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NEW H_∞ CONTROL DESIGN FOR POLYTOPIC SYSTEMS WITH MIXED TIME-VARYING DELAYS IN STATE AND INPUT

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Abstract. *This paper concerns with the problem of state-feedback H_∞ control design for a class of linear systems with polytopic uncertainties and mixed time-varying delays in state and input. Our approach can be described as follows. We first construct a state-feedback controller based on the idea of parameter-dependent controller design. By constructing a new parameter-dependent Lyapunov-Krasovskii functional (LKF), we then derive new delay-dependent conditions in terms of linear matrix inequalities ensuring the exponential stability of the corresponding closed-loop system with a H_∞ disturbance attenuation level. The effectiveness and applicability of the obtained results are demonstrated by practical examples.*

Keywords: Polytopic uncertainties, H_∞ control, Time-varying delays, Input delayed, Linear matrix inequalities

1. **Introduction.** It it well known that time delay frequently occurs in engineering systems and usually is a source of bad performance, oscillations or instability [1, 2]. The problems of stability analysis and controller synthesis for time delay systems are essential and of great importance for theoretical and practical reasons [3], which have been extensively studied in the past decades, see, for example, [4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

In many models of control systems such as chemical, hydraulic and pneumatic systems, digital control or communication networks, a time-varying input delay arises due to many reasons. Its presence is usually motivated by a physical nature of a plant or being introduced artificially to model a sampling effect [13, 14, 15]. Control of a system with input delay is an important problem treated in the literature, see, for example, [15, 16, 17, 18, 19] and the references within.

Beside, external disturbances are usually unavoidable in modeling a wide range of phenomena in practical and engineering systems due to data transformations, modeling inaccuracies, linearization approximations, unknown disturbances and measurement errors [20]. Therefore, the problem of control for dynamical systems subject to time-delay/input delayed and external disturbance has been an important topic in control engineering [20, 21, 22]. Recently, considerable attention has been paid to address the problem of H_∞ control especially for systems with prescribed ranges of uncertainties [23, 24, 25, 26, 27, 28]. Roughly speaking, the main objective of the H_∞ control problem is to design a stabilizing feedback controller for such system subjected to norm-bounded disturbances. This controller is usually robust with respect to prescribed ranges of parameter uncertainties. Thus, the H_∞ control for time-delay systems is of practical and theoretical interest in many industrial and engineering processes [15, 18, 29, 30].

Among the models for describing the realistic parameter uncertainty, the polytopic uncertainty has been recognized to be more general, which can cover the well-known interval and linear parameter uncertainty as well as multimodel structures [21, 31]. An advanced research topic for time-delay systems with parameter uncertainties residing in a polytope is to develop robust delay-dependent stability conditions using parameter-dependent approach. Many attempts have been made in the past few years to realize the parameter-dependent idea in stability analysis and control for time-delay systems, and some less conservative robust stability conditions have been proposed [32, 33, 34, 35, 36, 37, 38, 39]. Particularly, in [34], the problem of exponential stabilization via state feedback controller for linear polytopic systems with constant delay was studied which was later extended to polytopic systems with mixed discrete and distributed constant delays in state and input in [37]. The problem of H_∞ control for a class of mixed time-varying delays in the state was considered in [35]. By using a parameter-dependent approach in designing a state feedback controller and in constructing an improved LKF, delay-dependent conditions were derived in terms of some linear matrix inequalities.

However, it should be pointed out that, the problem of H_∞ control for polytopic systems with mixed time-varying delays in both state and control input would be interesting. Theoretically, analyzing the stability of systems with mixed delays in both state and control input are quite complicated, especially for the case where the system matrices belong to some convex polytopes. In practice, systems with distributed delays in both state and input have many important applications in various areas as discussed in the preceding paragraphs. Although many important results in the field of stability analysis and control have been devoted to polytopic systems with delays, the problem of H_∞ control for polytopic systems with mixed discrete and distributed time-varying delays in state and control input has not yet been fully investigated.

Motivated by the above discussions, in this paper, we consider the problem of H_∞ control for a class of linear polytopic systems with mixed time-varying delays in state and control input. The novel features of the results obtained in this paper are twofold. Firstly, the system considered in this paper is subjected to polytopic uncertainties and mixed discrete and distributed time-varying delays in both state and control input. Secondly, by constructing an improved parameter-dependent LKF, new delay-dependent conditions are derived in terms of linear matrix inequalities in order to design a parameter-dependent state feedback controller guaranteeing exponential stability of the closed-loop system with an H_∞ disturbance attenuation level. The derived conditions in this paper do not require any assumption on the controllability of the nominal system. The approach also allows us to compute simultaneously the two bounds that characterize the exponential stability of the closed-loop system.

Notations: Throughout this paper, we let $\mathbb{R}$, $\mathbb{N}$ denote the set of real numbers and natural numbers, respectively. For given $p \in \mathbb{N}$, we denote $\underline{p} = \{1, 2, \dots, p\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with standard norm $\|.\|$ and scalar product $\langle , \rangle$. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$-matrices. For a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes the transpose of A. Let $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda(A)$ the set of eigenvalues of A and $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ the maximal and minimal real part of the eigenvalues of A, respectively. A matrix $Q \in \mathbb{R}^{n\times n}$ is symmetric if $Q = Q^{\mathsf{T}}$, semi-positive definite, write $Q \geq 0$, if it is symmetric and $\langle Qx, x\rangle \geq 0$, $\forall x \in \mathbb{R}^n$ and positive definite, write $Q > 0$, if it is symmetric and $\langle Qx, x\rangle > 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$. For any $A, B \in \mathbb{R}^{n\times n}$, $A \geq B$, $A > B$ mean that $A - B \geq 0$ and $A - B > 0$, respectively.

2. **Problem Statement and Preliminaries.** Consider the following control system with mixed time-varying delays in state and input

$$\begin{cases} \dot{x}(t) = A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) + A_2(\xi)\displaystyle\int_{t-r(t)}^{t} x(s)ds + C(\xi)w(t) \\ \qquad\qquad + B_0(\xi)u(t) + B_1(\xi)u(t-h(t)) + B_2(\xi)\displaystyle\int_{t-r(t)}^{t} u(s)ds, \quad t \geq 0, \\ z(t) = E(\xi)x(t) + F(\xi)u(t), \\ x(t) = \phi(t), \quad t \in [-\overline{h}, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state vector and control input, respectively, $w(t) \in \mathbb{R}^s$ is unknown disturbance, $z(t) \in \mathbb{R}^q$ is the observation vector, $h(t)$, $r(t)$ are time-varying delays satisfying $0 \leq h(t) \leq h$, $0 \leq r(t) \leq r$, $\dot{h}(t) \leq \delta$, $\dot{r}(t) \leq \delta$, where $\delta < 1$ is a constant, $\overline{h} = \max\{h, r\}$ and $\phi \in C([-\overline{h}, 0], \mathbb{R}^n)$ is the initial function with the norm $\|\phi\| = \sup_{-\overline{h}\leq t\leq 0} \|\phi(t)\|$. The system matrices are assumed belonging to a polytope Ω defined by

$$\Omega = \left\{ [A_k, B_k, C, E, F](\xi) = \sum_{i=1}^{p} \xi_i [A_{ki}, B_{ki}, C_i, E_i, F_i],\ k = 0, 1, 2,\ \xi_i \geq 0,\ \sum_{i=1}^{p} \xi_i = 1 \right\},$$

where $p \in \mathbb{N}$ is the number of vertices of Ω, A_{ki}, B_{ki}, C_i, E_i, F_i, $k = 0, 1, 2$, $i \in \underline{p} := \{1, 2, \ldots, p\}$, are given real matrices with appropriate dimensions.

