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RESEARCH ON THE INVESTMENT PROPERTIES OF THE INTRINSIC VALUE FUNCTION DETERMINED BY GENERAL CONTINUOUS CASH FLOW

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Abstract. *This paper examines the change rules of the intrinsic value over time from the aspect of function, which in fact is the intrinsic value acquired through the discounted cash flow model in every moment. This paper focuses on the general cash flow that has no limitation to plus or minus, deduce the formula of the intrinsic value function of assets from the continuous yield equation of the rate of return, discuss the relationship between the intrinsic value function of assets and discount rate, explains the abnormal relationship between the intrinsic value function and discount rate that may exist in the general cash flow through some examples, makes an elaborate analysis into the situation that the intrinsic value function equals zero, puts forward the concept and the definition formula of the duration function determined by the continuous cash flow on the basis of the intrinsic value function, gains the differential equation that connects the duration function to the intrinsic value function, and explores the investment meaning when the duration function equals to several particular values. In addition, this paper also adopts the concept of the high-order duration and gets the Taylor series of the intrinsic value function that is represented by the high-order duration.*

Keywords: Discounted cash flow model, Cash earnings function, Intrinsic value function, Duration, Duration function

1. Introduction.

The discounted cash flow model is the most important mathematical model that is used to evaluate the value of asset in the investment analysis. Accordingly, the dividend discount model is used to evaluate the stock value and the general discounted free cash flow of equity model is used to evaluate the values of bonds and other financial assets. The future cash revenue of assets, the asset price in the final moment, and the future level of the discount rate are the three important factors determining the intrinsic value of assets in the present moment. Using the discounted cash flow model, people can easily carry on the reasonable assessment of the asset value and study the quantitative relationship between the changes in interest rate and the changes of the intrinsic value of the asset. People can also regard the market price of the asset as its intrinsic value, and use the discounted cash flow model to solve the corresponding internal rate of return. Comparing the calculated internal rate of return and the market interest rate and considering the risk premium factors required by investors for risky assets, people can assess whether asset pricing by the market are overvalued or undervalued and guide the investment action of investors.

In general, commercial banks or institutional investors have a lot of assets and each asset has a special cash flow. Some of the cash flows arrive early, while some arrive late, some take a year, a half of year or a quarter as the unit of time, and some take a months, a week, or a day as the unit of time. If those cash flows get together, the total cash flow can be approximately described in a continuous function that takes time as the variable. Liabilities of commercial Banks or institutional investors can also be analyzed in the same way. For every enterprise, because of the uninterrupted production process and the uninterrupted marketing process, and the constant capital financing and the debt repayment activities, enterprises' monetary income flows and expenditure flows can be approximately described in some continuous function respectively. If a company's future monetary income flows (with plus sign) and expenditure flows (with minus sign) are combined, the following description can be concluded: in certain period of time of the future, the total monetary flows takes plus (embodied as net income); in some other periods of time, the total monetary flows takes minus (embodied as net expenditure); at some points of time, the total monetary flows takes zero (embodied that income and expenditure are equal). At this time, the continuous cash flow represented by a continuous function has no limitation to plus or minus. According to the above analysis, it is concluded that researching the investment properties of the general continuous cash flow is not just the theoretical research interest, and that it has a broad practical application background.

The discounted cash flow model can be further discussed in some aspects. The discrete discounted cash flow model is familiar to financial experts, while the continuous discounted cash flow model has not been systematically studied. It needs people to define the continuous cash earnings function and deduce the formula of the continuous discounted cash flow. It is ignored by financial experts that for the cash earnings in the future period, people use the discounted cash flow model to examine the intrinsic value of different time in the future, and use the idea and method of function to study various properties of the intrinsic value function consisting of all intrinsic values of assets of different time. The author of this paper has obtained the intrinsic value function formula of the integral form for the non-negative continuous cash flow in [7], systematically studies the various properties of the intrinsic value function, and puts forward the application pattern of the intrinsic value function in the aspect of the value assessment of risk assets. The author of this paper has also studied the duration properties for the non-negative continuous cash flow in [8]. On the basis of existing researches, this paper discusses the various properties of the intrinsic value function to which the general continuous cash flow corresponds, the income flow of holding assets when the cash flow is plus, and the expenditure flow of holding liabilities when the cash flow is minus. Because the discounted cash flow model and the duration model that are existing implicitly assumes that the sign of the future cash flow is changeless, some new problems will come up when there is no limitation to the sign of cash earnings function and it is completely necessary to discuss the meaning of investment of those problems. Based on the general cash earnings functions, this paper researches on the intrinsic value function and the duration function of assets, systematically discusses the relationship among the intrinsic value function and the duration function and the discount rate, these research works do have some theoretical value and application value. The calculus tools are adopted in the deduction of those formulas and the proof of theorems and this research has a clear background of investment.

In Section 2, the continuous yield equation of assets is deduced from the definition formula of the rate of return of assets for general continuous cash flow. The mathematical formula of the intrinsic value function of assets is acquired by means of the continuous yield equation, including the indefinite period and the definite period. The relationship between the intrinsic value and the discount rate of assets is discussed in some simple case. In Section 3, on the basis of the comparison of the intrinsic value function and the market price of assets, some principles of judging the feasibility of investment are put forward. In Section 4, the relationship between the discount rate and the intrinsic value function of assets that is determined by the general cash flow is discussed, and proposed the concepts of the abnormal interval of asset and the abnormal interval of liability. In Section 5, we discuss the properties of assets on the condition that the intrinsic value function takes the value of zero. In Section 6, based on the concept of the intrinsic value function, the definition of the duration function is given, the fact is illustrated that the duration function can take any real-value, and some intrinsic value functions and some duration functions are given to which some frequently-used cash earnings functions correspond. In Section 7, several differential equations that the duration function and intrinsic value function of assets conform are obtained, and the meaning of investment on the condition that the duration function takes zero and negative value are acquired. In Section 8, we adopt the concept of the high-order duration and get the Taylor series of the intrinsic value function which is represented by the high-order duration. The last part is the conclusion of this paper.

2. **The Intrinsic Value Function of Assets Determined by the Yield Equation for General Continuous Cash Flow.** Suppose there is an asset A whose cash revenue is continuously achieved. $C(t)$ is used to represent the cash revenue of the asset per unit time in the moment t, and $C(t)$ is called the cash earnings function or the cash flow. In this paper, the asset A should be understood as the asset in a broader sense, because $C(t)$ is only required to be a continuous function of real-values and there is no limitation to its plus and minus. For the traditional asset, it is required that there is scheduled income cash flow in the given period of time in future, that is $C(t) \geq 0$. For the traditional liability, it is required that there is scheduled expenditure cash flow in the given period of time in future, that is $C(t) \leq 0$. The asset corresponds to the future monetary income flow and the liability corresponds to the future monetary expenditure flow. In order to state conveniently, when the cash flow $C(t)$ has no limitation to the plus or minus, the corresponding financial tool is also called an asset, which is one in a broader sense. Because $C(t) < 0$ means that there is monetary expenditure in the moment t, the saying that $C(t)$ is the cash earnings function is also made in a broader sense. When $C(t) \geq 0$, it indicates the general revenue. When $C(t) < 0$, it indicates the revenue in a broader sense, and it is the expenditure flow and the minus revenue.

One of the real background factors of the cash earnings function $C(t)$ having no limitation to plus and minus is to combine several income flows and expenditure flows into a total cash flow. At this time, $C(t)$ cannot be limited by plus-minus and $C(t)$ can be represented by a general continuous function. Another background factor is that the properties of the revenue of many derivative financial instruments are embodied in the fact that they take positive values in some time and negative values in some other time. In addition, there is another situation that the future revenue of a complicated investment portfolio can be represented by general continuous function, but not the cash earnings function with changeless sign.

