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OPTIMAL TYPE-2 FUZZY ADAPTIVE CONTROL FOR A CLASS OF UNCERTAIN NONLINEAR SYSTEMS USING AN LMI APPROACH

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Abstract. *In this paper we propose an optimal type-2 fuzzy adaptive controller for a class of uncertain nonlinear system with unknown dynamic. As it is known, one of the most famous control strategies that deals with uncertainties is Fuzzy Control. In the last decade, scientists have shown that standard fuzzy logic cannot handle uncertainties because the measurements that activate a standard FLS (Fuzzy Logic System) may be noisy and therefore uncertain. Therefore, the method is based on type-2 FLS to approximate unknown non-linear functions. Thus we introduce a type-2 FLS, which can handle rule uncertainties. The implementation of this type-2 FLS involves operations of fuzzification, inference, and output processing. The design of the on-line adaptive scheme of the proposed controller is based on linear matrix inequalities (LMI) technique. We exploit the linear structure of a Takagi-Sugeno fuzzy system with constant conclusion to design an indirect adaptive fuzzy controller. However, the control law cannot ensure the conditions of Lyapunov stability. To overcome this problem, attenuate the influence of external disturbances, and remove fuzzy approximation error, we add a component called the supervision control that will force the time derivative of the function of Lyapunov to be negative. Simulation results are given to illustrate the effectiveness of the proposed approach.*
Keywords: Fuzzy set type-2, LMI, Lyapunov approach, Nonlinear system, Takagi-Sugeno fuzzy model

1. Introduction.

The control of nonlinear systems has been an important topic of research [1,2]. Traditionally, control system design has been tackled using mathematical models derived from physical laws. In fact the structure and most of the parameters of the system are unknown. To overcome this problem in the design of control systems several techniques have emerged in the recent years especially techniques based on the intelligent technology such as neural networks, fuzzy logic, genetic algorithms, and evolutionary computation [3-6]. In particular, FLSs have been successfully applied to control complex or ill-defined processes whose mathematical models are difficult to obtain [7,8].

In the past few years, fuzzy control of nonlinear systems has been implemented successfully in many applications. In most of these applications, the so called Takagi-Sugeno type fuzzy model is used to represent a nonlinear system [6]. Then based on this model, a fuzzy controller was designed. Fuzzy logic control has found promising applications for a wide range of industrial systems specifically applicable to plants that are mathematically poorly modeled [9]. Based on the universal approximation capability, many effective adaptive fuzzy control schemes have been developed to incorporate with information and knowledge of human experts in a systematic way and can also guarantee stability and performance criteria [7,8,10,26].

However, FLSs have a major drawback which is expressed in the fact that the fuzzy rules must be previously tuned by time-consuming trial-and-error procedures. This is due to the lack of adequate analysis and design techniques. To overcome this problem, some researches have been focusing on the Lyapunov synthesis approach to construct stable adaptive fuzzy controllers. The basic idea of most of these works is that with the universal approximation ability of FLSs, the system uncertainties can be represented by linearly parameterized uncertainties so the standard parametric adaptive techniques can be utilized [11,12].

Feedback linearization based on adaptive control is suitable for the control of nonlinear systems with accurate nominal models or linearly parametrical dynamical models. However, due to modelling errors, these controls may not be very effective without proper compensation to overcome the modelling error effects [13,14,26].

The design procedure aims at rendering the fuzzy controllers stable. More significantly, the stability analysis and control design problems are reduced to LMI problem. The so-called linear matrix inequality (LMI) approach has been widely used to solve problems in linear robust control, gain-scheduling and in multi-objective control [15]. Since the work [16] that showed a solution to the nonlinear problems using LMIs, researchers have proposed different solutions to nonlinear robust H_∞ control problems, e.g., [17].

Numerically, the LMI problems can be solved very efficiently by means of some of the most powerful tools available to date in the literature of mathematical programming. Therefore, recasting the stability analysis and control design problems as LMI problems is equivalent to finding solutions to original problems [18,19].