In this paper, we will design a parameter-dependent state feedback controller of the form

$$u(t) = K(\xi)x(t), \quad t \geq 0, \tag{2}$$

solve H_∞ control problem for system (1) given in the following definition.

Definition 2.1. *Given $\beta > 0$, $\gamma > 0$. System* (1) *is said to be H_∞ stabilizable if there exists a controller* (2) *satisfying the two following requirements*

(i) *The closed-loop system of* (1) *without disturbance, i.e., $w(t) = 0$, is β-exponentially stable, that means, there exists a positive number σ such that every solution $x(t, \phi)$ of the closed-loop system satisfies*

$$\|x(t, \phi)\| \leq \sigma \|\phi\| e^{-\beta t}, \quad t \geq 0.$$

(ii) *There is a number $c_0 > 0$ such that*

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0\|\phi\|^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma^2,$$

where the supremum is taken over all $\phi \in C([-\overline{h}, 0], \mathbb{R}^n)$ and nonzero $w \in L_2([0, \infty), \mathbb{R}^s)$.

The main goal of this paper is to derive delay-dependent conditions in terms of linear matrix inequalities for designing the parameter-dependent controller (2) to solve the H_∞ control problem for system (1).

Remark 2.1. *In this paper, the idea of parameter-dependent approach is employed to derive conditions for designing H_∞ stabilizer* (2) *which depends on the parameter ξ. As discussed in* [32]*, the polytopic-type uncertainty can describe the parametric uncertainty more precisely, and thus less conservative than the norm-bounded uncertainty. Moreover, in many practical applications, parameters can be measured on-line without difficulty and the parameter-dependent controller* (2) *can lead to less conservative conditions.*

Let us introduce some auxiliary lemmas as follows.

Lemma 2.1 (Non-strict Schur complement lemma [20]). *For any matrices X, Y with appropriate dimensions, $X = X^\top$, $Z = Z^\top > 0$, then $\begin{bmatrix} X & Y \\ Y^\top & -Z \end{bmatrix} \le 0$ if and only if $X + YZ^{-1}Y^\top \le 0$.*

Lemma 2.2 (Completing square [35]). *For any matrices P, Q with appropriate dimensions, $Q = Q^\top > 0$, then*

$$2\langle Py, x\rangle - \langle Qy, y\rangle \le \langle PQ^{-1}P^\top x, x\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Lemma 2.3 (Jensen's inequality [4]). *For given $\nu > 0$, symmetric positive definite matrix W, then for any vector function $\omega(.)$ such that the concerned integrals are well defined, the following inequality holds*

$$\left[\int_0^\nu \omega(s)ds\right]^\top W \left[\int_0^\nu \omega(s)ds\right] \le \nu \int_0^\nu \omega^\top(s) W \omega(s) ds.$$

3. **Main Results.** The following notations are specifically used in this paper. For symmetric positive definite matrices P_j, Q_j, R_j, semi-positive definite matrices S_1, S_2 and matrices Y_j, we denote $\mu = 1 - \delta$ and, for $i, j \in \underline{p}$,

$$\begin{aligned}
&\mathcal{A}_{ij} = A_{0i}P_j + P_j A_{0i}^\top, \\
&\mathcal{B}_{ij} = B_{0i}Y_j + Y_j^\top B_{0i}^\top + e^{2\beta r}\left(\mu^{-1}B_{1i}B_{1j}^\top + rB_{2i}B_{2j}^\top\right), \\
&\Gamma_{ij} = \mathcal{A}_{ij} + \mathcal{B}_{ij} + 2\beta P_j + Q_j + hR_j, \\
&\mathcal{H}_{ij} = \begin{bmatrix} A_{1i}P_j & \sqrt{h}A_{2i}P_j & \sqrt{1+r}Y_j^\top \end{bmatrix}, \\
&\mathcal{D}_j = \operatorname{diag}\left\{\mu e^{-2\beta h}Q_j, e^{-2\beta r}R_j, I_m\right\}, \quad \mathcal{M}_{ij} = \begin{bmatrix} \Gamma_{ij} & \mathcal{H}_{ij} \\ * & -\mathcal{D}_j \end{bmatrix}, \\
&\mathcal{N}_{ij} = \begin{bmatrix} -2\beta P_j + 1/\gamma^2 C_i C_j^\top & P_j E_i^\top + Y_j^\top F_i^\top \\ * & -I_q \end{bmatrix}, \\
&\mathbb{S}_1 = \begin{bmatrix} S_1 & 0_{n\times(2n+m)} \\ * & 0_{(2n+m)\times(2n+m)} \end{bmatrix}, \quad \mathbb{S}_2 = \begin{bmatrix} S_2 & 0_{n\times q} \\ * & 0_{q\times q} \end{bmatrix}.
\end{aligned}$$

For sake of brevity, we use the following notations $P = \sum_{i=1}^p \xi_i P_i$, $Q = \sum_{i=1}^p \xi_i Q_i$, $R = \sum_{i=1}^p \xi_i R_i$, $Y = \sum_{i=1}^p \xi_i Y_i$ and

$$\begin{aligned}
\lambda_{\min}(P) &= \min_{i\in\underline{p}} \lambda_{\min}(P_i), \quad \lambda_{\max}(P) = \max_{i\in\underline{p}} \lambda_{\max}(P_i), \\
\lambda_{\max}(Q) &= \max_{i\in\underline{p}} \lambda_{\max}(Q_i), \quad \lambda_{\max}(R) = \max_{i\in\underline{p}} \lambda_{\max}(R_i),
\end{aligned}$$

$$
\begin{aligned}
\lambda_{\max}(Y^\mathsf{T}Y) &= \max_{i\in\underline{p}}\lambda_{\max}(Y_i^\mathsf{T}Y_i), \quad \lambda_1 = [\lambda_{\max}(P)]^{-1},\\
\lambda_2 = &[\lambda_{\min}(P)]^{-1} + \left\{\frac{1-e^{-2\beta h}}{2\beta}\lambda_{\max}(Q) + \frac{1-e^{-2\beta r}}{2\beta}\lambda_{\max}(Y^\mathsf{T}Y)\right.\\
&+\frac{2\beta h + e^{-2\beta h}-1}{4\beta^2}\lambda_{\max}(R)\\
&\left.+\frac{2\beta r + e^{-2\beta r}-1}{4\beta^2}\lambda_{\max}(Y^\mathsf{T}Y)\right\}[\lambda_{\min}(P)]^{-2}.
\end{aligned}
$$

We are now in the position to state our first result as given in the following theorem.