Let $P(t)$ refer to the price of asset in moment t, and have no limitation to plus and minus. The typical example of the positive intrinsic value is the discount value of the future income cash flow that assets correspond to, while the typical example of the negative intrinsic value is the discount value of the future expenditure cash flow that liabilities correspond to. According to the definition of the cash earnings function $C(t)$, the cash revenue acquired in the short time interval $[t, t + \Delta t]$ approximately equals to $C(t)\Delta t$. If $r(t)$ is used to represent the instantaneous rate of return at time t, the rate of return in the time interval $[t, t + \Delta t]$ is $r(t)\Delta t$. When the discounted cash flow model is studied, $r(t)$ is the discount rate and $r(t) > 0$. For the time interval $[t, t + \Delta t]$, the following formula of the rate of return is workable:

$$r(t)\Delta t = \frac{C(t)\Delta t + [P(t+\Delta t) - P(t)]}{P(t)}$$

If the numerator of right side of the above formula takes negative value, it is infallible that $P(t) < 0$ because $r(t) > 0$. The plus-minus problem of $P(t)$ will be discussed in detail later. $P(t+\Delta t) - P(t)$ is the capital gain in the time interval $[t, t+\Delta t]$. When $\Delta t \to 0$, we obtain the following differential equation:

$$\frac{dP(t)}{dt} = r(t)P(t) - C(t) \tag{1}$$

Equation (1) is called the continuous yield equation and it is a differential equation about the asset price $P(t)$. For the given discount rate function $r(t)$, if the cash earnings function is $C(t)$, it is feasible to solve the asset price function $P(t)$ through Equation (1). Under some conditions, the asset price function $P(t)$ conforming to Equation (1) is a fair and reasonable price and it should be the intrinsic value function of the asset A. Likewise, if the intrinsic value function of assets $P(t)$ is known, it is feasible to solve the cash earnings function $C(t)$ that the asset A should achieve through Equation (1).

Next, we deduce the mathematic formula of the intrinsic value function $P(t)$ from Equation (1). The situation of the infinite time is first considered, that is, the cash earnings function $C(t)$, $t \in [0, +\infty)$. Suppose when $t \in [0, +\infty)$, $r(t) > 0$ on the basis of the real background of the investment. Equation (1) is a first-order differential equation. For the general first-order differential equation $y' + p(x)\,y = q(x)$, its general solution is:

$$y = e^{-\int p(x)dx}\left[\int q(x)\, e^{\int p(x)dx} dx + c\right]$$

The independent variable of Equation (1) is the time parameter t. Accordingly, $p(t) = -r(t)$, $q(t) = -C(t)$. $R(t)$ is used to represent the primitive function of $r(t)$, that is, $R'(t) = r(t)$. $R(t)$ is substituted into the expression of the general solution:

$$P(t) = e^{R(t)}\left[\int_0^t -C(\xi)\, e^{-R(\xi)} d\xi + C_0\right] \tag{2}$$

In expression (2), C_0 is a constant. Let $t = 0$, it is derived from the above expression that $C_0 = P(0)$, that is,

$$P(t) = e^{R(t)}\left[\int_0^t -C(\xi)\, e^{-R(\xi)} d\xi + P(0)\right]$$

The reasonable assumption is that $0 < r(t) < 1$, and the solution $P(t)$ can be acquired on if the value of $P(0)$ can be determined. Only if the increasing speed of $|C(t)|$ is much smaller than $e^{R(t)}$ when $t \to +\infty$, the integral $\int_0^\infty C(\xi)\, e^{-R(\xi)} d\xi$ must show its convergence. Suppose the integral $\int_0^\infty C(\xi)\, e^{-R(\xi)} d\xi$ exists. If $P(0) = \int_0^\infty C(\xi)\, e^{-R(\xi)} d\xi + C$ in which C is an undetermined constant. The expression of $P(0)$ is substituted into the expression of $P(t)$ and it is gained that:

$$\begin{aligned} P(t) &= e^{R(t)}\left[\int_0^{+\infty} C(\xi)\, e^{-R(\xi)} d\xi - \int_0^t C(\xi)\, e^{-R(\xi)} d\xi + C\right] \\ &= \int_t^{+\infty} e^{-(R(\xi)-R(t))} C(\xi)\, d\xi + e^{R(t)} C \end{aligned}$$

When $C(t) \equiv 0$, $P(t)$ should be equal to 0 and we have $P(t) = e^{R(t)} \cdot C$, so $C = 0$. In this way, the solution $P(t)$ of the differential Equation (1) conforming to the meaning of investment is acquired. Obviously, $P(t)$ contains the discount rate function $r(t)$. $P(t, r(t))$

is used to represent the solution of Equation (1) and the intrinsic value function $P(t, r(t))$ of the asset A is given by the following formula:

$$P(t, r(t)) = \int_t^{+\infty} e^{-(R(\xi)-R(t))} C(\xi)\, d\xi \tag{3}$$

The expression (3) is the formula of the intrinsic value function of assets for general continuous cash flow. Especially when $t = 0$, we have:

$$P(0, r(0)) = e^{R(0)} \int_0^{+\infty} C(\xi)\, e^{-R(\xi)} d\xi$$

This is the discount formula of the intrinsic value of assets that the continuous cash flow corresponds to.

For the fixed discount rate r, $r(t) = r$, $R(t) = \int_0^t r(\xi) d\xi = rt$, $R(0) = 0$, the following mathematic formula of the intrinsic value function is workable:

$$P(t, r) = \int_t^{+\infty} e^{-r(\xi - t)} C(\xi)\, d\xi \tag{4}$$

Especially when $t = 0$, we have:

$$P(0, r) = \int_0^{+\infty} e^{-r\xi} C(\xi)\, d\xi$$

This is the discount formula for general continuous cash flow when $r(t) = r$.

Now, the situation of finite time limit is considered, that is, $t \in [0, T]$. Let the C_0 in the expression (2) of the general solution be:

$$C_0 = \int_0^T C(\xi)\, e^{-R(\xi)} d\xi + C$$

C is an undetermined constant. Substituting the above expression into the expression (2) of the general solution, it is gained that:

$$\begin{aligned} P(t) &= e^{R(t)} \left[\int_0^T C(\xi)\, e^{-R(\xi)} d\xi - \int_0^t C(\xi)\, e^{-R(\xi)} d\xi + C \right] \\ &= \int_t^T e^{-(R(\xi)-R(t))} C(\xi)\, d\xi + e^{R(t)} C \end{aligned}$$

Suppose $C(t)$ is the continuous bounded function in $[0, T]$. It must be workable that:

$$\lim_{t \to T} \int_t^T e^{-(R(\xi)-R(t))} C(\xi)\, d\xi = 0$$

Therefore, $P(t) = Ce^{R(T)}$, that is, $C = e^{-R(T)} P(t)$ in which $P(t)$ is the price of the asset A in the moment T and $C = e^{-R(T)} P(t)$ is the value of the price of the asset A discounted from the moment T to 0. So we have:

$$P(t, r(t)) = \int_t^T e^{-(R(\xi)-R(t))} C(\xi)\, d\xi + e^{-(R(T)-R(t))} P(t) \tag{5}$$

The expression (5) is the formula of the intrinsic value function of assets that the continuous cash flow corresponds to in the situation of infinite time limit. Especially, when $t = 0$, there exists the following discount formula of the cash flow of the continuous intrinsic value in the moment 0:

$$P(0, r(0)) = e^{R(0)} \left(\int_0^T C(\xi)\, e^{-R(\xi)} d\xi + e^{-R(T)} P(t) \right)$$

For the fixed discount rate r, $r(t) = r$, $R(t) = rt$, $R(0) = 0$, and the following formula of the intrinsic value function is workable:

$$P\left(t, r(t)\right) = \int_t^T e^{-r(\xi - t)} C\left(\xi\right) d\xi + e^{-r(T-t)} P(t) \tag{6}$$

Particularly, when $t = 0$, there exists

$$P\left(0, r\right) = \int_0^T e^{-r\xi} C\left(\xi\right) d\xi + e^{-rT} P(t)$$

This paper only takes the situation of the fixed discount rate in the follow-up discussion, that is, $r(t) \equiv r$. The related discussion about the properties of the intrinsic value function for the general discount rate function $r(t)$ will be made in the following studies.

According to the expression (4) of $P\left(t, r\right)$, there exists:

$$\frac{\partial P\left(t, r\right)}{\partial r} = -\int_t^{+\infty} \left(\xi - t\right) C\left(\xi\right) e^{-r(\xi - t)} d\xi$$

Because $\xi \geq t$, $e^{-r(\xi - t)} > 0$ and the integrand in the right of the above expression takes the non-negative values when $C\left(\xi\right) \geq 0$, there exists $\frac{\partial P(t,r)}{\partial r} \leq 0$, and it must be workable that $\frac{\partial P(t,r)}{\partial r} < 0$ when $C\left(\xi\right) \geq 0$ does not vanish identically, which indicates that for pure income cash flow ($C(t) \geq 0$), the intrinsic value function must be the decreasing function of the discount rate. However, the integrand in the right of the above expression takes the negative values when $C\left(\xi\right) \leq 0$, there exists $\frac{\partial P(t,r)}{\partial r} \geq 0$, which indicates that for pure expenditure cash flow ($C(t) \leq 0$), the intrinsic value function must be the increasing function of the discount rate. On the basis of the expression (6), the same conclusion can be drawn, and it is required that $P(t) \geq 0$ when $C(t) \geq 0$ and that $P(t) \leq 0$ when $C(t) \leq 0$.