Many researches have shown that type-1 FLSs have difficulties in modelling and minimizing the effect of uncertainties [4,20,24,25]. One reason is that a type-1 fuzzy set is certain in the sense that the membership grade for a particular input is a crisp value. Recently, type-2 fuzzy sets, characterized by membership functions that are themselves fuzzy, have been attracting interest [4,20-22]. For such sets, each input has a unity of secondary membership grade defined by two type-1 membership functions, upper membership function and lower membership function. The concept of a type-2 fuzzy set was introduced by Zadeh as an extension of the concept of an ordinary fuzzy set called a type-1 fuzzy set [23].

An FLS using at least one type-2 fuzzy set is called a type-2 FLS. The wide range of applications of type-2 FLSs has shown that it provides good solutions, especially in the presence of uncertainties [4]. Similar to the conventional adaptive control, adaptive fuzzy control can be categorized into direct, indirect and composite schemes according to the type of fuzzy rules [12,26].

In this paper, we present an optimal type-2 adaptive fuzzy control for a class of nonlinear systems with unknown dynamic. The basic idea is that first the type-2 FLS is utilized to approximate the unknown nonlinear function, and then the fuzzy parameters are adjusted on-line by the adaptive laws with stability and convergence analysis using the Lyapunov approach in order to achieve the specified tracking performance. The auxiliary compensation control is designed to attenuate the influence of external disturbances and the fuzzy approximation error. The design of this signal depends on the well-known upper bounds of both the approximation error and the external disturbances, which is a restrictive assumption due to the fact that these bonds are generally unknown.

The structure of a type-2 Fuzzy Logic Controller (FLC) is given in the figure below. It is very similar to the structure of a type-1 FLS. For a type-1 FLS, the output processing block contains only the defuzzifier. The fuzzifier maps the crisp input into a fuzzy set.

This fuzzy set can, in general, be a type-2 set. In the type-1 case, we generally have "IF-THEN" rules. The distinction between type-1 and type-2 is associated with the nature of the membership functions, which is not important while forming rules. Hence, the structure of the rules remains exactly the same in the type-2 case. The unique difference is that now some or all of the sets involved are of type-2. It is not necessary that all the antecedents and the consequent are type-2 fuzzy sets. As long as one antecedent or the consequent set is type-2, we will have a type-2 FLS.

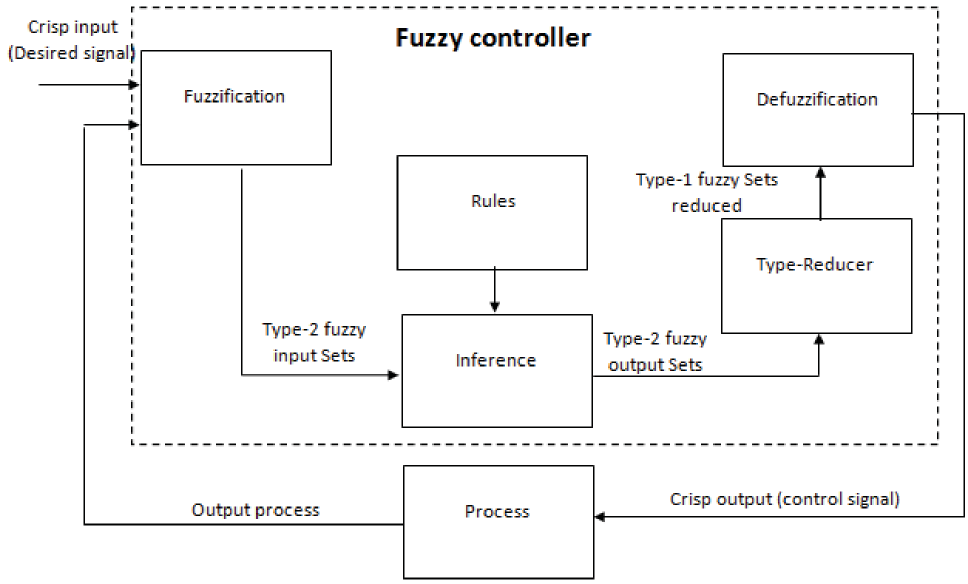

Figure 1. Type-2 fuzzy logic system

This paper is organized as follows. Section 2 describes the type-2 FLSs. In Section 3, we define the problem and we propose the optimal adaptive type-2 fuzzy control. Section 4 presents numerical results which validate the proposed approach. Concluding remarks are given in Section 5.