Theorem 3.1. *Given $\beta > 0$, $\gamma > 0$. System (1) is H_∞ stabilizable if there exist symmetric positive definite matrices P_i, Q_i, $R_i \in \mathbb{R}^{n\times n}$, $i \in \underline{p}$, semi-positive definite matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$ and matrices $Y_i \in \mathbb{R}^{m\times n}$, $i \in \underline{p}$, satisfy the following LMIs*

$$\mathcal{M}_{ii} + \mathbb{S}_1 \le 0, \quad i \in \underline{p}, \tag{3a}$$

$$\mathcal{M}_{ij} + \mathcal{M}_{ji} - \frac{2}{p-1}\mathbb{S}_1 \le 0, \quad 1 \le i < j \le p, \tag{3b}$$

$$\mathcal{N}_{ii} + \mathbb{S}_2 \le 0, \quad i \in \underline{p}, \tag{3c}$$

$$\mathcal{N}_{ij} + \mathcal{N}_{ji} - \frac{2}{p-1}\mathbb{S}_2 \le 0, \quad 1 \le i < j \le p. \tag{3d}$$

The stabilizing feedback controller is given by (2) with

$$K(\xi) = \left(\sum_{i=1}^{p}\xi_i Y_i\right)\left(\sum_{i=1}^{p}\xi_i P_i\right)^{-1}. \tag{4}$$

Moreover, every solution $x(t,\phi)$ of the closed-loop system of (1) without disturbance, i.e., $w(t) = 0$, satisfies the following exponential estimate

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\beta t}, \quad t \ge 0.$$

Proof: Since P_i, $i \in \underline{p}$, are symmetric positive definite matrices and $\xi_i \ge 0$, matrix $P = \sum_{i=1}^{p}\xi_i P_i$ is symmetric positive definite, and thus, $\overline{P} = P^{-1}$ is also a symmetric positive definite matrix. Let $\overline{Q} = \overline{P}Q\overline{P}$, $\overline{R} = \overline{P}R\overline{P}$ and $K = Y\overline{P}$. We will prove that system (1) is H_∞ stabilizable by controller $u(t) = K(\xi)x(t)$, where $K(\xi) = Y(\xi)P(\xi)^{-1}$. For this, we construct the following LKF for the closed-loop system of (1)

$$V(x_t) = V_1 + V_2 + V_3 + V_4 + V_5, \tag{5}$$

where

$$
\begin{aligned}
V_1 &= \langle \overline{P}x(t), x(t)\rangle,\\
V_2 &= \int_{t-h(t)}^{t} e^{2\beta(\theta-t)}\langle \overline{Q}x(\theta), x(\theta)\rangle d\theta,\\
V_3 &= \int_{t-h}^{t}\int_{s}^{t} e^{2\beta(\theta-t)}\langle \overline{R}x(\theta), x(\theta)\rangle d\theta ds,\\
V_4 &= \int_{t-r(t)}^{t} e^{2\beta(\theta-t)}\langle K^\mathsf{T}Kx(\theta), x(\theta)\rangle d\theta,\\
V_5 &= \int_{t-r}^{t}\int_{s}^{t} e^{2\beta(\theta-t)}\langle K^\mathsf{T}Kx(\theta), x(\theta)\rangle d\theta ds.
\end{aligned}
$$

It follows from (5) that

$$\lambda_1\|x(t)\|^2 \le V(t,x_t) \le \lambda_2\|x_t\|^2, \quad t \ge 0. \tag{6}$$

Taking derivative of V_1 along solution of the closed-loop system of (1) we obtain

$$\begin{aligned}\dot{V}_1 =& 2\langle \overline{P}[A_0x(t) + A_1x(t-h(t))], x(t)\rangle \\ &+ 2\langle \overline{P}[B_0u(t) + B_1u(t-r(t)) + Cw(t)], x(t)\rangle \\ &+ 2\langle \overline{P}A_2\int_{t-h(t)}^t x(s)ds, x(t)\rangle \\ &+ 2\langle \overline{P}B_2\int_{t-r(t)}^t u(s)ds, x(t)\rangle.\end{aligned}$$

Let $y(t) = \overline{P}x(t)$, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}2\langle A_1x(t-h(t)), y(t)\rangle &\le e^{2\beta h}\mu^{-1}\langle A_1\overline{Q}^{-1}A_1^{\mathsf{T}}y(t), y(t)\rangle \\ &\quad + \mu e^{-2\beta h}\langle \overline{Q}x(t-h(t)), x(t-h(t))\rangle; \\ 2\langle B_1u(t-r(t)), y(t)\rangle &\le e^{2\beta r}\mu^{-1}\langle B_1B_1^{\mathsf{T}}y(t), y(t)\rangle + \mu e^{-2\beta r}\|u(t-r(t))\|^2; \\ 2\left\langle A_2\int_{t-h(t)}^t x(s)ds, y(t)\right\rangle &\le he^{2\beta h}\langle A_2\overline{R}^{-1}A_2^{\mathsf{T}}y(t), y(t)\rangle \\ &\quad + \frac{1}{h}e^{-2\beta h}\left[\int_{t-h(t)}^t x(s)ds\right]^{\mathsf{T}}\overline{R}\left[\int_{t-h(t)}^t x(s)ds\right] \\ &\le he^{2\beta h}\langle A_2\overline{R}^{-1}A_2^{\mathsf{T}}y(t), y(t)\rangle + e^{-2\beta h}\int_{t-h}^t x^{\mathsf{T}}(s)\overline{R}x(s)ds;\end{aligned}$$

and

$$\begin{aligned}2\left\langle B_2\int_{t-r(t)}^t u(s)ds, y(t)\right\rangle &\le re^{2\beta r}\langle B_2B_2^{\mathsf{T}}y(t), y(t)\rangle \\ &\quad + \frac{1}{r}e^{-2\beta r}\left[\int_{t-r(t)}^t u(s)ds\right]^{\mathsf{T}}\left[\int_{t-r(t)}^t u(s)ds\right] \\ &\le re^{2\beta r}\langle B_2B_2^{\mathsf{T}}y(t), y(t)\rangle + e^{-2\beta r}\int_{t-r}^t \|u(s)\|^2ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}_1 \le& \langle (A_0P + PA_0^{\mathsf{T}})y(t), y(t)\rangle \\ &+ e^{2\beta h}\mu^{-1}\langle A_1\overline{Q}^{-1}A_1^{\mathsf{T}}y(t), y(t)\rangle + \langle (B_0Y + YB_0^{\mathsf{T}})y(t), y(t)\rangle \\ &+ he^{2\beta h}\langle A_2\overline{R}^{-1}A_2^{\mathsf{T}}y(t), y(t)\rangle + e^{2\beta r}\mu^{-1}\langle B_1B_1^{\mathsf{T}}y(t), y(t)\rangle \\ &+ re^{2\beta r}\langle B_2B_2^{\mathsf{T}}y(t), y(t)\rangle + 2\langle Cw(t), y(t)\rangle \\ &+ \mu e^{-2\beta h}\langle \overline{Q}x(t-h(t)), x(t-h(t))\rangle \\ &+ \mu e^{-2\beta r}\langle K^{\mathsf{T}}Kx(t-r(t)), x(t-r(t))\rangle \\ &+ e^{-2\beta h}\int_{t-h}^t x^{\mathsf{T}}(s)\overline{R}x(s)ds + e^{-2\beta r}\int_{t-r}^t \langle K^{\mathsf{T}}Kx(s), x(s)\rangle ds.\end{aligned} \tag{7}$$