3. **Judging the Feasibility of Investment with the Help of the Intrinsic Value Function.** According to the mathematic Formulae (4) and (6) of the intrinsic value function, the value of the intrinsic value function is positive when the cash earnings function is positive and the value of the intrinsic value function is negative when the cash earnings function is negative. For a general continuous cash earnings function, if the value of the intrinsic value function is positive, the price of the assets would be positive; if the value of the intrinsic value function is negative, the price of the assets would be negative. When the value of the intrinsic value function is positive, it is necessary to pay a certain sum of money to gain the assets and the price of the assets would be positive; when the value of the intrinsic value function is negative, investors can obtain the assets if they gain a certain sum of money and the price of the assets would be negative.

In order to state conveniently, we only take the condition of the infinite time into consideration in this section, that is, $t \in [0, +\infty)$. If the cash earnings function of the asset A is non-negative ($C(t) \geq 0$), the asset A is a pure asset and holding it means that there will be a continuous cash inflow with no need of any cash expenditure. When $C(t) \geq 0$, the intrinsic value function determined by the expression (4) is non-negative. We consider the situation that $t = 0$, it is obviously that it is profitable to invest in the asset A only if the market price of the asset is lower than its intrinsic value which corresponds to the situation where the asset A is obtained through the payment of money in accordance with the market price. Because of the negative correlation between the intrinsic value function of assets and the discount rate, when the market price of the asset is lower than its intrinsic value, the internal rate of return gained through replacing the intrinsic value with the market price must be greater than the discount rate. If the market price of the asset is higher than its intrinsic value, it is unprofitable to invest in the asset A. At this situation, the internal rate of return gained through replacing the intrinsic value with the market price must be less than the discount rate. For the pure asset (with positive cash flow), the feasibility of investment can be judged by comparison between the internal rate of return and the discount rate: it is profitable when the internal rate of return is greater than the discount rate, and on the contrary, it is unprofitable when the internal rate of return is less than the discount rate.

If the cash earnings function of the asset A is non-positive ($C(t) \leq 0$), the asset A is indeed a pure liability and holding it means that there will be a continuous cash outflow with no need of any cash income. When $C(t) \leq 0$, the intrinsic value function determined by the expression (4) must be non-positive. We consider the situation that $t = 0$, holding the asset A (with non-positive cash flow $C(t) \leq 0$) is worthwhile or not depends on whether the original monetary income is enough to pay the future cash flow or not. The monetary value determined by the intrinsic value function exactly can pay the future cash flow. Only if the value of the original monetary income (the absolute value of the market price of assets) is higher than the absolute value of the intrinsic value of the asset, it is profitable to hold it. Or other, only if the market price of the asset is lower than its intrinsic value (the intrinsic value of the asset is negative) in the initial moment, it is profitable to invest in the asset A, which corresponds to the situation where the asset A is obtained through the income of money in accordance with the market price when $t = 0$. In Section 2, it is explained that for the pure liability (with negative cash flow), because of the positive correlation between the intrinsic value function of assets and the discount rate, the internal rate of return gained through replacing the market price with the intrinsic value must be less than the discount rate when the market price of the asset is lower than its intrinsic value. If the market price of the asset is higher than its intrinsic value, it is unprofitable to invest in the asset A. At this time, the internal rate of return gained through replacing the market price with the intrinsic value must be greater than the discount rate. The greater the absolute value of the intrinsic value is, the less the internal rate of return is. In other words, on the contrary for asset, for the pure liability, that the internal rate of return is less means that the absolute value of the market price is greater and that the original monetary income is greater, so it is profitable for investors.

For the positive cash flow (pure asset), the feasibility of investment can be judged by comparison between the internal rate of return and the discount rate: it is profitable when the internal rate of return is greater than the discount rate, and on the contrary, it is unprofitable when the internal rate of return is less than the discount rate. For the negative cash flow (pure liability), the feasibility of investment can also be judged by comparison between the internal rate of return and the discount rate: it is profitable when the internal rate of return is less than the discount rate, and on the contrary, it is unprofitable when the internal rate of return is greater than the discount rate.

For the general cash flow $C(t)$, it can take any real-value in different time intervals, sometimes it is positive, while sometimes negative. We denote that:

$$T^{+} = \{t \,|t \in [0, +\infty), C(t) \geq 0\}$$

$$T^{-} = \{t \,|t \in [0, +\infty), C(t) < 0\}$$

A general cash flow means combining all the cash flows that the assets and liabilities held into a total cash flow. And at this time, the intrinsic value function of the cash flow $C(t)$ can still be obtained, that is, the intrinsic value function $P(t, r)$ determined by the expression (4):

$$P(t, r) = \int_{t}^{+\infty} e^{-r(\xi - t)} C(\xi)\, d\xi$$

The mixture of asset and liability that the cash flow $C(t)$ corresponds to is also called an asset, which is an asset in a broad sense. This kind of asset will have different characteristic in different time intervals of future; sometimes it is the income flow ($t \in T^+$), sometimes the expenditure flow ($t \in T^-$).

For the asset that the general cash flow $C(t)$ corresponds to, the internal rate of return determined by the market price have not the function of judging whether the investment is profitable or not yet. At this situation, it can be judged according to the comparison between the intrinsic value of assets and the market price. The following two criteria are used to judge the feasibility of investment:

(1) When $P(t,r) > 0$, if the market price of an asset is less than $P(t,r)$ including the situation where the market price is less than zero, the investment is profitable; if the market price of an asset is greater than $P(t,r)$, the investment is unprofitable.

(2) When $P(t,r) < 0$, if the market price of an asset is less than $P(t,r)$ (it means the absolute value of the market price of an asset is greater than that of the absolute value of the intrinsic value), the investment is profitable; if the market price of an asset is greater than $P(t,r)$, the investment is unprofitable.

4. **The Relationship between the Discount Rate and the Intrinsic Value Determined by General Cash Flow.** In this section, the relationship between the general cash earnings function $C(t)$ and the discount rate r will be discussed. As is known, if $C(t) \geq 0$, there exists the negative correlation between the intrinsic value function $P(t,r)$ and the discount rate r, that is, $\frac{\partial P(t,r)}{\partial r} \leq 0$, $P(t,r)$ is a monotonic decreasing function of r and $P(t,r) \geq 0$. If $C(t) \leq 0$, there exists the positive correlation between the intrinsic value function $P(t,r)$ and the discount rate r, that is, $\frac{\partial P(t,r)}{\partial r} \geq 0$, $P(t,r)$ is a monotonic increasing function of r and $P(t,r) \leq 0$.

The points that the shadow areas of Figure 1 and Figure 2 correspond to are the profitable investment pattern. In any case, the area under the curve is profitable, while the area above the curve is unprofitable. It is unreasonable to choose the investment pattern above the curve.

When the sign of the cash earnings function $C(t)$ stay the same, the intrinsic value function $P(t,r)$ is the monotonic function of r and its sign remains unchanged. At this time, it is feasible to judge whether the investment is profitable or not according to the internal rate of return. For the general cash earnings function $C(t)$, because it can take any real value, the intrinsic value function $P(t,r)$ dose not keep monotonic function of r and its signs do not remain unchanged any more.

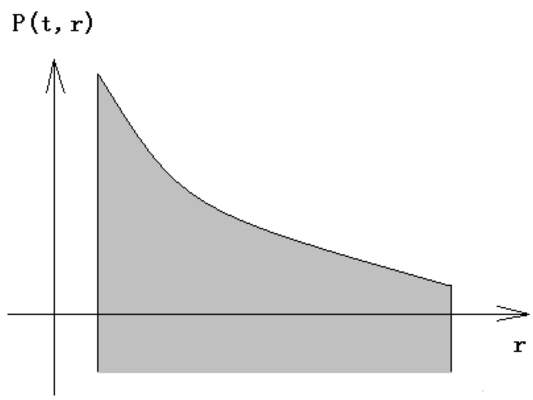

FIGURE 1. When $C(t) \geq 0$, the relationship between the intrinsic value function $P(t,r)$ and the discount rate r

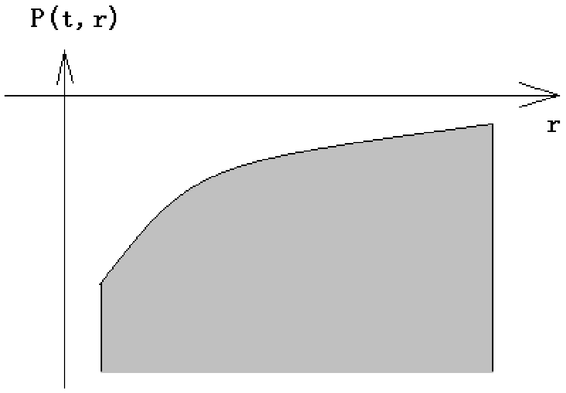

FIGURE 2. When $C(t) \leq 0$, the relationship between the intrinsic value function $P(t,r)$ and the discount rate r