2. **Type-2 Fuzzy Logic Systems.** The basic configuration of an FLS consists of a fuzzifier, a fuzzy rule base, a fuzzy inference engine and a defuzzifier. The structure of a type-2 FLS is similar to type-1 counterpart. The major difference is that at least one of the fuzzy sets in the rule base is type-2. A type-2 fuzzy set is characterized by membership functions that are themselves fuzzy. The key concept is "footprint of uncertainty", which models the uncertainties in the shape and position of the type-1 fuzzy set. In a type-1 FLS, the inference engine combines rules and gives a mapping from inputting type-1 fuzzy sets to output type-1 fuzzy sets. In the type-2 case, the inference process is very similar. The inference engine combines rules and gives a mapping from input type-2 fuzzy sets to output type-2 fuzzy sets. To do this one needs to find unions and intersections of type-2 sets, as well as compositions of type-2 relations.

Figure 2 illustrates an example of a type-2 fuzzy membership functions with footprint of uncertainty [21,22]. Uncertainty in the primary membership grades of a type-2 membership function consists of a region that we call the footprint of uncertainty of a type-2 MF. The footprint of uncertainty represents the union of all primary memberships.

The output of inference engine for a type-2 FLS is type-2 sets. We focus on "output processing" which consists of type reduction and defuzzification. Type-reduction methods are extended versions of type-1 defuzzification methods. Type reduction captures more

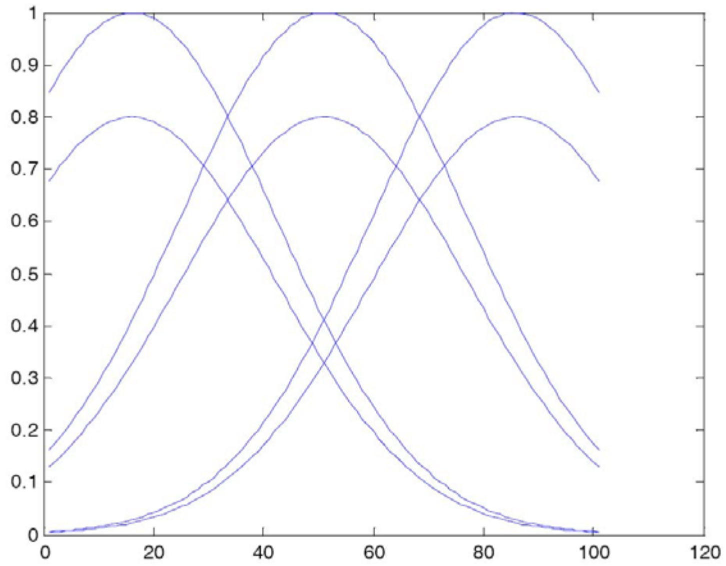

FIGURE 2. Type-2 fuzzy sets

information about rule uncertainties than does the defuzzified value (a crisp number), but it is computationally intensive, except for interval type-2 fuzzy sets for which we provide a simple type-reduction computation procedure. Hence a type-reducer is needed to convert them into type-1 sets before defuzzification can be carried out.

2.1. **Inference in type-2 fuzzy logic systems.** The inference engine uses the fuzzy IF-THEN rules to perform a mapping from an input vector $\underline{x} = (x_1, x_2, \ldots, x_n)^T$ to an output scalar y. We introduce a class of FLSs, type-2 FLSs, in which the antecedent or consequent membership functions are type-2 fuzzy sets.