Next, taking derivative of V_k, $k=2,\dots,5$, give

$$\begin{aligned}
\dot V_2 &\le \langle \overline{Q}x(t),x(t)\rangle - 2\beta V_2 - \mu e^{-2\alpha h}\langle \overline{Q}x(t-h(t)),x(t-h(t))\rangle;\\
\dot V_3 &= h\langle \overline{R}x(t),x(t)\rangle - 2\beta V_3 - e^{-2\beta h}\int_{t-h}^{t}\langle \overline{R}x(s),x(s)\rangle ds;\\
\dot V_4 &\le \langle K^\top Kx(t),x(t)\rangle - 2\beta V_4 - \mu e^{-2\beta r}\langle K^\top Kx(t-r(t)),x(t-r(t))\rangle;\\
\dot V_5 &= r\langle K^\top Kx(t),x(t)\rangle - 2\beta V_5 - e^{-2\beta r}\int_{t-r}^{t}\langle K^\top Kx(s),x(s)\rangle ds.
\end{aligned} \tag{8}$$

From (7) and (8) we readily obtain

$$\dot V(x_t) + 2\beta V(x_t) \le \langle \Xi y(t),y(t)\rangle + 2\langle Cw(t),y(t)\rangle, \tag{9}$$

where $\Xi = \mathcal{A}+\mathcal{B}+2\beta P+Q+hR+\mathcal{H}\mathcal{D}^{-1}\mathcal{H}^\top$ and

$$\begin{aligned}
\mathcal{A} &= A_0P + PA_0^\top, \quad \mathcal{H} = \begin{bmatrix} A_1P & \sqrt{h}A_2P & \sqrt{1+r}Y^\top \end{bmatrix},\\
\mathcal{B} &= B_0Y + Y_0^\top B^\top + e^{2\beta r}(\mu^{-1}B_1B_1^\top + rB_2B_2^\top),\\
\mathcal{D} &= \operatorname{diag}\left\{\mu e^{-2\beta h}Q, e^{-2\beta h}R, I_m\right\}.
\end{aligned}$$

Let $\Gamma = \mathcal{A}+\mathcal{B}+2\beta P+Q+hR$. Using properties $P=\sum_{i=1}^p \xi_iP_i$, $Q=\sum_{i=1}^p \xi_iQ_i$, $R=\sum_{i=1}^p \xi_iR_i$ and $\sum_{i=1}^p \xi_i = 1$, we have

$$\begin{aligned}
\begin{bmatrix} \Gamma & \mathcal{H}\\ * & -\mathcal{D}\end{bmatrix} &= \sum_{i=1}^p \xi_i^2 \begin{bmatrix} \Gamma_{ii} & \mathcal{H}_{ii}\\ * & -\mathcal{D}_i\end{bmatrix} + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\left\{\begin{bmatrix} \Gamma_{ij} & \mathcal{H}_{ij}\\ * & -\mathcal{D}_i\end{bmatrix} + \begin{bmatrix} \Gamma_{ji} & \mathcal{H}_{ji}\\ * & -\mathcal{D}_j\end{bmatrix}\right\}\\
&= \sum_{i=1}^p \xi_i^2\mathcal{M}_{ii} + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j(\mathcal{M}_{ij}+\mathcal{M}_{ji}).
\end{aligned}$$

It follows from (3a) and (3b)

$$\begin{aligned}
\begin{bmatrix} \Gamma & \mathcal{H}\\ * & -\mathcal{D}\end{bmatrix} &\le -\sum_{i=1}^p \xi_i^2\mathbb{S}_1 + \frac{2}{p-1}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\mathbb{S}_1\\
&\le -\frac{1}{p-1}\left[(p-1)\sum_{i=1}^p \xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\right]\mathbb{S}_1.
\end{aligned}$$

Using the fact

$$(p-1)\sum_{i=1}^p \xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j = \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(\xi_i-\xi_j)^2 \ge 0$$

we have $\begin{bmatrix} \Gamma & \mathcal{H}\\ * & -\mathcal{D}\end{bmatrix} \le 0$ and, thus, by Lemma 2.1

$$\Xi = \Gamma + \mathcal{H}\mathcal{D}^{-1}\mathcal{H}^\top \le 0. \tag{10}$$

Therefore, under conditions (3a)-(3b), from (9), (10) we have

$$\dot V(x_t) + 2\beta V(x_t) \le 2\langle Cw(t),y(t)\rangle, \quad t\ge 0.$$

Let $w(t)=0$ then $\dot V(x_t)+2\beta V(x_t)\le 0$ which yields

$$V(x_t) \le V(x_0)e^{-2\beta t}, \quad t\ge 0.$$

Taking (6) into account we obtain

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\beta t}, \quad t \ge 0.$$

This shows that the closed-loop system of (1) is β-exponentially stable. It remains to show the H_∞ disturbance attenuation level, requirement (ii) in Definition 2.1, holds.

Note that

$$2\langle Cw(t), y(t)\rangle \le \gamma^2\|w(t)\|^2 + 1/\gamma^2\langle CC^{\mathsf{T}}y(t), y(t)\rangle, \quad t \ge 0,$$

and $V(x_t) \ge 2\beta\langle Py(t), y(t)\rangle$, it follows from (9)

$$\begin{aligned}\dot V(x_t) &\le -2\beta V(x_t) + 1/\gamma^2\langle CC^{\mathsf{T}}y(t), y(t)\rangle + \gamma^2\|w(t)\|^2\\ &\le \langle(-2\beta P + 1/\gamma^2 CC^{\mathsf{T}})y(t), y(t)\rangle + \gamma^2\|w(t)\|^2.\end{aligned}$$

For any $T > 0$,

$$\begin{aligned}\int_0^T \left[\|z(t)\|^2 - \gamma^2\|w(t)\|^2\right] dt &\le \int_0^T \left[\|z(t)\|^2 - \gamma^2\|w(t)\|^2 + \dot V(x_t)\right] dt + V(x_0)\\ &\le \int_0^T \left[\|z(t)\|^2 + \langle\left(-2\beta P + 1/\gamma^2 CC^{\mathsf{T}}\right) y(t), y(t)\rangle\right] dt\\ &\quad + \lambda_2\|\phi\|^2.\end{aligned}$$