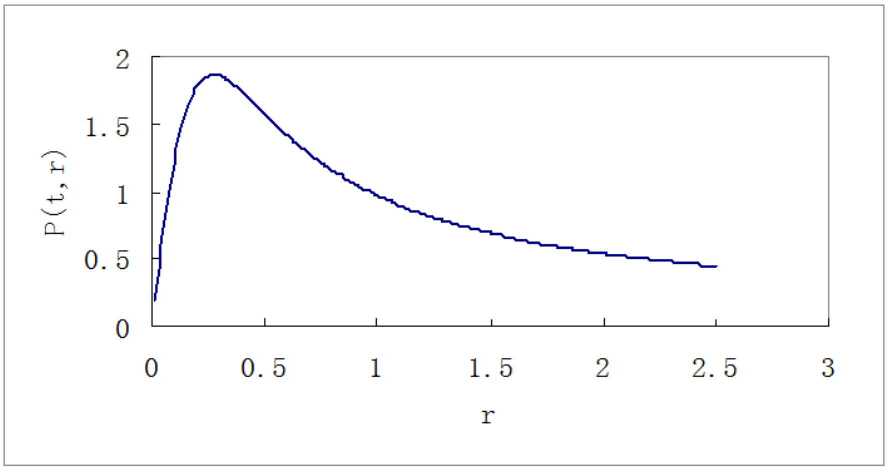

FIGURE 3. The curve of the intrinsic value function $P(t,r)$ of the discount rate r for the cash flow $C(t) = \sin bt$ when $b = 0.27$ and $t = 5$

For instance, let $C(t) = \sin bt$, and the corresponding intrinsic value function is:

$$P(t,r) = \int_t^{+\infty} e^{-r(\xi - t)} \sin b\xi d\xi = \frac{1}{r^2 + b^2}(b \cos bt + r \sin bt)$$

The relationship between the discount rate r and the intrinsic value $P(t,r)$ is showed in Figure 3 (the cash flow is $C(t) = \sin bt$ when $b = 0.27$ and $t = 5$). In Figure 3, the y-axis represents the intrinsic value, the x-axis represents the discount rate, and the intrinsic value reaches the maximum value when the discount rate $r = 0.27$ and the intrinsic value function takes non-negative values. It can be concluded from the expression $P(t,r)$ of the intrinsic value function of $C(t) = \sin bt$ that for the given b and r, $P(t,r)$ is the periodic function of t and it fluctuates circularly around 0.

The relationship between the discount rate r and the intrinsic value $P(t,r)$ of the cash flow $C(t) = -\sin bt$ when $b = 0.27$ and $t = 3$ is showed in Figure 4. In Figure 4, the intrinsic value reaches the minimum value when the discount rate $r = 0.27$ and the intrinsic value function takes negative values.

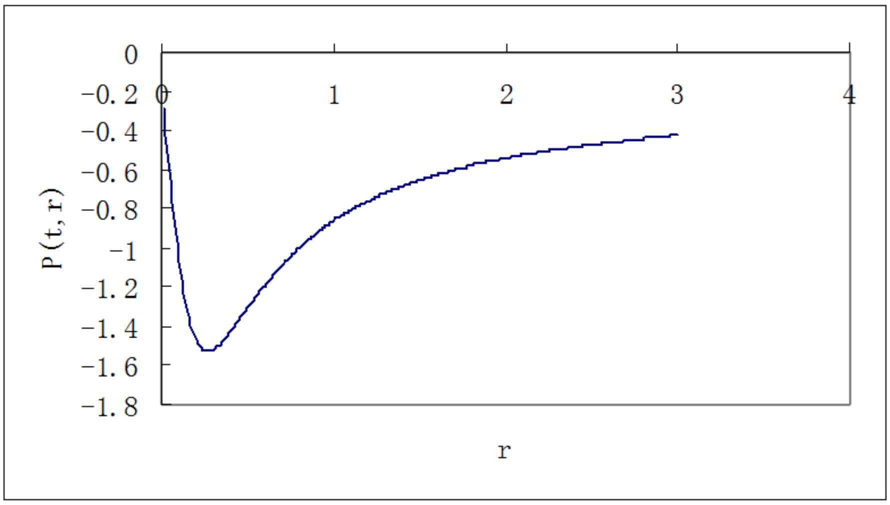

FIGURE 4. The curve of the intrinsic value function $P(t,r)$ of the discount rate r for the cash flow $C(t) = -\sin bt$ when $b = 0.27$ and $t = 3$

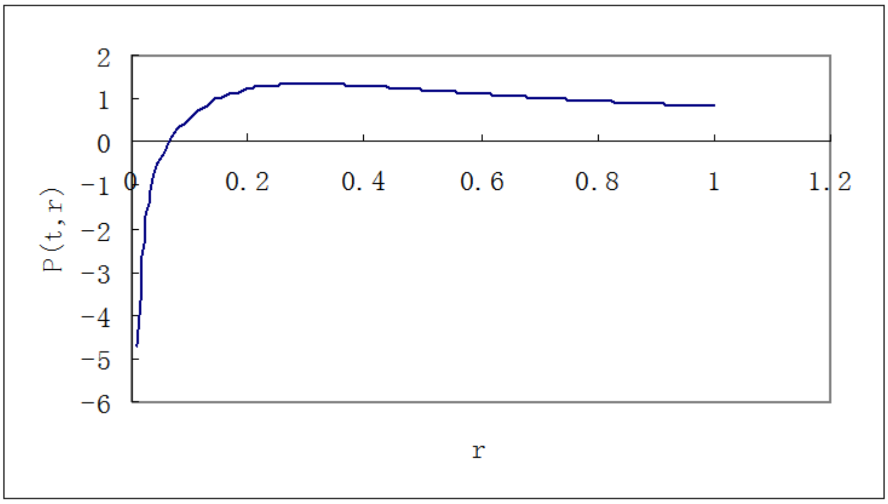

FIGURE 5. The curve of the intrinsic value function $P(t,r)$ of the discount rate r for the cash flow $C(t) = H + \sin bt$ when $H = -0.05$, $b = 0.27$ and $t = 3$

For an other instance, when $C(t) = H + \sin bt$, the corresponding intrinsic value function is:

$$P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)}(H + \sin b\xi)d\xi = \frac{H}{r} + \frac{1}{r^2+b^2}(b\cos bt + r\sin bt)$$

The relationship between the discount rate r and the intrinsic value $P(t,r)$ for the cash flow $C(t) = H + \sin bt$ when $H = -0.05$, $b = 0.27$ and $t = 3$ is showed in Figure 5. In Figure 5, the intrinsic value reaches the maximum value when the discount rate $r = 0.31$. The intrinsic value function takes negative values when $r < 0.07$ and positive values when $r \geq 0.07$, which indicates that for a general cash flow, with the change of the discount rate, the sign of the intrinsic value cannot stay the same.

Because $P(t,r)$ of the discount rate r is not monotonic function any more, it is possible that there are several internal rates of return corresponding to the same intrinsic value. However, around the extreme point of $P(t,r)$, there exist monotonic increasing intervals and monotonic decreasing intervals of the discount rate r. At this situation, it is infeasible to judge whether the investment is profitable or not with the use of the relationship of the internal rate of return and the discount rate. This is a main difference between the general cash flow and the cash flow with constant signs. So the intrinsic value of the general cash flow must have more complicated investment properties.

According to the common understanding, if the intrinsic value function satisfies $P(t,r) > 0$ to which the cash earnings function $C(t)$ corresponds, the income flow of the asset is greater than its expenditure flow, and the asset is approximately equivalent to a pure asset. At this time, there may be $\frac{\partial P(t,r)}{\partial r} \leq 0$. However, if $\frac{\partial P(t,r)}{\partial r} > 0$, which means that the intrinsic value of the asset whose price is positive will increase with the increasing of the discount rate, and it is abnormal. That situation will happen according to the previous examples. If $C(t) = \sin(0.27t), t \in [0, +\infty)$, there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ when $t = 5$ and $r \in (0, 0.27)$. The curve of the intrinsic value is showed in Figure 3.