These rules represent a type-2 fuzzy relation between the input space and the output space of the FLS. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules in the following form:

$$R^i : \text{ If } x_1 \text{ is } \tilde{F}_1^i \text{ and } x_2 \text{ is } \tilde{F}_2^i \ldots \text{ and } x_n \text{ is } \tilde{F}_n^i \text{ Then } y \text{ is } \tilde{G}^i \tag{1}$$

where $\tilde{F}_j^i$ are the antecedent sets ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$), $\tilde{G}^i$ are the consequent sets and m is the number of rules. We denote the membership function of this type-2 relation as:

$$\mu_{\tilde{F}_1^i \times \tilde{F}_2^i \times \tilde{F}_3^i \times \ldots \times \tilde{F}_n^i \to \tilde{G}^i}(x_1, x_2, x_3, \ldots, x_n, y)$$

where $\tilde{F}_1^i \times \tilde{F}_2^i \times \tilde{F}_3^i \times \ldots \times \tilde{F}_n^i$ denotes the Cartesian product of $\tilde{F}_1^i$, $\tilde{F}_2^i$, $\tilde{F}_3^i \ldots \tilde{F}_n^i$.

The first step in the extended sup-star operation is to obtain the firing set $F^i(\underline{x})$ by performing the input and antecedent operations.

$$F^i(\underline{x}) = \prod\nolimits_{j=1}^{n} \mu_{\tilde{F}_j^i}(x_j) \tag{2}$$

A general type-2 FLS is very complicated because of type reduction. Things are more simplified a lot when the secondary membership functions are interval sets in which case we call the type-2 FLS an interval type-2 FLS. As only interval type-2 fuzzy sets are used and the met operation is implemented by the product t-norm, the firing set is the following type-1 interval set:

$$F^i(\underline{x}) = \left[\underline{f}^i(\underline{x}), \overline{f}^i(\underline{x})\right] \tag{3}$$

where: $\underline{f}^i(\underline{x}) = \underline{\mu}_{\tilde{F}_1^i}(x_1) * \underline{\mu}_{\tilde{F}_2^i}(x_2) * \cdots * \underline{\mu}_{\tilde{F}_n^i}(x_n)$ and $\overline{f}^i(\underline{x}) = \overline{\mu}_{\tilde{F}_1^i}(x_1) * \overline{\mu}_{\tilde{F}_2^i}(x_2) * \cdots * \overline{\mu}_{\tilde{F}_n^i}(x_n)$.

The term $\underline{\mu}_{\tilde{F}_j^i}(x_j)$ and $\overline{\mu}_{\tilde{F}_j^i}(x_j)$ are the lower and upper membership grades of $\mu_{\tilde{F}_j^i}(x_j)$, respectively.

Next, the firing set $f^i(\underline{x})$ is combined with the consequent fuzzy set of the i^{th} rule using the product t-norm to derive the fired output consequent sets.

2.2. **Type-reduction for interval type-2 fuzzy logic systems and defuzzification.** Since the output of the inference engine is a type-2 fuzzy set, it must be type-reduced before the defuzzifier can be used to generate a crisp output. This is the main structural difference between type-1 and type-2 FLC. The most commonly used type-reduction method is the center-of-sets type-reducer. The center-of-sets type reducer replaces each consequent set by its centroid (which itself is a type-1 set if the consequent set is type-2) and finds a weighted average of these centroids, where the weight associated with the i^{th} centroid is the degree of firing corresponding to the i^{th} rule, namely $\prod_{j=1}^{n}\mu_{\tilde{F}_j^i}(x_j)$. The expression of the type-reduced set is given by [12]:

$$\begin{aligned} &Y_{\cos}(Y^1,\dots,Y^m,F^1,\dots,F^m)\\ &=\int_{y^1}\cdots\int_{y^m}\cdots\int_{f^1}\cdots\int_{f^m}\tau_i^m\mu_{Y^i}(y^i)*\tau_i^m\mu_{F^i}(f^i)\Big/\frac{\sum_{i=1}^m f^iy^i}{\sum_{i=1}^m f^i} \end{aligned} \tag{4}$$

where τ and $*$ denote the chosen t-norm, $y^i\in C_i=C_{\tilde{G}^i}$ is the centroid of the i^{th} consequent set, and $f^i\in F^i(\underline{x})=\prod_{j=1}^{n}\mu_{\tilde{F}_j^i}(x_j)$ is the degree of firing associated with the i^{th} consequent set for $i=1,2,\dots,m$.

