FIGURE 7. Probability for normal distribution

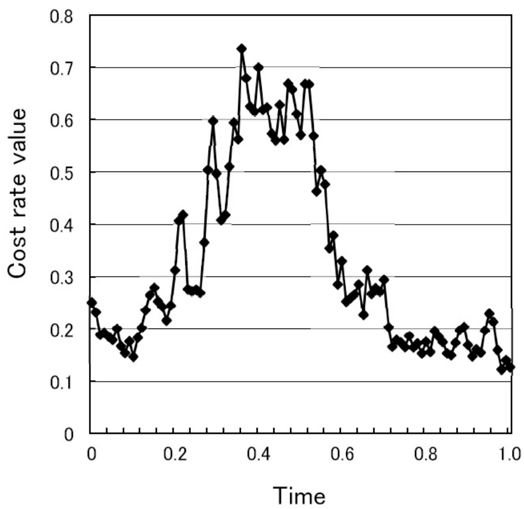

FIGURE 8. Solution process of Equation (9)

With respect to an entropy value, Figure 12 shows a high value which represents the sum of entropy because of considerable variation and corresponds to Testrun1. Figure 13 shows a low value which represents the sum of entropy because of small variation and corresponds to Testrun2. Figure 14 shows the lowest value which represents the sum of entropy because of small variation, and corresponds to Testrun3.

Definition 6.1. *Entropy function*

$$\mathcal{H} \equiv - \int Prob(y > \theta) \ln Prob(y > \theta) \tag{25}$$

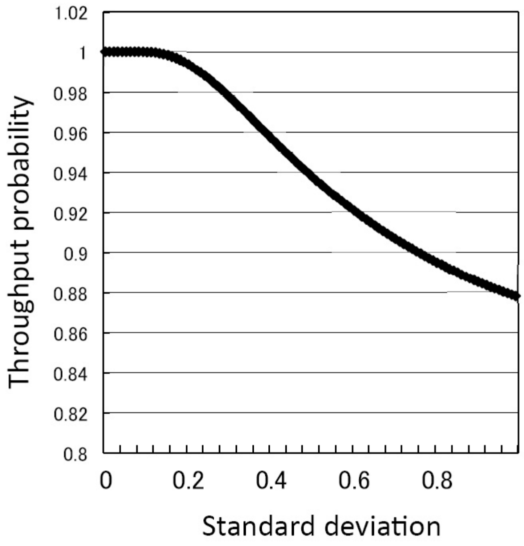

FIGURE 9. Throughput probability (→ Testrun1)

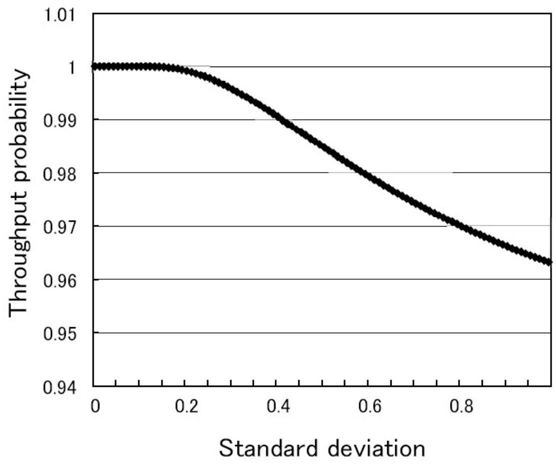

FIGURE 10. Throughput probability (→ Testrun2)

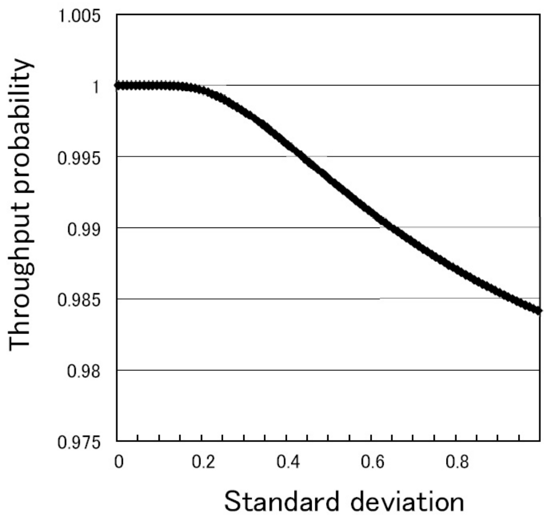

FIGURE 11. Throughput probability (→ Testrun3)

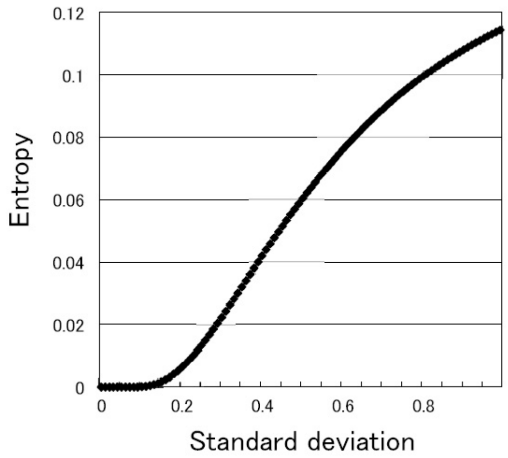

FIGURE 12. Entropy value (→ Testrun1)