Similarly, if the intrinsic value function satisfies $P(t,r) < 0$, to which the cash earnings function $C(t)$ corresponds, the income flow of the asset is less than its expenditure flow and the asset is approximately equivalent to a pure liability. At this time, there may be $\frac{\partial P(t,r)}{\partial r} \geq 0$. However, if $\frac{\partial P(t,r)}{\partial r} < 0$, which means that the intrinsic value of the asset whose price is negative will decrease with the increasing of the discount rate, and it is abnormal. That situation will happen according to the previous examples. If $C(t) = -\sin(0.27t)$, $t \in [0, +\infty)$, there exist $P(t,r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$ when $t = 3$ and $r \in (0, 0.27)$. The curve of the intrinsic value is showed in Figure 4.

According to the above discussion, the following definitions are introduced.

Definition 4.1. *For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t,r)$. For the given moment t, if there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ when the discount rate is in $[r_1, r_2]$, then the interval $[r_1, r_2]$ is called the abnormal interval of the asset.*

Definition 4.2. *For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t,r)$. For the given moment t, if there exist $P(t,r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$ when the discount rate is in $[r_1, r_2]$, then the interval $[r_1, r_2]$ is called the abnormal interval of the liability.*

Definition 4.3. *For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t,r)$. In the given moment t, r^* is the local extreme point of $P(t,r)$ for r. If $P(t,r^*) > 0$, then r^* is called the abnormal point of asset in the moment t; if $P(t,r^*) < 0$, then r^* is called the abnormal point of liability in the moment t.*

According to the above discussion, if the sign of the cash earnings function $C(t)$ remains unchanged, the corresponding intrinsic value function does not have the abnormal interval of asset or the abnormal interval of liability.

Theorem 4.1. *For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t,r)$. If $[r_1, r_2]$ is the abnormal interval of asset, that is, there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ in $[r_1, r_2]$, then for the cash earnings function $-C(t)$, $[r_1, r_2]$ is the abnormal interval of liability.*

Proof: The intrinsic value function that the cash flow $-C(t)$ corresponds to is represented by $P^-(t,r)$ and obviously there exists $P^-(t,r) = -P(t,r)$. According to the conditions the theorem contains, there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ in $[r_1, r_2]$. There exist:

$$P^-(t,r) = -P(t,r) < 0$$
$$\frac{\partial P^-(t,r)}{\partial r} = -\frac{\partial P(t,r)}{\partial r} < 0$$

Therefore, they conform to the conditions of Definition 4.2 and $[r_1, r_2]$ is the abnormal interval of liability of the cash earnings function $-C(t)$. The theorem is proved.

Theorem 4.2. *For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t, r)$. If $[r_1, r_2]$ is the abnormal interval of liability, that is, there exist $P(t, r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$ in $[r_1, r_2]$, then for the cash earnings function $-C(t)$, $[r_1, r_2]$ is the abnormal interval of asset.*

Proof: The intrinsic value function that the cash flow $-C(t)$ corresponds to is represented by $P^{-}(t, r)$ and obviously there exists $P^{-}(t, r) = -P(t, r)$. According to the conditions the theorem contains, there exist $P(t, r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$ in $[r_1, r_2]$. There exist:

$$P^{-}(t, r) = -P(t, r) > 0$$
$$\frac{\partial P^{-}(t,r)}{\partial r} = -\frac{\partial P(t,r)}{\partial r} > 0$$

Therefore, they conform to the conditions of Definition 4.1 and $[r_1, r_2]$ is the abnormal interval of asset of the cash earnings function $-C(t)$. The theorem is proved.

For the general cash earnings function $C(t)$, the corresponding intrinsic value function is $P(t, r)$. Under the normal condition, that is, $P(t, r) > 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$, to hold the asset whose cash flow is $C(t)$, investors pay money according to the price $P(t, r)$. When r increases, $P(t, r)$ will decrease, which is bad for investors, that is, the increasing of the interest rate is the risk event that investors face. For the abnormal interval of asset, there exist $P(t, r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$, when r increases, $P(t, r)$ will increase, which is good for investors; on the contrary. When r decreases, $P(t, r)$ will decrease, which is bad for investors. In other words, in the abnormal interval of asset, the decreasing of the interest rate is the risk event that investors face.

Under another normal condition, that is, $P(t, r) < 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$, to hold the asset whose cash flow is $C(t)$, investors receipt money according to the price $P(t, r)$. When r increases, $P(t, r)$ will increase, which is good for investors; on the contrary, when r decreases, $P(t, r)$ will decrease which is bad for investors. That is, the decreasing of the interest rate is the risk event that investors face. For the abnormal interval of liability, there exist $P(t, r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$. When r increases, $P(t, r)$ will decrease, which is bad for investors; on the contrary, when r decreases, $P(t, r)$ will increase, which is good for investors. In other words, in the abnormal interval of liability, the increasing of the interest rate is the risk event that investors face.

It is thus clear that in the abnormal interval of asset and the abnormal interval of liability, the relationship between the discount rate and the intrinsic value of assets is exactly opposite to that under the normal situation, which are the especial property that the general cash flow have.

5. **The Property of Assets When the Intrinsic Value is Zero.** In this section we discuss the situation where the continuous cash flow function conform to $P(t, r) = 0$, which means when the discount rate is r and in the moment t, the future cash income can offset the future cash expenditure. In the moment t, in order to hold the asset, investors do not need any cash expenditure and it will not bring losses to investors. Under the condition of $P(t, r) = 0$, if the asset is held after a certain sum of money is obtained in the moment t, it is profitable for investors. However, if the asset is held after a certain sum of money is paid in the moment t, it is unprofitable for investors.

Firstly, the situation where the intrinsic value function $P(t, r)$ changes with the fluctuation of the discount rate r is examined. Suppose that $P(t, r_0) = 0$ and $\frac{\partial P(t,r)}{\partial r}|_{r=r_0} \neq 0$. If $\frac{\partial P(t,r)}{\partial r}|_{r=r_0} < 0$, with the increasing of r around r_0, the intrinsic value of an asset changes from positive to negative in the moment t. At this time, the asset changes from the one that is similar to a pure asset to the one that is similar to a pure liability. When $r < r_0$, there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$, the asset is under the normal condition. However, when $r > r_0$, there exist $P(t,r) < 0$ and $\frac{\partial P(t,r)}{\partial r} < 0$, and the asset is in the abnormal interval of liability.

If $P(t,r_0) = 0$ and $\frac{\partial P(t,r)}{\partial r}|_{r=r_0} > 0$, with the increasing of r around r_0, the intrinsic value of an asset changes from negative to positive in the moment t. At this time, the asset changes from the one that is similar to a pure liability to the one that is similar to a pure asset. When $r < r_0$, there exist $P(t,r) < 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ and the asset is under the normal condition. However, when $r > r_0$, there exist $P(t,r) > 0$ and $\frac{\partial P(t,r)}{\partial r} > 0$ and the asset is in the abnormal interval of asset.

We discuss the situation where the intrinsic value function $P(t,r)$ changes with the t. Suppose $P(t,r) = 0$ and $\frac{\partial P(t,r)}{\partial t} \neq 0$, if $\frac{\partial P(t,r)}{\partial t} > 0$, the intrinsic value of the asset increases progressively with time going on. It is profitable for investors, so this asset should be held. If $\frac{\partial P(t,r)}{\partial t} < 0$, the intrinsic value of the asset decreases progressively with time going on. It is unprofitable for investors, so this asset should be dumped.

From the perspective of mathematics, if the equation $P(t,r) = 0$ has continuous solutions, these solutions must determine a curve which is $t = t(r)$, that is, $P(t(r),r) = 0$. For any given discount rate r, in the moment $t(r)$, the intrinsic value of the asset becomes zero. From the perspective the geometry, in the meaningful area of the plane (t,r), generally the curve $t = t(r)$ divides the area into two parts: one guarantees $P(t,r) > 0$, while the other one guarantees $P(t,r) < 0$. The reversal of the property of an asset occurs when the point (t,r) changes from one area to the other area and this reversal must cross the curve $t = t(r)$. The curve $t = t(r)$ is called the critical curve of the continuous cash flow $C(t)$. One side of the curve meets the condition that $P(t,r) > 0$, which is similar to a pure asset. The other side meets the condition that $P(t,r) < 0$, which is similar to a pure liability. It is undoubtedly meaningful to conduct an in-depth study on the curve $t = t(r)$ from the perspective of investment. This paper will not study the property of the curve $t = t(r)$ any longer.