For an interval type-2 FLS, each Y^i and F^i is an interval type-1 set, then $\mu_{Y^i}(y^i)=\mu_{F^i}(f^i)=1$. Equation (4) can be rewritten as:

$$Y_{\cos}(Y^1,\dots,Y^m,F^1,\dots,F^m)=\int_{y^1}\cdots\int_{y^m}\cdots\int_{f^1}\cdots\int_{f^m}1\Big/\frac{\sum_{i=1}^m f^iy^i}{\sum_{i=1}^m f^i}=[y_l,y_r] \tag{5}$$

The fuzzifier maps a crisp point $\underline{x}=(x_1,x_2,\dots,x_n)^T$ into a fuzzy set. The defuzzifier maps fuzzy set in $\Re$ to crisp points in $\Re$.

By using the singleton fuzzification, product inference, and centre-average defuzzification, the output value of fuzzy system is:

$$y(\underline{x})=\underline{\theta}^T\xi(\underline{x},t) \tag{6}$$

where $\underline{\theta}=(\theta^1,\theta^2,\dots,\theta^m)^T=(\overline{y}^1,\overline{y}^2,\dots,\overline{y}^m)^T$ is the parameter vector, and $\xi(\underline{x},t)=(\xi^1,\xi^2,\dots,\xi^m)^T$ is the vector of fuzzy basis functions.

3. **Problem Formulation and Design Optimal Type-2 Fuzzy Adaptive Controller.** Consider a general class of SISO n^{th} order nonlinear systems. The dynamics equation can be described as:

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \vdots\\ \dot{x}_n=f(\underline{x},t)+bu+d(t)\\ y=x_1\end{cases}$$

or equivalently

$$\begin{cases}x^{(n)}=f(\underline{x},t)+bu+d(t)\\ y=x\end{cases} \tag{7}$$

where $\underline{x}=(x_1,x_2,\dots,x_n)^T=(x,\dot{x},\dots,x^{(n-1)})^T\in\Re^n$ is the state vector of the system which is assumed to be available for measurement. $u\in\Re$ and $y\in\Re$ are respectively the input and the output of the system. f is a nonlinear system function representing the dynamic system behaviour. b is a positive constant and $d(t)$ is the unknown external disturbance. To satisfy the controllability of system (7), the input gain $b \neq 0$ is necessary.

In many real application, $f(\underline{x}, t)$ may not be exactly known, which can be split into two parts as:

$$f(\underline{x}, t) = \bar{f}(\underline{x}, t) + \Delta f(\underline{x}, t) \tag{8}$$

where $\bar{f}(\underline{x}, t)$ denotes the nominal part and $\Delta f(\underline{x}, t)$ represents its uncertain part.

Without loss of generality, we assume the following assumptions.

Assumption 3.1. *$\underline{x}$ belongs to a compact set $U_x = \{x \in \Re^n : |\underline{x}| \leq M_x\}$, where M_x is a positive constant.*

Assumption 3.2. *$\Delta f(\underline{x}, t)$ is bounded as follows: $|\Delta f(\underline{x}, t)| \leq F(\underline{x}, t), \forall x \in U_x \subset \Re^n$.*

Assumption 3.3. *$|d(t)| \leq d_m$, where d_m is upper bound.*

A type-1 FLCs are unable to handle rule uncertainties directly, because they use type-1 fuzzy sets that are certain. On the other hand, type-2 FLCs are very useful in circumstances where it is difficult to determine an exact measurement of uncertainties. It is known that type-2 fuzzy set let us model and minimize the effects of uncertainties in rule base FLC. Unfortunately, type-2 fuzzy sets are more difficult to use and understand than type-1 fuzzy sets; hence, their use is not widespread yet.

Knowing the limits of the standard fuzzy logic in the treatment of uncertainties, the last decade started a new direction in FLS. Scientists “led” by Mendel “dusted off” the ideas of Zadeh that he had on enhanced fuzzy sets (type-2 fuzzy sets). In fuzzy type-2 sets the uncertainty is represented as an extra dimension and advantages can be gained.

To consider the treatment of uncertainty in the dynamic of a system, we use type-2 FLS (6) to approximate the unknown nonlinear function $\Delta f(\underline{x}, t)$:

$$\Delta \hat{f}(\underline{x}/\underline{\theta}_f) = \underline{\theta}_f^T \xi_f(\underline{x}, t) \tag{9}$$

where $\underline{\theta}_f$ is free to be tuned adaptively and $\xi_f(\underline{x}, t)$ is a regressive vector.