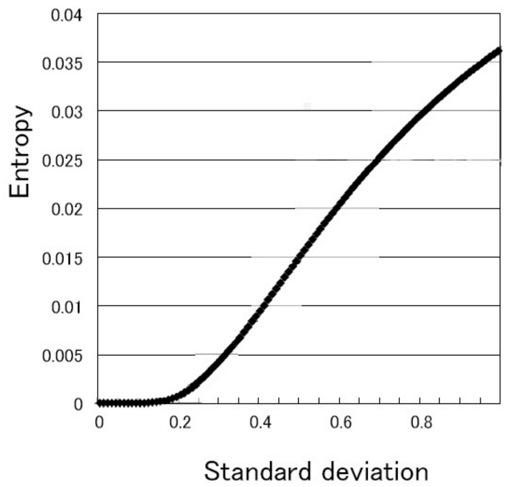

FIGURE 13. Entropy value (→ Testrun2)

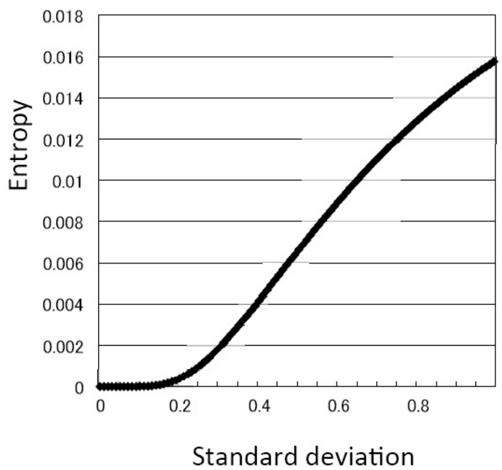

FIGURE 14. Entropy value (→ Testrun3)

TABLE 2. Entropy value calculation using regression model

Fig	Figure 12	Figure 13	Figure 14
M	54	54	54
D	25	5	6
P	20	5	3
Average of throughput probability	0.77	0.95	0.94
Bias rate	0.37	0.13	0.05
Sum of entropy	5.61	1.55	0.68

TABLE 3. Pieces of equipment of Testrun1-Testrun3 in practice

	Testrun1	Testrun2	Testrun3
Average	0.73	0.92	0.92
STD	0.29	0.06	0.03
Pieces of equipment	4.4	5.5	5.5

TABLE 4. Throughput value of Testrun1-Testrun3 by the calculation

	Testrun1	Testrun2	Testrun3
Fig	Figure 9	Figure 10	Figure 11
Throughput probability	0.77	0.95	0.94
Throughput	$6 \times 0.77 = 4.6$	$6 \times 0.95 = 5.7$	$6 \times 0.94 = 5.6$

7. **Validation of Evaluation.** Here, the trend coefficient, which is the ratio of actual number to the target number of pieces of equipment, represents a factor that indicates the trend for the number of pieces of production equipment, as shown in Table 3.

Here, the throughput values calculated from the throughput probability in Testrun1-Testrun3, are as follows. The unit of throughput in Table 3 is pieces of equipment. The throughput data of Table 4 are almost equal to the pieces of equipment which is produced in practice (Table 3). Therefore, the validity of this approach can be verified.

8. **Conclusions.** As shown in Table 4, the throughput decreased in the order Testrun2 $>$ Testrun3 $>$ Testrun1. Testrun2 and Testrun3 were both synchronous processes while Testrun1 was asynchronous. The regression model and entropy analysis showed that the synchronous process is the best method for improving the process throughput, suggesting that the financial model proposed in this study was valid.

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Appendix A. **Induction of Vasicek Model Equation.** Equation (6) is the throughput model as follows.

$$dC(t) = \mu C(t)dt + \sigma_s C(t)dW(t)$$

We obtain from Equation (6) as follows [20].

$$\mu C(t) = \frac{\partial C(t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C(t)}{\partial S^2(t)} + m\frac{\partial C(t)}{\partial S(t)} \tag{26}$$

$$\sigma_C C(t) = \sigma \frac{\partial C(t)}{\partial S(t)} \tag{27}$$

Because the market is risk free, the production price of value risk is constant, the same as the market price of interest rate risk. Then, we obtain as follows.

$$\frac{\mu - S(t)}{\sigma_C} = \lambda(constant) \tag{28}$$

We obtain as follows same as Vasicek model.

$$\lambda = \frac{\mu - S(t)}{\sigma_C} = \frac{m - a(W - S(t))}{\sigma} \tag{29}$$

Using the above equation of Equation (29), we calculate σ_C, and then we substitute σ_C into Equation (28). We obtain as follows.

$$\frac{\mu - S(t)}{\frac{\sigma}{C(t)} \cdot \frac{\partial C(t)}{\partial S(t)}} = \frac{m - a(W - S(t))}{\sigma} \tag{30}$$

Rearranging Equation (30), we obtain as follows.

$$\mu C(t) = (m - a(W - S(t))\frac{\partial C(t)}{\partial S(t)} + S(t)C(t) \tag{31}$$

We substitute Equation (31) into Equation (26). Then, Equation (7) is derived as follows.

$$\frac{\partial C(t)}{\partial t} + a(W - S(t))\frac{\partial C(t)}{\partial S(t)} + \frac{1}{2}\sigma^2\frac{\partial^2 C(t)}{\partial S^2(t)} - S(t)C(t) = 0$$

The production throughput $S(t)$ is assumed as follows.

$$dS(t) = a(W - S(t))dt + \sigma dB(t) \tag{32}$$

where $B(t)$ represents the Wiener process. Equation (32) is the so-called Vasicek equation.