6. **The Concept of Duration Function for General Cash Flow and Its Property.**

The concept of the duration of the future cash flow (in discrete form) is widely applied in the measurement of the interest rate risk of assets and the management of the investment portfolio. And study on the properties of the duration of the general continuous cash flow is undoubtedly an important part in investment theory. Referring to the definition of Macaulay duration when the cash flow is discrete, this section introduces the concept of the duration function for the general cash earnings function $C(t)$, puts forward the definition formula in integral form of the duration function for continuous cash flow, and discusses the relationships between duration function and intrinsic value function and cash flow function $C(t)$.

Definition 6.1. *Suppose $\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi \neq 0$. For the continuous cash flow $C(t), t \in [0,+\infty)$, the duration $D_\infty(t,r)$ in the moment t is determined by the following expression:*

$$D_\infty(t,r) = \frac{\int_t^{+\infty} (\xi - t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi} \tag{7}$$

In the expression (7), $\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi$ is the intrinsic value function $P(t,r)$ determined by the cash flow $C(t)$ which is put forward in the expression (4), $\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}$ $C(\xi)d\xi$ is the weighted average of the $C(t)$ about the time parameter $(\xi - t)$ in $[t,+\infty)$.

$D_\infty(t,r)$ is a function of the time parameter t and the discount rate r. We call $D_\infty(t,r)$ the duration function determined by the cash flow $C(t), t \in [0,+\infty)$.

Definition 6.2. *Suppose $\int_t^T e^{-r(\xi-t)}C(\xi)d\xi + e^{-r(T-t)}P_{face} \neq 0$. For the continuous cash flow $C(t)$, $t \in [0,T]$, the duration $D_T(t,r)$ in the moment t is determined by the following expression:*

$$D_T(t,r) = \frac{\int_t^T (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi + (T-t)e^{-r(T-t)}P_{face}}{\int_t^T e^{-r(\xi-t)}C(\xi)d\xi + e^{-r(T-t)}P_{face}} \tag{8}$$

In the expression (8), $\int_t^T e^{-r(\xi-t)}C(\xi)d\xi + e^{-r(T-t)}P_{face}$ is the intrinsic value function $P(t,r)$ determined by the cash flow $C(t)$ which is put forward in the expression (6). $D_T(t,r)$ is a function of the time parameter t and the discount rate r. We call $D_T(t,r)$ the duration function determined by the cash flow $C(t)$, $t \in [0,T]$.

Theorem 6.1. *If the cash flow function $C(t)$ does not vanish identically and its sign stays the same, the value of the duration function $D_\infty(t,r)$ or $D_T(t,r)$ must be positive.*

Proof: When $C(t) \geq 0$, if it is the one of the indefinite duration, that is, $t \in [0,+\infty)$, then there exists:

$$D_\infty(t,r) = \frac{\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi}$$

In the above expression, the numerator and denominator of its right side take positive values, so $D_\infty(t,r) > 0$.

If it is the one of the definite duration, that is, $t \in [0,T]$, there exists:

$$D_T(t,r) = \frac{\int_t^T (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi + (T-t)e^{-r(T-t)}P_{face}}{\int_t^T e^{-r(\xi-t)}C(\xi)d\xi + e^{-r(T-t)}P_{face}}$$

Because $C(t) \geq 0$, there should exist $P_{face} > 0$. In the above expression, the numerator and denominator of its right side take positive values, so $D_T(t,r) > 0$.

When $C(t) \leq 0$, if it is the one of the indefinite duration, that is, $t \in [0,+\infty)$, then there exists:

$$D_\infty(t,r) = \frac{\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi}$$

In the above expression, the numerator and denominator of its right side take negative values, so $D_\infty(t,r) > 0$.

If it is the one of the definite duration, that is, $t \in [0,T]$, there exists:

$$D_T(t,r) = \frac{\int_t^T (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi + (T-t)e^{-r(T-t)}P_{face}}{\int_t^T e^{-r(\xi-t)}C(\xi)d\xi + e^{-r(T-t)}P_{face}}$$

Because $C(t) \leq 0$, there should exist $P_{face} < 0$. In the above expression, the numerator and denominator of its right side take negative values, so $D_T(t,r) > 0$. The theorem is proved.

Let $C(t) = \sin bt$, according to Definition 6.1, it is not difficult to obtain the duration function to which the cash flow $C(t) = \sin bt$ corresponds. We have:

$$D_\infty(t,r) = \frac{\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}\sin b\xi d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}\sin b\xi d\xi} = \frac{2r}{r^2+b^2} - \frac{\sin bt}{b\cos bt + r\sin bt}$$

When $b = 0.27$ and $t = 5$, the duration curve of $D_\infty(t,r)$ about the discount rate r is showed in Figure 6.

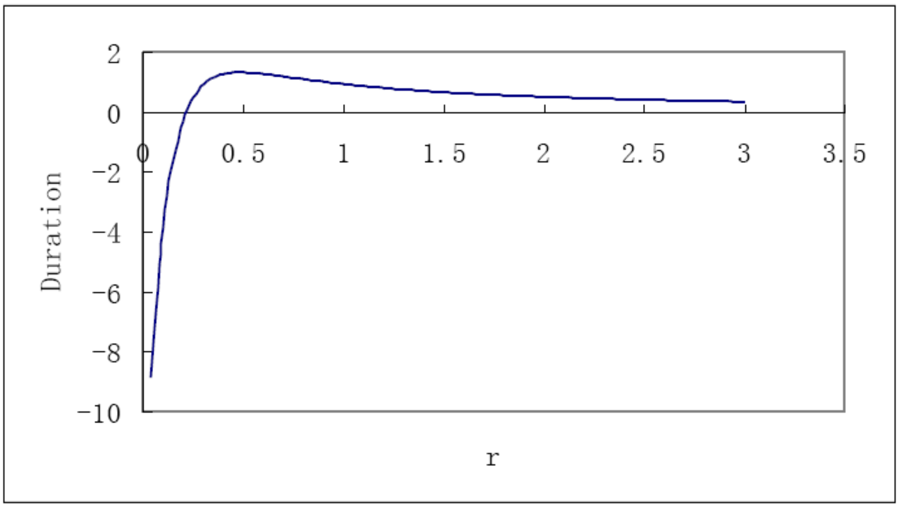

FIGURE 6. The duration curve of the cash flow $C(t) = \sin bt$ when $b = 0.27$ and $t = 5$

When $r < 0.27$, the duration function $D_\infty(t, r)$ takes negative values; when $r > 0.27$, the duration function $D_\infty(t, r)$ takes positive values. It is thus clear for the general cash flow function that it is easy to find an example that the duration function takes negative values.

According to the original meaning of the concept of the duration, the duration is a weighted average of the cash flow about the time in the payoff period. All the examined cash earnings functions are the cash flows in different moments in the future, that is, $t \geq 0$. Therefore, it should be guaranteed that the duration value is positive. The conclusions of above theorems also indicate that for non-negative or non-positive cash earnings function $C(t)$, the value the duration function must be positive. However, for a general continuous cash flow, the definition Formulae (7) and (8) of the duration function cannot guarantee the signs of their numerator and denominator which are the same. The above example indicates that there exists the situation where the value of the duration function is negative. The meaning of investment of this situation will be discussed in next section.

From the definition Formulae (7) and (8) of the duration function, it can be concluded that when $P(t, r) = 0$, the definition Formulae (7) and (8) fail to define the duration function. Considering the theoretical demand, we believe that there should exist the duration value in any case. It is set that when the intrinsic value of cash flows is zero, the corresponding duration value is ∞.

From Definitions 6.1 and 6.2, it can be concluded that if the numerator of the expression of the duration function is zero, that is,

$$\int_t^{+\infty} (\xi - t)e^{-r(\xi - t)}C(\xi)d\xi = 0$$

or

$$\int_t^T (\xi - t)e^{-r(\xi - t)}C(\xi)d\xi + (T - t)e^{-r(T-t)}P_{face} = 0$$

Then the value of the duration function is zero.

It is thus clear that for a general continuous cash earnings function, the duration function can take any real value including positive numbers, negative numbers, zero and ∞.