Define the optimal parameters of type-2 fuzzy system:

$$\underline{\theta}_f^* = \arg \min_{\underline{\theta}_f \in \Omega_f} \left[\sup_{\underline{x} \in U_x} \left| \Delta f(\underline{x}, t) - \underline{\theta}_f^T \xi_f(\underline{x}, t) \right| \right] \tag{10}$$

It is assumed that the optimal vector $\underline{\theta}_f^*$ that minimizes the modelling error lies in a convex region:

$$\Omega_f = \left\{ \underline{\theta}_f \in \Re^m / \left| \underline{\theta}_f \right| \leq M_f \right\} \tag{11}$$

where the radii M_f is a positive constant.

Since $\Delta \hat{f}(\underline{x}/\underline{\theta}_f)$ is interval type-2 fuzzy system, then the type-reduced sets will be given respectively by:

$$\tilde{F}_{\cos}\left(\theta_f^1, \dots, \theta_f^m, f^1, \dots, f^m\right) = \int_{\theta_f^1} \cdots \int_{\theta_f^m} \int_{f^1} \cdots \int_{f^m} 1 \Bigg/ \frac{\sum_{i=1}^m f^i \theta_f^i}{\sum_{i=1}^m f^i} = \left[\hat{f}_l, \hat{f}_r\right] \tag{12}$$

f^i is the degree of firing associated with the i^{th} rule of the FLS $\Delta \hat{f}(\underline{x}/\underline{\theta}_f)$.

Equation (12) may be computed using the Karnik-Mendel iterative method [12]. It has been proven that this iterative procedure can converge in at most N iterations.

Once $\hat{f}_l$ and $\hat{f}_r$ are obtained, they can be used to calculate the crisp output. Since the type-reduced set is an interval type-1 set, the defuzzified output is:

$$\Delta\hat{f}(\underline{x}/\underline{\theta}_f) = \frac{\hat{f}_l + \hat{f}_r}{2} \tag{13}$$

$$\hat{f}_l = \frac{\sum\limits_{i=1}^{m} f_l^i \theta_f^i}{\sum\limits_{i=1}^{m} f_l^i}, \quad \hat{f}_r = \frac{\sum\limits_{i=1}^{m} f_r^i \theta_f^i}{\sum\limits_{i=1}^{m} f_r^i} \tag{14}$$

where: $\xi_{fl}(\underline{x}) = (\xi_{fl}^1, \xi_{fl}^2, \ldots, \xi_{fl}^m)^T$ and $\xi_{fr}(\underline{x}) = (\xi_{fr}^1, \xi_{fr}^2, \ldots, \xi_{fr}^m)^T$ are the vectors of the fuzzy basis functions with:

$$\begin{aligned} \xi_{fl}^i &= \frac{f_l^i}{\sum\limits_{i=1}^{m} f_l^i} \\ \xi_{fr}^i &= \frac{f_r^i}{\sum\limits_{i=1}^{m} f_r^i} \end{aligned} \tag{15}$$

f_l^i and f_r^i denote the firing values used to compute the left point $\hat{f}_l$ and right point $\hat{f}_r$, respectively.

$\underline{\theta}_f = (\theta_f^1, \theta_f^2, \ldots, \theta_f^m)$ is the adjustable parameter vector of type-2 fuzzy systems $\Delta\hat{f}(\underline{x}/\underline{\theta}_f)$. Then:

$$\begin{aligned} \hat{f}_l &= \underline{\theta}_f^T \xi_{fl}(\underline{x}) \\ \hat{f}_r &= \underline{\theta}_f^T \xi_{fr}(\underline{x}) \end{aligned} \tag{16}$$

Therefore:

$$\begin{aligned} \Delta\hat{f}(\underline{x}, \underline{\theta}_f) &= \frac{\underline{\theta}_f^T \xi_{fl}(\underline{x}) + \underline{\theta}_f^T \xi_{fr}(\underline{x})}{2} \\ &= \underline{\theta}_f^T \left[\frac{\xi_{fl}(\underline{x}) + \xi_{fr}(\underline{x})}{2} \right] \\ &= \underline{\theta}_f^T \xi_f(\underline{x}) \end{aligned} \tag{17}$$

Based on the universal approximation theorem [11], the above FLS is capable of uniformly approximating any well-defined nonlinear function over a compact set U_x to any degree of accuracy.