Next, we consider the following for induction of Equation (7).

$$f(t, S(t)) = \exp(at)S(t) \tag{33}$$

$f = S(0)$ as $t = 0$, then using Theorem of Ito, we obtain as follows.

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \tag{34}$$

Here,

$$\frac{\partial f}{\partial t} = a\exp(at)S(t),\ \frac{\partial f}{\partial S} = \exp(at),\ \frac{1}{2}\frac{\partial^2 f}{\partial S^2} = 0 \tag{35}$$

We substitute Equation (35) into Equation (34). Then, we obtain as follows.

$$\begin{aligned} df &= a\exp(at)S(t)dt + \exp(at)\Big(a(W - S(t))dt + \sigma dB(t)\Big) \\ &= aW\exp(at)dt + \sigma\exp(at)dB(t)(\textit{from Ito's integral}) \\ \exp(at)S(t) &= \int_0^t aW\exp(au)du + \int_0^t \sigma\exp(au)dB(u) + S(0)(\textit{from Ito's integral}) \\ S(t) &= \exp(-at)\Big\{S(0) + W(\exp(at) - 1) + \sigma\int_0^t \exp(au)dB(u)\Big\} \end{aligned} \tag{36}$$

Appendix B. **Testrun1 through Testrun3 Results Using Production Flow Process.** In Table 5, the circle mark represents the working delay by comparing with WS data (working standard).

TABLE 5. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

TABLE 6. Standard deviation of Table 5

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

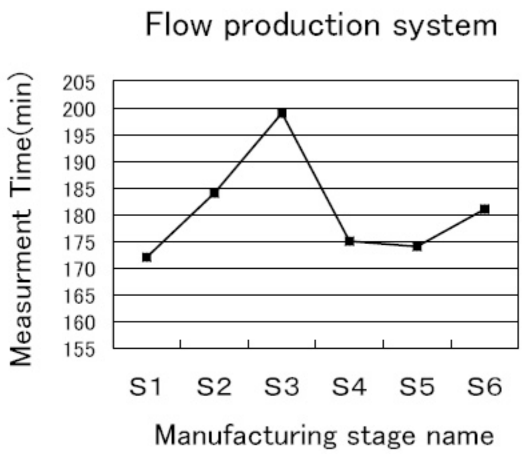

FIGURE 15. Total work time for each stage (S1-S6) in Table 5

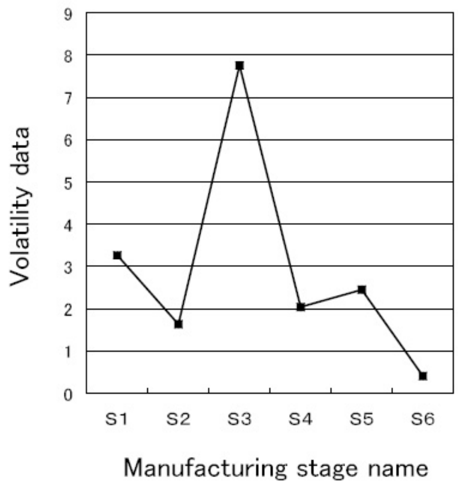

FIGURE 16. STD data for each stage (S1-S6) in Table 5

TABLE 7. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

TABLE 8. Deviation of Table 7

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 9. Total manufacturing time at each stage for each worker, K5 (*): Previous process

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	18	18	18
K2	20	18	18	18	18	18	18
K3	20	21	21	21	21	21	21
K4	16	13	11	11	13	13	13
K5	16	*	*	*	*	*	*
K6	16	18	18	18	18	18	18
K7	16	14	14	13	14	14	13
K8	20	22	22	22	22	22	22
K9	20	20	20	20	20	20	20
Total	148	144	143	141	144	144	143

TABLE 10. Standard deviation of values stated in Table 9, K5: Previous process

K1	0.67	0.33	0.67	0.67	0.67	0.67
K2	0.67	0.67	0.67	0.67	0.67	0.67
K3	0.33	0.33	0.33	0.33	0.33	0.33
K4	1	1.67	1.67	1	1	1
K5	*	*	*	*	*	*
K6	0.67	0.67	0.67	0.67	0.67	0.67
K7	0.67	0.67	1	0.67	0.67	1
K8	0.67	0.67	0.67	0.67	0.67	0.67
K9	0	0	0	0	0	0