TABLE 1. The intrinsic value functions and duration functions to which some cash earnings functions correspond

The cash earnings function $C(t)$	The intrinsic value function $P(t,r)$	The duration function $D_\infty(t,r)$
$C(t)=D_0$ $D_0>0,\ t\in[0,+\infty)$	$\frac{D_0}{r}$	$\frac{1}{r}$
$C(t)=a+bt$ $a>0,\ b>0,\ t\in[0,+\infty)$	$\frac{a+bt}{r}+\frac{b}{r^2}$	$\frac{1}{r}\left(1+\frac{b}{(a+bt)r+b}\right)$
$C(t)=Ae^{\lambda(t-t_0)}$ $\lambda<r,\ t\in[0,+\infty)$	$\frac{Ae^{\lambda(t-t_0)}}{r-\lambda}$	$\frac{1}{r-\lambda}$
$C(t)=\begin{cases}0 & 0\le t\le t_0\\ A & t>t_0\end{cases}$ $t\in[0,+\infty)$	$\begin{cases}\frac{A}{r}e^{r(t-t_0)} & 0\le t\le t_0\\ \frac{A}{r} & t>t_0\end{cases}$	$\begin{cases}\frac{1}{r}+(t_0-t) & 0\le t\le t_0\\ \frac{1}{r} & t>t_0\end{cases}$
$C(t)=\sin bt$ $t\in[0,+\infty)$	$\frac{1}{r^2+b^2}(b\cos bt+r\sin bt)$	$\frac{2r}{r^2+b^2}-\frac{\sin bt}{b\cos bt+r\sin bt}$

The meaning of investment of this situation where the duration value is zero will be discussed in next section. In Table 1, some common functions are taken as cash earnings functions and the corresponding intrinsic value functions and duration functions are listed.

7. The Relationship between the Intrinsic Value Function and the Duration Function.

Next, the relationship between the intrinsic value function and the duration function will be derived from Definitions 6.1 and 6.2. There are following theorems.

Theorem 7.1. *For the continuous cash flow $C(t)$, $t\in[0,+\infty)$, the corresponding intrinsic value function of $C(t)$ is $P(t,r)$, $P(t,r)\neq 0$. The intrinsic value function $P(t,r)$ conforms to the following differential equation:*

$$\frac{\partial P(t,r)}{\partial r}=-D_\infty(t,r)P(t,r) \tag{9}$$

In this equation, $D_\infty(t,r)$ is the duration function determined by the definition Formula (7).

Proof: According to the expression (4), there exists

$$P(t,r)=\int_t^{+\infty}e^{-r(\xi-t)}C(\xi)d\xi$$

Take the partial derivative of the discount rate r about the above expression, and there exists:

$$\frac{\partial P(t,r)}{\partial r}=\frac{\partial}{\partial r}\int_t^{+\infty}e^{-r(\xi-t)}C(\xi)d\xi=\int_t^{+\infty}-(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi$$

According to the definition Formula (7), there exists

$$\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi=D_\infty(t,r)\int_t^{+\infty}e^{-r(\xi-t)}C(\xi)d\xi=D_\infty(t,r)P(t,r)$$

Therefore, there exists

$$\frac{\partial P(t,r)}{\partial r}=-D_\infty(t,r)P(t,r)$$

The theorem is proved.

Theorem 7.2. *For the continuous cash flow $C(t)$, $t \in [0, T]$, the intrinsic value function $P(t, r)$ determined by the expression (6) conforms to the following differential equation:*

$$\frac{\partial P(t,r)}{\partial r} = -D_T(t,r)P(t,r) \tag{10}$$

In this expression, $D_T(t, r)$ is the duration function determined by the expression (8).

Proof: The process of proof is the same as that of Theorem 7.1.

Theorem 7.3. *If the intrinsic value function $P(t, r)$ and duration function $D_\infty(t, r)(D_T(t, r))$ of the continuous cash flow $C(t)$ maintains the same sign in the term of validity, then the intrinsic value function $P(t, r)$ of the continuous cash flow $C(t)$ must be a monotonic function of r.*

Proof: According to the conclusion of Theorem 7.1 and Theorem 7.2, we have the formula $\frac{\partial P(t,r)}{\partial r} = -D_\infty(t,r)P(t,r) \left(\frac{\partial P(t,r)}{\partial r} = -D_T(t,r)P(t,r)\right)$. We know that if the intrinsic value function $P(t, r)$ and duration function $D_\infty(t, r)(D_T(t, r))$ of the continuous cash flow $C(t)$ maintains the same sign in the term of validity, then the partial derivative of r of the intrinsic value function $P(t, r)$ maintains the same sign in the term of validity. If $\frac{\partial P(t,r)}{\partial r}$ is positive, $P(t, r)$ is a monotonic increasing function of r; if $\frac{\partial P(t,r)}{\partial r}$ is negative, $P(t, r)$ is a monotonic decreasing function of r. The theorem is proved.

If $P(t, r) > 0$ and $D_\infty(t, r) > 0$, according to Theorem 7.1, there exists $\frac{\partial P(t,r)}{\partial r} = -D_\infty(t, r)P(t, r) < 0$, which is the generally negative correlation between the intrinsic value of assets and the discount rate; if $P(t, r) > 0$ and $D_\infty(t, r) < 0$, according to Theorem 7.1, there exists $\frac{\partial P(t,r)}{\partial r} = -D_\infty(t, r)P(t, r) > 0$, which indicates that r is in the abnormal interval of asset. If $P(t, r) < 0$ and $D_\infty(t, r) > 0$, according to Theorem 7.1, there exists $\frac{\partial P(t,r)}{\partial r} = -D_\infty(t, r)P(t, r) > 0$, which is the generally positive correlation between the intrinsic value of liabilities and the discount rate; if $P(t, r) < 0$ and $D_\infty(t, r) < 0$, according to Theorem 7.1, there exists $\frac{\partial P(t,r)}{\partial r} = -D_\infty(t, r)P(t, r) < 0$, which indicates that r is in the abnormal interval of liability. The conclusion of Theorem 7.1 shows that when the duration function of the continuous cash flow $C(t)$ takes negative values, the corresponding discount rate parameter must be in the abnormal interval (the abnormal interval of asset or the abnormal interval of liability).

If r^* is the local extreme point of the intrinsic value function $P(t, r)$ of r, and $P(t, r^*) \neq 0$, then there exists $\frac{\partial P(t,r)}{\partial r}|_{r=r^*} = 0$. According to the conclusion of Theorem 7.1, the value of the duration function must be zero when $r = r^*$. Some side of r^* must be in the abnormal interval of asset ($P(t, r) > 0$ and $D_\infty(t, r) < 0$) or the abnormal interval of liability ($P(t, r) < 0$ and $D_\infty(t, r) < 0$).

Theorems 7.1 and 7.2 give the relationship among the intrinsic value and the duration and the discount rate for general continuous cash flow. The conclusions of Theorems 7.1 and 7.2 show that for the continuous cash flows of the indefinite and definite durations the relationship between the intrinsic value function of assets and the corresponding duration function can be represented with the same differential equation. The rate of relative change of the intrinsic value function of assets about the discount rate is exactly the product of the duration value and -1.

The conclusions of Theorems 7.1 and 7.2 provide people another perspective to understand the meaning of the duration value. Duration is not only a weighted average of the time during which the cash flow is achieved, but also the factor for measuring the rate of relative change of the intrinsic value function of assets about the discount rate r. When the duration function takes negative value, the rate of relative change of the intrinsic value of assets about the discount rate r is positive, that is, $\frac{\partial P(t,r)}{\partial r}/P(t, r) \geq 0$, and the signs of $\frac{\partial P(t,r)}{\partial r}$ and $P(t,r)$ are the same. Of course, the greater the absolute value of the duration is, the greater the rate of relative change of the intrinsic value of assets caused by r is.

According to Theorems 7.1 and 7.2, the formula of the partial derivative of the duration function about the discount rate can be obtained. There exist

$$\frac{\partial D_\infty(t,r)}{\partial r} = -\frac{P(t,r)\frac{\partial^2 P(t,r)}{\partial r^2} - \left(\frac{\partial P(t,r)}{\partial r}\right)^2}{P^2(t,r)} \tag{11}$$

In this expression, the intrinsic value function $P(t,r)$ is determined by the expression (4).

$$\frac{\partial D_T(t,r)}{\partial r} = -\frac{P(t,r)\frac{\partial^2 P(t,r)}{\partial r^2} - \left(\frac{\partial P(t,r)}{\partial r}\right)^2}{P^2(t,r)} \tag{12}$$

In this expression, the intrinsic value function $P(t,r)$ is determined by the expression (6).

Next, the rate of change of the duration function about the time parameter will be deduced. There are following theorems.