The control problem is to obtain the state $\underline{x}$ in order to track a desired state $\underline{x}_d$ in the presence of model uncertainties and external disturbance with the tracking error:

$$\underline{e} = \underline{x} - \underline{x}_d = (e, \dot{e}, \ldots, e^{(n-1)})^T \tag{18}$$

If the function $f(\underline{x}, t)$ is completely known and $d(t) = 0$, we can solve the control problem stated above by the so-called feedback linearization method. In this method, the function $f(\underline{x}, t)$ is used to construct the following feedback control law:

$$u = \frac{1}{b}\left(-f(\underline{x}, t) + x_d^{(n)} - \sum_{i=0}^{n-1} c_i e^{(i)} \right) \tag{19}$$

with defining $\underline{C} = (c_0, c_1, \ldots, c_{n-1}, 1)^T$ provided that the polynomial: $h(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$ is a polynomial Hurwitiz, i.e., all the roots are in the open left half-plane.

However, since $f(\underline{x}, t)$ and disturbance are actually unknown in practical systems, we cannot use them for constructing the control law (19). Therefore, to solve these problems, we replace them by their estimates with the type-2 FLS $\bar{f}(\underline{x}, t) + \Delta\hat{f}(\underline{x}/\underline{\theta}_f)$ to construct a self-tuning controller:

$$u_c = \frac{1}{b}\left(-\bar{f}(\underline{x}, t) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f) + x_d^{(n)} - \sum_{i=0}^{n-1} c_i e^{(i)}\right) \tag{20}$$

This control law synthesized by the type-2 fuzzy logic allows to take into account the uncertainties of dynamic we have in the system dynamics as opposed to standard fuzzy logic which cannot handle uncertainties.

In order to obtain good results in the tracking performance, we can use the auxiliary compensation control. In this article, we combined the contributions made by the fuzzy logical type-2 with the importance of LMI technique for analysis and synthesis of control systems. One of the main advantages of LMI techniques in control is that they allow us to save, in a numerical way, many interesting problems that have always been considered hard to tackle and that lack an analytical solution. It is, hence, a relevant engineering objective to assess whether these techniques, which are very appealing from a theoretical point of view, are effectively beneficial in designing controllers for real systems. The synthesis of this control relies on LMI and Lyapunov approach. We follow the Lyapunov synthesis method by constructing a Lyapunov function candidate V and then determining the conditions required to, indeed, make it a Lyapunov function of the closed-loop system.

The auxiliary control part is given as:

$$u_r = \frac{sat(\underline{e}^T PB)}{b}\left[F(x) + \left|\Delta\hat{f}(\underline{x}/\underline{\theta}_f^*)\right| + d_m\right] \tag{21}$$

with P and B being two matrices defined below.

The proposed fuzzy control scheme can guarantee that:

i) All the variables of the closed-loop system are bounded.

ii) The tracking performance is achieved.

The function $sat(\underline{e}^T PB)$ may be written as:

$$sat(\underline{e}^T PB) = \begin{cases} (\underline{e}^T PB) \text{ if } \left|\underline{e}^T PB\right| > \varepsilon \\ \frac{\left|\underline{e}^T PB\right|}{\varepsilon} \text{ if } \left|\underline{e}^T PB\right| \le \varepsilon \end{cases} \tag{22}$$

$$u = u_c + u_r \tag{23}$$

with adaptive law giving:

$$\begin{cases} \dot{\underline{\theta}}_f = \delta\underline{e}^T PB\xi_f(\underline{x}, t) \text{ if } \left(\left|\underline{\theta}_f\right| < M_f\right) \\ \text{or } \left(\left|\underline{\theta}_f\right| = M_f \text{ and } \underline{e}^T PB\underline{\theta}_f^T\xi_f(\underline{x}, t) \ge 0\right) \\ Proj(\delta\underline{e}^T PB\xi_f(\underline{x}, t)) \text{ otherwise} \end{cases} \tag{24}$$

The projection operator is given by:

$$Proj\left(\delta\underline{e}^T PB\xi_f(\underline{x}, t)\right) = \delta\underline{e}^T PB\xi_f(\underline{x}, t) - \delta\underline{e}^T PB\frac{\underline{\theta}_f^T\underline{\theta}_f\xi_f(\underline{x}, t)}{\left\|\underline{\theta}_f\right\|^2} \tag{25}$$

where δ is fixed adaptive gain.