On the other hand,

$$\|z(t)\|^2 = \langle(EP + FY)^{\mathsf{T}}(EP + FY)y(t), y(t)\rangle,$$

and thus,

$$\int_0^T \left[\|z(t)\|^2 - \gamma^2\|w(t)\|^2\right] dt \le \int_0^T \langle\Xi_1 y(t), y(t)\rangle dt + \lambda_2\|\phi\|^2,$$

where $\Xi_1 = -2\beta P + 1/\gamma^2 CC^{\mathsf{T}} + (EP + FY)^{\mathsf{T}}(EP + FY)$. By the same arguments used in deriving (10), from (3c)-(3d), it follows

$$\begin{aligned}\begin{bmatrix} -2\beta P + 1/\gamma^2 CC^{\mathsf{T}} & (EP + FY)^{\mathsf{T}}\\ * & -I_q\end{bmatrix} &= \sum_{i=1}^{p}\xi_i^2\mathcal{N}_{ii} + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\left(\mathcal{N}_{ij} + \mathcal{N}_{ji}\right)\\ &\le -\frac{1}{p-1}\left[(p-1)\sum_{i=1}^{p}\xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\right]\mathbb{S}_2\\ &\le 0.\end{aligned}$$

Applying Lemma 2.1 gives

$$\Xi_1 = -2\beta P + 1/\gamma^2 CC^{\mathsf{T}} + (EP + FY)^{\mathsf{T}}(EP + FY) \le 0$$

which yields

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2\left\{\int_0^T \|w(t)\|^2 dt + \frac{\lambda_2}{\gamma^2}\|\phi\|^2\right\}.$$

Let $T \longrightarrow \infty$ we finally obtain

$$\sup\frac{\int_0^\infty \|z(t)\|^2 dt}{c_0\|\phi\|^2 + \int_0^\infty \|w(t)\|^2 dt} \le \gamma^2$$

for all $w \in L_2([0,\infty), \mathbb{R}^s)$, $w \ne 0$, where $c_0 = \frac{\lambda_2}{\gamma^2}$. This shows that the H_∞ disturbance attenuation level holds. The proof is completed. □

Remark 3.1. *It can be seen in the proof of Theorem 3.1 that conditions* (3a)-(3d) *need not to be strict LMIs. Moreover, matrices S_1, S_2 can be relaxed by taking $S_1 = S_2 = 0$ to get a simpler form for* (3a)-(3d).

Remark 3.2. *The H_∞ performance index can be optimized by the following convex optimization procedure*

$$\min \gamma^2 \quad \text{subject to (3a)-(3d)}$$

and the corresponding parameter-dependent H_∞ controller gain (4) *can be computed.*

Remark 3.3. *As discussed in* [33], *an important feature of Theorem 3.1 is that no matrix variable needs to be fixed for different vertices of the polytope Ω, which can lead to less conservative conditions for the H_∞ stabilization of the system. Furthermore, the proposed conditions in Theorem 3.1 also guarantee an exponential convergence of the closed-loop system of* (1) *with explicit convergent rate which can be prescribed in practical applications.*

Remark 3.4. *In* [35, 36], *some delay-dependent conditions for H_∞ stabilization of polytopic systems with time-varying delays in the state were derived in terms of LMIs with a hard constraint that $E_i^\mathsf{T} F_i = 0$ and $F_i^\mathsf{T} F_i = I$ for any $i \in \underline{p}$. Furthermore, the transformation proposed in* [40] *cannot be used for polytopic systems to achieve this condition. Different from the aforementioned works, in this paper we derive LMIs conditions for the H_∞ control problem of polytopic systems with time-varying distributed delays in state and input without using this technical constraint which makes our conditions less conservative.*

Remark 3.5. *It is worth mentioning that Theorem 3.1 in this paper encompasses Theorem 3 in* [34] *as a special case without imposing any condition. More precisely, for the β-exponential stabilization of linear polytopic systems considered in* [34], *if we let $A_{2i} = 0$, $B_{1i} = B_{2i} = 0, i \in \underline{p}$, and by fixing $Y_i = -\frac{1}{2}B_{0i}, i \in \underline{p}$, then Theorem 3.1 recovers Theorem 3 in* [34].

To further demonstrate the efficiency of this paper, let us consider some special cases of (1). The first one is a class of linear polytopic systems with continuous discrete and distributed delays

$$\dot{x}(t) = A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) + A_2(\xi)\int_{t-r(t)}^{t} x(s)ds. \tag{11}$$

The following corollary gives a criterion for β-exponential stability of (11).

Corollary 3.1. *For given $\beta > 0$, system* (11) *is β-exponentially stable if there exist symmetric positive definite matrices $P_i, Q_i, R_i \in \mathbb{R}^{n\times n}$, $i \in \underline{p}$, and semi-positive definite matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$ satisfy the following LMIs*

$$\widetilde{\mathcal{M}}_{ii} + \mathbb{S}_1 \le 0, \quad i \in \underline{p}, \tag{12a}$$

$$\widetilde{\mathcal{M}}_{ij} + \widetilde{\mathcal{M}}_{ji} - \frac{2}{p-1}\mathbb{S}_1 \le 0, \quad 1 \le i < j \le p, \tag{12b}$$

where $\widetilde{\mathcal{M}}_{ij} = \begin{bmatrix} \widetilde{\Gamma}_{ij} & P_jA_{1i} & \sqrt{h}P_jA_{2i} \\ A_{1i}^\mathsf{T}P_j & -\mu e^{-2\beta h}Q_j & 0 \\ A_{2i}^\mathsf{T}P_j & 0 & -e^{-2\beta r}R_j \end{bmatrix}$, $\widetilde{\Gamma}_{ij} = A_{0i}^\mathsf{T}P_j + P_jA_{0i} + 2\beta P_j + Q_j + hR_j$.

Moreover, every solution $x(t,\phi)$ of (11) *satisfies*

$$\|x(t,\phi)\| \le \sqrt{\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}}\|\phi\|e^{-\beta t}, \quad t \ge 0,$$

where $\tilde{\lambda}_1 = \lambda_{\min}(P)$, $\tilde{\lambda}_2 = \lambda_{\max}(P) + \frac{1-e^{-2\beta h}}{2\beta}\lambda_{\max}(Q) + \frac{2\beta h + e^{-2\beta h}-1}{4\beta^2}\lambda_{\max}(R)$.

Remark 3.6. *A special class of* (11)*, where* $A_{2i} = 0$*,* $i \in \underline{p}$*, was considered in the existing works, for example,* [33, 34]*. It is worth pointing out that: (i) Corollary 3.1 covers Theorem 1 in* [34] *as a special case; and (ii) exponential stability condition given in Corollary 3.1 is derived directly from the LKF without using any free-weighting matrix as proposed in* [33]*. Although free-weighting matrix approach can lead to less conservative stability conditions, it increases significantly the computational cost.*

The second special class, when $p = 1$, (1) reduces to linear system with mixed time-varying delays of the form

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-h(t)) + A_2\displaystyle\int_{t-r(t)}^{t} x(s)ds + Cw(t) \\ \qquad\qquad + B_0u(t) + B_1u(t-h(t)) + B_2\displaystyle\int_{t-r(t)}^{t} u(s)ds, \quad t \geq 0, \\ z(t) = Ex(t) + Fu(t), \\ x(t) = \phi(t), \quad t \in [-\overline{h}, 0], \end{cases} \tag{13}$$

where A_k, B_k, $k = 0, 1, 2$, and C, E, F are known constant matrices.

The obtained result in Theorem 3.1 leads to conditions for the H_∞ stabilization of system (13) as presented in the following corollary.