Theorem 7.4. *For the continuous cash flow $C(t)$, $t \in [0,+\infty)$, $P(t,r)$ is the intrinsic value function determined by the expression (4). In the point (t,r) satisfying $P(t,r) \neq 0$, the duration function $D_\infty(t,r)$ determined by the expression (7) conforms to the following differential equation:*

$$\frac{\partial D_\infty(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_\infty(t,r) - 1 \tag{13}$$

Proof: According to Definition 6.1 (7), we deduce the partial derivative of $D_\infty(t,r)$ on the time parameter t. For a general parameter integral, the following rule of getting derivatives is workable:

$$\frac{d}{dt}\int_{\varphi(t)}^{+\infty} f(\xi,t)d\xi = \int_{\varphi(t)}^{+\infty} \frac{d}{dt} f(\xi,t)d\xi - f(\varphi(t),t)\varphi'(t)$$

The derivative of the denominator of $D_\infty(t,r)$ on the time parameter t is:

$$\frac{\partial}{\partial t}\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi = r\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi - C(t)$$

The derivative of the numerator of $D_\infty(t,r)$ on the time parameter t is:

$$\begin{aligned}\frac{\partial}{\partial t}\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi &= \int_t^{+\infty} ((\xi-t)r-1)e^{-r(\xi-t)}C(\xi)d\xi \\ &= r\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi - \int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi\end{aligned}$$

The denominator of $D_\infty(t,r)$ is represented by M and its numerator Z. There exists:

$$\begin{aligned}\frac{\partial D_\infty(t,r)}{\partial t} &= \frac{M\int_t^{+\infty}((\xi-t)r-1)e^{-r(\xi-t)}C(\xi)d\xi - Z(r\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi - C(t))}{M^2} \\ &= \frac{M(rZ-M) - Z(rM - C(t))}{M^2}\end{aligned}$$

Simplify the above expression, and notice $M = P(t,r)$ and $Z = D_\infty(t,r)P(t,r)$. There exist:

$$\frac{\partial D_\infty(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_\infty(t,r) - 1$$

The theorem is proved.

Theorem 7.5. *For the continuous cash flow $C(t)$, $t \in [0,T]$, $P(t,r)$ is the intrinsic value function determined by the expression (6). In the point (t,r) satisfying $P(t,r) \neq 0$, the duration function $D_T(t,r)$ determined by the expression (8) conforms to the following differential equation:*

$$\frac{\partial D_T(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_T(t,r) - 1 \tag{14}$$

Proof: The process of proof is the same as that of Theorem 7.4.

The conclusions of Theorems 7.4 and 7.5 show that for the continuous cash flows of the indefinite and definite durations, the partial derivative of the duration function on the time parameter conforms to the same differential equation.

8. **The Taylor Expansion of the Intrinsic Value Function and the High-Order Duration.** Next, the mathematic relationship in which the intrinsic value function changes with the change of the discount rate will be discussed. According to the definition Formula (4) of the intrinsic value function, the Taylor expansion of the intrinsic value function $P(t,r)$ on any r can be solved. The values of each-order derivative of $P(t,r)$ on r can be used to precisely calculate the value of the intrinsic value function on $r + \Delta r$. According to the concept of the higher-order moments of random variable, we introduce the concept of the high-order duration.

Definition 8.1. *For the continuous cash flow $C(t)$, $t \in [0,+\infty)$, when the intrinsic value function determined by the expression (4) takes the non-zero value, the n-order duration $D^n_\infty(t,r)$ in any moment t can be defined by the following expression:*

$$D^n_\infty(t,r) = \frac{\int_t^{+\infty} (\xi - t)^n e^{-r(\xi - t)} C(\xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) d\xi} \tag{15}$$

$D^n_\infty(t,r)$ is a function of the time parameter t and the discount rate r, we call $D^n_\infty(t,r)$ the n-order duration function determined by the cash flow $C(t)$, $t \in [0,+\infty)$.

Obviously, there is $D_\infty(t,r) = D^1_\infty(t,r)$. According to the expression (4), $P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi$. Solve the derivative of $P(t,r)$ to r, and there exists:

$$\frac{\partial P}{\partial r} = \int_t^{+\infty} -(\xi - t) e^{-r(\xi-t)} C(\xi) d\xi$$

In general, for $k \geq 1$, there exists:

$$\frac{\partial^k P}{\partial r^k} = \int_t^{+\infty} (-1)^k (\xi - t)^k e^{-r(\xi-t)} C(\xi) d\xi$$

According to the definition of the n-order duration $D^n_\infty(t,r)$, there exists:

$$\frac{\partial^k P}{\partial r^k} = (-1)^k D^k_\infty(t,r) P(t,r)$$

There exists the following Taylor expansion:

$$P(t, r + \Delta r) = P(t,r) + \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} D^k_\infty(t,r) P(t,r) (\Delta r)^k$$

Notice that $P(t,r) \neq 0$, simplify the above expression and we have:

$$\frac{\Delta P}{P} = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} D^k_\infty(t,r) (\Delta r)^k \tag{16}$$

In this expression, each-order duration $D^n_\infty(t,r)$ of the cash flow $C(t)$ and the change Δr of the discount rate are used to represent the relative change of the intrinsic value function. Take the first term of the expression (16), and there exists the following formula:

$$\frac{\Delta P}{P} = -D_\infty(t,r)\Delta r$$

This expression is the same as the expression (9). Take the first two terms of the expression (16), and there exists the following formula:

$$\frac{\Delta P}{P} = -D_\infty(t,r)\Delta r + \frac{1}{2}D^2_\infty(t,r)(\Delta r)^2$$

According to the expression (16), for $\Delta r > 0$, if $D_\infty(t,r) < 0$, then $\frac{\Delta P}{P}$ takes positive values for the sufficiently small Δr; if $D_\infty(t,r) > 0$, then $\frac{\Delta P}{P}$ takes negative values for the sufficiently small Δr; if $D_\infty(t,r) = 0$, then $\frac{\partial P(t,r)}{\partial r} = 0$ and the sign of $\frac{\Delta P}{P}$ is the same as that of the 2-order duration $D^2_\infty(t,r)$ for the sufficiently small Δr. In the case of the continuous cash flow, for any $k \geq 1$, the first k terms of the expression can be used to precisely calculate $\frac{\Delta P}{P}$. For the continuous cash flow $C(t)$, $t \in [0,T]$, the concept of the n-order duration can also be introduced to get the same formula as that of the expression (16), which is not repeated here.

9. **Conclusion.** This paper focuses on the general continuous cash flow that has no limitation on the sign to plus or minus. We deduce the formula of the intrinsic value function of assets from the continuous yield equation; discuss the relationship between the intrinsic value function of assets and discount rate. In the normal case (the sign of the cash flow stays the same), when the intrinsic value of assets is greater than zero, there exists the negative correlation between the intrinsic value and the discount rate; when the intrinsic value of assets is less than zero, there exists the positive correlation between the intrinsic value and the discount rate. Through some examples, this paper explains that the abnormal relationship between the intrinsic value function and discount rate may exist in the general cash flow: one is that when the intrinsic value of assets is greater than zero, there exists the positive correlation between the intrinsic value and the discount rate for some discount rate; the other is that when the intrinsic value of assets is less than zero, there exists the negative correlation between the intrinsic value and the discount rate for some discount rate. This paper makes an elaborate analysis into the situation that the intrinsic value function equals zero, and there must exist the abnormal relationship between the intrinsic value and the discount rate when the discount rate near the zero point. This paper puts forward the definition formula of the duration function determined by the continuous cash flow on the basis of the concept of the intrinsic value function, gains the differential equation that connects the duration function to the intrinsic value function, and explores the investment meaning when the duration equals to several particular values (positive value, negative value, zero and ∞). In addition, this paper also introduces the concept of the high-order duration and gets the Taylor series of the discount rate of the intrinsic value function. And the Taylor series can be used to study the sensitiveness of the intrinsic value of assets on the interest rate.

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REFERENCES

[1] F. R. Macauley, Some theoretical problem suggested by movements of interest rate, bond yields, and stock prices in the united states since 1856, *National Bureau of Economic Research*, vol.4, 1938.
[2] J. Chua, A closed-form formula for calculating bond duration, *Financial Analysts Journal*, pp.76-78, 1984.
[3] G. O. Bierway, G. Kaufman and A. Toevs, Duration: Its development and use in bond portfolio management, *Financial Analysts Journal*, pp.15-37, 1983.
[4] L. Fisher, Determination of risk premiums on corporate bonds, *Journal of Political Economy*, pp.217-237, 1959.
[5] B. Malkiel, Expectation, bond prices, and the term structure of interest rate, *Quarterly Journal of Economics*, pp.197-218, 1962.
[6] F. Reilly and R. Sidus, The many uses of bond duration, *Financial Analysts Journal*, pp.58-72, 1980.
[7] X. Wang and Z. Chen, Intrinsic value function of assets determined by the yield equation and its properties, *International Journal of Innovative Computing, Information and Control*, vol.8, no.5(A), pp.3223-3238, 2012.
[8] X. Wang, Properties of the intrinsic value function and the duration function determined by continuous cash flow, *International Journal of Innovative Computing, Information and Control*, vol.9, no.10, pp.4023-4043, 2013.