Corollary 3.2. *For given* $\beta > 0$*,* $\gamma > 0$*, system* (13) *is* H_∞ *stabilizable if there exist symmetric positive definite matrices* P*,* Q*,* R *and a matrix* Y *satisfying the following LMIs*

$$\begin{bmatrix} \Pi & A_1P & \sqrt{h}A_2P & \sqrt{1+r}Y^\top \\ * & -\mu e^{-2\beta h}Q & 0 & 0 \\ * & * & -e^{-2\beta r}R & 0 \\ * & * & * & -I_m \end{bmatrix} \leq 0, \tag{14a}$$

$$\begin{bmatrix} -2\beta P + 1/\gamma^2 CC^\top & PE^\top + Y^\top F^\top \\ * & -I_q \end{bmatrix} \leq 0, \tag{14b}$$

where $\Pi = A_0P + PA_0^\top + B_0Y + Y^\top B_0^\top + e^{2\beta r}\left(\mu^{-1}B_1B_1^\top + rB_2B_2^\top\right) + 2\beta P + Q + hR.$

The H_∞ *stabilizing controller is given by*

$$u(t) = YP^{-1}x(t), \quad t \geq 0.$$

Likewise, the H_∞ disturbance attenuation bound γ can be optimized by the following convex optimization procedure

$$\min\ \gamma^2 \quad \text{subject to (14a) and (14b).}$$

Then, the corresponding H_∞ controller gain is defined by $K = YP^{-1}$.

4. Application Examples.

In this section, we present some examples to demonstrate the effectiveness of the results obtained in this paper. The first example is to verify our conditions for general systems in the form of (1). The next few examples are some applications of our results to practical systems. A comparative example is also provided.

Example 4.1. *Consider a three vertices polytopic system* (1)*, where,*

$$A_{01} = \begin{bmatrix} -10 & 4 \\ 1 & -6 \end{bmatrix}, A_{02} = \begin{bmatrix} -9 & 1 \\ 2 & -10 \end{bmatrix}, A_{03} = \begin{bmatrix} -10 & 5 \\ 3 & -11 \end{bmatrix}, A_{11} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, A_{13} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, A_{21} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, B_{02} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, B_{03} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}, B_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{13} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{23} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, F_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, F_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $h(t) = \sin^2(0.5t)$, $r(t) = \cos^2(0.5t)$, $t \geq 0$.

It should be noted that the matching conditions, $E_i^\mathsf{T} F_i = 0$, $F_i^\mathsf{T} F_i = I$, $i = 1, 2, 3$, *are not satisfied. Therefore, the* H_∞ *stabilization conditions proposed in* [35, 36] *are not applicable to this system. In this case, we have* $h = r = 1$ *and* $\delta = 0.5$. *We take* $\beta = 1$, $\gamma = 1$, *using Matlab LMI toolbox, from* (3a)-(3d) *we obtain the following gain matrices for the feedback controller (some other matrices are omitted here)*

$$P_1 = \begin{bmatrix} 3.0516 & -0.8463 \\ -0.8463 & 2.8189 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 5.3385 & -1.6351 \\ -1.6351 & 7.4512 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1.6639 & -0.7443 \\ -0.7443 & 4.5751 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} -1.4295 & -3.1380 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -0.8114 & -5.6834 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -0.4284 & -1.6267 \end{bmatrix}.$$

Therefore, by Theorem 3.1, the system is H_∞ *stabilizable. To obtain the gain matrix,* $K(\xi)$, *we compute*

$$P = \sum_{i=1}^{3} \xi_i P_i = \begin{bmatrix} p_{11} & p_{12} \\ * & p_{22} \end{bmatrix}, \quad Y = \sum_{i=1}^{3} \xi_i Y_i = \begin{bmatrix} y_1 & y_2 \end{bmatrix},$$

where

$$\begin{aligned}
p_{11} &= 3.0516\xi_1 + 5.3385\xi_2 + 1.6639\xi_3, \\
p_{12} &= -0.8463\xi_1 - 1.6351\xi_2 - 0.7443\xi_3, \\
p_{22} &= 2.8189\xi_1 + 7.4512\xi_2 + 4.5751\xi_3, \\
y_1 &= -1.4295\xi_1 - 0.8114\xi_2 - 0.4284\xi_3, \\
y_2 &= -3.1380\xi_1 - 5.6834\xi_2 - 1.6267\xi_3,
\end{aligned}$$

and, thus

$$\det(P) = 7.8859\xi_1^2 + 37.1048\xi_2^2 + 7.0585\xi_3^2 + 35.0192\xi_1\xi_2 + 17.3919\xi_1\xi_3 + 34.3882\xi_2\xi_3.$$

The stabilizing feedback controller is given by

$$u(t) = \frac{1}{\det(P)} \begin{bmatrix} p_{22}y_1 - p_{12}y_2 & p_{11}y_2 - p_{12}y_1 \end{bmatrix} x(t).$$

In addition, it can be found that the minimum guaranteed closed-loop H_∞ *performance index* $\gamma_{\min}$ *is 0.04561.*

For $\beta = 0.5$, $\gamma = 1$, *the upper bounds* $h_{\max}$ *and* $r_{\max}$ *of* $h(t)$ *and* $r(t)$ *for different values of* δ *are given in Table 1 and Table 2, respectively.*

In the next example, we consider a mechanical model to illustrate the applicability of the theoretical results developed in this paper.

TABLE 1. Upper bounds of $h(t)$ with $\beta = 0.5$, $\gamma = 1$, $r = 0.5$

δ	0	0.1	0.3	0.5	0.7	0.9
$h_{\max}$	2.3768	2.3479	2.2712	2.1492	1.9110	0.9882

TABLE 2. Upper bounds of $r(t)$ with $\beta = 0.5$, $\gamma = 1$, $h = 0.5$

δ	0	0.1	0.3	0.5	0.7	0.9
$r_{\max}$	2.1395	2.1329	2.1081	2.0167	1.7890	0.9128

Example 4.2. *Consider an inverted pendulum in Figure 1. The cart motor is travelling on a plane under the force F induced by a control. M and m denote the cart mass and the blob mass, respectively; x is the distance traveled by the cart; θ is the angle of the blob from the vertical and ℓ is length of the massless rigid connector of the pendulum. The equations of motion (EqM) are as follows*

$$\begin{cases} m\ell\ddot{\theta} - mg\sin\theta + m\ddot{x}\cos\theta = 0 \\ (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = F. \end{cases} \tag{15}$$

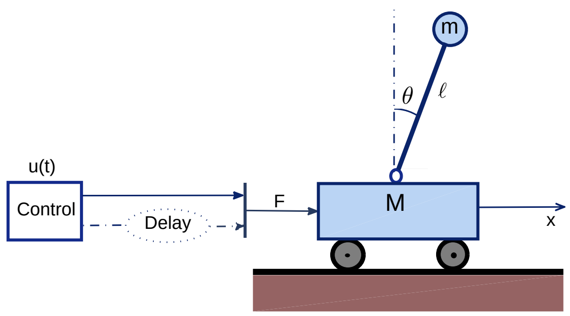

FIGURE 1. Inverted pendulum control

A commonly used approach in analyzing behavior of practical systems is the linear approximation, see, for example, [15, 18]*. An approximation of* (15)*, in regard to uncertainties, can be described by the following system*

$$\begin{cases} \ell\ddot{\theta} - g(1+\rho)\theta + \ddot{x} = 0 \\ (M+m)\ddot{x} + m\ell\ddot{\theta} - m\ell w = F, \end{cases} \tag{16}$$

where $w = \dot{\theta}^2\sin\theta$ denotes the disturbance input and ρ is a scalar parameter involving errors. We choose the set of variables $x = [x_1\ x_2\ x_2\ x_4]^{\mathsf{T}} = [x\ \dot{x}\ \theta\ \dot{\theta}]^{\mathsf{T}}$ then (16) *can be written as*

$$\dot{x}(t) = A_\rho x(t) + BF(t) + Cw(t), \tag{17}$$

where

$$A_\rho = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M}(1+\rho) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell}(1+\rho) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ \frac{m\ell}{M} \\ 0 \\ -\frac{m}{M} \end{bmatrix}.$$

Couple with (17) *we consider the output $z(t) = E_\rho x(t)$, where $E_\rho = \begin{bmatrix} 1+\rho & 0 & 1+\rho & 0 \end{bmatrix}$. For illustrative purpose, we consider the following two cases of delayed signal force*

(I) $F(t) = 0.7(1+\rho)u(t) + 0.3(1+\rho)u(t-h)$ *and*
(II) $F(t) = 0.7(1+\rho)u(t) + 0.3(1+\rho)\int_{t-r}^{t} u(s)ds$.

Let $|\rho| \le \overline{\rho}$. *We present* (17) *in the form of two-vertices polytopic system which was shown to be better than a norm-bounded representation* [30, 32] *as follows*

$$\begin{cases} \dot{x}(t) = A_0(\xi)x(t) + B_0(\xi)u(t) + B_1(\xi)u(t-h) + B_2(\xi)\int_{t-r}^{t} u(s)ds + C(\xi)w(t), \\ z(t) = E(\xi)x(t), \end{cases} \tag{18}$$

where

$$A_{01} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M}(1-\overline{\rho}) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell}(1-\overline{\rho}) & 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M}(1+\overline{\rho}) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell}(1+\overline{\rho}) & 0 \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} 0 & \frac{0.7(1-\overline{\rho})}{M} & 0 & -\frac{0.7(1-\overline{\rho})}{M\ell} \end{bmatrix}^{\mathsf{T}}, \quad B_{02} = \begin{bmatrix} 0 & \frac{0.7(1+\overline{\rho})}{M} & 0 & -\frac{0.7(1+\overline{\rho})}{M\ell} \end{bmatrix}^{\mathsf{T}},$$

$$C_1 = C_2 = C, \quad E_1 = \begin{bmatrix} 1-\overline{\rho} & 0 & 1-\overline{\rho} & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1+\overline{\rho} & 0 & 1+\overline{\rho} & 0 \end{bmatrix},$$

and $B_{11} = 3/7B_{01}$, $B_{12} = 3/7B_{02}$, $B_{21} = B_{22} = 0$ *for case* (I), $B_{11} = B_{12} = 0$, $B_{21} = 3/7B_{01}$, $B_{22} = 3/7B_{02}$ *for case* (II).

In the simulation, some parameters are listed in Table 3.

TABLE 3. Parameters in the simulation

g	ℓ	m	M	$\overline{\rho}$	β
9.8 m/s^2	0.5 m	0.1 kg	2 kg	0.1	0.1

Note that $A_0(\xi) = \xi_1 A_{01} + \xi_2 A_{02} = A_\rho$, *where* $\rho = (\xi_2 - \xi_1)\overline{\rho}$, $\xi_1, \xi_2 \ge 0$, $\xi_1 + \xi_2 = 1$. *By direct computation,* $\det(\lambda I_4 - A_\rho) = \lambda^2 \left[\lambda^2 - \frac{(M+m)g}{M\ell}(1+\rho)\right]$. *For* $\overline{\rho} < 1$, *then* $1 + \rho > 0$ *for all* $|\rho| \le \overline{\rho}$. *Therefore,*

$$\lambda(A_\rho) = \left\{0,\ 0,\ -\sqrt{\frac{(M+m)g}{M\ell}(1+\rho)},\ \sqrt{\frac{(M+m)g}{M\ell}(1+\rho)}\right\}.$$

This shows that the open system of (18) *is unstable. It is worth noting that, for both two cases* (I) *and* (II), *apply the proposed controller design in this paper, it is found that system* (18) *is* H_∞ *stabilizable with exponential convergence.*

Case (I): *For* $h = 1$, *it can be found that the minimum guaranteed closed-loop* H_∞ *performance index* $\gamma_{\min}$ *is* 0.3781. *Solving* (3a)-(3d) *gives*

$$P_1^I = \begin{bmatrix} 0.4171 & -0.1769 & -0.2718 & 0.3898 \\ -0.1769 & 0.1546 & 0.1828 & -0.3301 \\ -0.2718 & 0.1828 & 0.3733 & -0.5340 \\ 0.3898 & -0.3301 & -0.5340 & 0.8690 \end{bmatrix},$$

$$P_2^I = \begin{bmatrix} 0.2461 & -0.1619 & -0.2334 & 0.3677 \\ -0.1619 & 0.1661 & 0.1696 & -0.3423 \\ -0.2334 & 0.1696 & 0.3434 & -0.5093 \\ 0.3677 & -0.3423 & -0.5093 & 0.8841 \end{bmatrix},$$

$$Y_1^I = \begin{bmatrix} -0.1702 & -0.1645 & 0.3263 & 0.1479 \end{bmatrix}, \ Y_2^I = \begin{bmatrix} -0.0564 & -0.2743 & 0.1515 & 0.4336 \end{bmatrix}.$$

The H_∞ *stabilizing controller with disturbance attenuation* $\gamma_{\min} = 0.3781$ *is given by* $u(t) = K^I(\xi)x(t)$, *where*

$$K^I(\xi) = \left(\xi_1 Y_1^I + \xi_2 Y_2^I\right)\left(\xi_1 P_1^I + \xi_2 P_2^I\right)^{-1}, \ \xi_1 \ge 0,\ \xi_2 \ge 0,\ \xi_1 + \xi_2 = 1.$$

Case (II): *For $r = 1$, it is found that the minimum guaranteed closed-loop H_∞ performance index $\gamma_{\min}$ is* 0.6368. *In this case we have*

$$P_1^{II} = \begin{bmatrix} 0.3361 & -0.1083 & -0.1664 & 0.2356 \\ -0.1083 & 0.0739 & 0.1057 & -0.1730 \\ -0.1664 & 0.1057 & 0.2437 & -0.3422 \\ 0.2356 & -0.1730 & -0.3422 & 0.5168 \end{bmatrix},$$

$$P_2^{II} = \begin{bmatrix} 0.1089 & -0.1005 & -0.1542 & 0.2389 \\ -0.1005 & 0.0834 & 0.1097 & -0.1940 \\ -0.1542 & 0.1097 & 0.2461 & -0.3614 \\ 0.2389 & -0.1940 & -0.3614 & 0.5748 \end{bmatrix},$$

$$Y_1^{II} = \begin{bmatrix} -0.1220 & -0.0548 & 0.2517 & -0.0449 \end{bmatrix},$$
$$Y_2^{II} = \begin{bmatrix} -0.0583 & -0.1097 & 0.1508 & 0.1068 \end{bmatrix}.$$

The H_∞ stabilizing controller with disturbance attenuation $\gamma_{\min} = 0.6368$ is given by $u(t) = K^{II}(\xi)x(t)$, where

$$K^{II}(\xi) = \left(\xi_1 Y_1^{II} + \xi_2 Y_2^{II}\right)\left(\xi_1 P_1^{II} + \xi_2 P_2^{II}\right)^{-1}, \ \xi_1 \geq 0, \ \xi_2 \geq 0, \ \xi_1 + \xi_2 = 1.$$

In the following example, we are interested in applying our theoretical results to a crane model as considered in [15].

Example 4.3. *Consider a hanging crane structure as in Figure 2. The cart motor and the hoist motor are powerful to drive the cart to the destination and keep the payload angle steady. M and m denote the cart mass and the payload mass, respectively; x is the distance traveled by the cart; θ is the angle of the payload from the vertical; F is a force to pull the cart; the massless rigid connector has length ℓ. The right is the positive direction of the force and the displacement. We choose the set of variables $x = [x_1\ x_2\ x_3\ x_4]^\mathsf{T} = [x\ \theta\ \dot{x}\ \dot{\theta}]^\mathsf{T}$.*

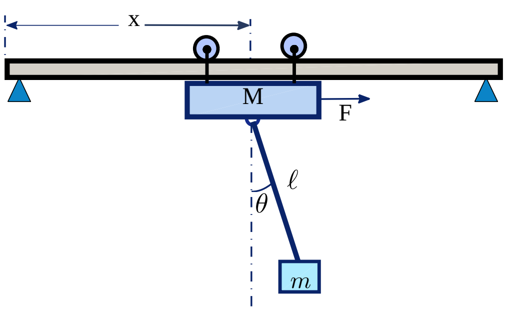

FIGURE 2. Crane structure

The state equations of the crane system are as follows, see [15] *for details*

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \tag{19}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mgr^2}{Mr^2+J} & -\frac{k_e k_t}{R_a(Mr^2+J)} & 0 \\ 0 & -\frac{\left((M+m)r^2+J\right)g}{(Mr^2+J)\ell} & -\frac{k_e k_t}{R_a(Mr^2+J)\ell} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{rk_t}{R_a(Mr^2+J)} \\ \frac{rk_t}{R_a(Mr^2+J)\ell} \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0.05 \\ 0.1 \end{bmatrix}.$$

The following parameters are borrowed from [15]*: Motor load* $J = 0.0001\mathrm{kgm}^2$*, back-EMF coefficient* $K_e = 0.4758\mathrm{Vs}$*, armature resistance* $R_a = 13.5\Omega$*, moment coefficient* $K_t = 0.0491\mathrm{kgm/A}$*, radius of the transport wheel* $r = 0.0227\mathrm{m}$*,* $m = 0.3\mathrm{kg}$*,* $M = 0.4\mathrm{kg}$ *and* $\ell = 0.205\mathrm{m}$*. Then we have*

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9575 & -5.6330 & 0 \\ 0 & -71.9877 & -27.4781 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.2695 \\ 1.3144 \end{bmatrix}.$$

The output variables are x *and* θ*, and thus,* $z(t) = Ex(t)$*, where* $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

In [15]*, using the network-based output feedback approach, an observer-based controller was designed to ensure that system* (19) *under stochastic disturbance is robustly asymptotically stable in the mean square with an* H_∞ *disturbance attenuation level* γ*. It was found that the minimum guaranteed* H_∞ *performance index is* $\gamma^* = 0.4510$*.*

We let $\beta = 0.1$ *(the exponential convergence rate of the closed-loop system), and apply the proposed controller design in Corollary 3.2. System* (19) *is* H_∞ *stablizable with a minimum guaranteed* H_∞ *performance index* $\gamma_{\min} = 0.2644$*. For* $\gamma = \gamma_{\min}$*, by Corollary 3.2, we obtain*

$$P = \begin{bmatrix} 0.0904 & 0.0524 & -0.3205 & -0.4318 \\ 0.0524 & 0.1750 & -0.2196 & -0.3472 \\ -0.3205 & -0.2196 & 1.3630 & 2.5935 \\ -0.4318 & -0.3472 & 2.5935 & 14.3937 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.0074 & -0.0076 & -0.2362 & -1.2253 \end{bmatrix},$$

and thus, the gain matrix controller is given by

$$K = YP^{-1} = \begin{bmatrix} -0.0074 & -0.0076 & -0.2362 & -1.2253 \end{bmatrix}.$$

The last example is to compare H_∞ stabilization conditions proposed in this paper to those reported in [36].

Example 4.4. *Consider the following three vertices polytopic system with state delay*

$$\begin{cases} \dot{x}(t) = A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) + B(\xi)u(t) + C(\xi)w(t), \\ z(t) = E(\xi)x(t) + F(\xi)u(t), \end{cases} \tag{20}$$

where

$$A_{01} = \begin{bmatrix} -10 & 4 \\ 1 & -6 \end{bmatrix}, \ A_{02} = \begin{bmatrix} -9 & 1 \\ 2 & -10 \end{bmatrix}, \ A_{03} = \begin{bmatrix} -10 & 5 \\ 3 & -11 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \ A_{13} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \ B_2 = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \ B_3 = \begin{bmatrix} 12 \\ 2 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \ E_i = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ F_i = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \ i = 1, 2, 3.$$

We take $\beta = 0.5$*,* $\gamma = 1$*. Table 4 shows a comparison of the upper bound* $h_{\max}$ *in terms of the feasibility obtained by the method proposed in* [36] *and by our method.*

It can be seen that the upper bounds $h_{\max}$ *of* $h(t)$ *obtained in this paper are larger than those in* [36]*. This shows that our conditions are less conservative than those in* [36]*.*

TABLE 4. Upper bounds of $h(t)$ with $\beta = 0.5$, $\gamma = 1$

δ	0	0.1	0.3	0.5	0.7	0.9	0.95	0.99
Our method	2.3095	2.2415	2.0818	1.8740	1.5718	0.9783	0.6554	0.0384
[36]	1.5971	1.4918	1.2405	0.9040	0.3932	–	–	–

5. **Conclusion.** In this paper, the problem of H_∞ control for a class of linear systems with polytopic uncertainties and time-varying discrete and distributed delays in state and input has been studied. By using an improved parameter-dependent Lyapunov-Krasovskii functional, new delay-dependent conditions have been derived for designing a state feedback controller guaranteeing the robust exponential stabilization of the system with an H_∞ performance. Some practical and comparative examples have been presented to show the effectiveness and applicability of the theoretical results obtained in this paper.

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