Theorem 3.1. *Consider the controlled system (7) with type-2 FLS to approximate the unknown nonlinear function (6). If Assumptions 3.1-3.3 are true and if there is a matrix $P > 0$ satisfying the LMI: $A^T P + PA < 0$, then the closed-loop control system with optimal control signal defined by (23) and adaptive law defined by (24) and (25) are globally stable. Thus the proposed optimal type-2 fuzzy control scheme can guarantee that:*

i) All the variable of the closed-loop system are bounded.
ii) The tracking performance is achieved.

Proof: Consider the system (7)

$$x^{(n)} = f(\underline{x}, t) + bu(t) + d(t)$$

Substituting (20) and (23) into (7), the output error dynamics can be expressed as:

$$x^{(n)} - y_r^{(n)} + \sum_{i=0}^{n-1} c_i e^{(i)} = f(\underline{x}, t) - \hat{f}(\underline{x}, \underline{\theta}) + bu_r + d(t) \tag{26}$$

Then

$$e^{(n)} = -\sum_{i=0}^{n-1} c_i e^{(i)} + f(\underline{x}, t) - \hat{f}\,(\underline{x}, \underline{\theta}) + bu_r + d(t) \tag{27}$$

After some manipulations, the error dynamic can be represented by:

$$\dot{\underline{e}} = A\underline{e} + B(\Delta f(\underline{x}, t) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f) + bu_r + d(t)) \tag{28}$$

with $B \in \Re^n$, defined by:

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{29}$$

We use the Lyapunov approach in which u_r and the adaptive law are chosen so as to make the Lyapunov function decrease along the trajectories of the adaptive system. The controller design is based on the type-2 FLS and on the LMI optimization techniques.

If there exists a matrix $P > 0$, satisfying the following LMI:

$$A^T P + PA < 0 \tag{30}$$

where A, a stable matrix, is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & \cdots & \cdots & -c_{n-1} \end{bmatrix} \tag{31}$$

we consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\underline{e}^T P \underline{e} + \frac{1}{2\delta}\Phi^T \Phi \tag{32}$$

where $\Phi = \theta_f^* - \theta_f$ and δ is a positive constant specified by the designer.

The time derivative of V along the system trajectory is:

$$\dot{V} = \frac{1}{2}\underline{e}^T(A^TP + PA)\underline{e} + \underline{e}^TPB\Big[\Delta f(\underline{x}, t) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f^*) + \Delta\hat{f}(\underline{x}/\underline{\theta}_f^*) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f) + bu_r + d(t)\Big] + \frac{1}{\gamma}\Phi^T\dot{\Phi}$$

$$\dot{V} = \frac{1}{2}\underline{e}^T(A^TP + PA)\underline{e} + \underline{e}^TPB\Big[\Delta f(\underline{x}, t) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f^*) + \Phi^T\xi(\underline{x}) + bu_r + d(t)\Big] + \frac{1}{\gamma}\Phi^T\dot{\Phi}$$

$$\dot{V} = \frac{1}{2}\underline{e}^T(A^TP + PA)\underline{e} + \underline{e}^TPB\Big[\Delta f(\underline{x}, t) - \Delta\hat{f}(\underline{x}/\underline{\theta}_f^*) + bu_r + d(t)\Big] + \frac{1}{\delta}\Phi^T(\dot{\Phi} + \delta\underline{e}^TPB\xi(\underline{x}))$$

By consideration of the update law (24) and satisfying assumptions, $\dot{V}$ can be written as:

$$\dot{V} \le \frac{1}{2}\underline{e}^T(A^TP + PA)\underline{e} + \left|\underline{e}^TPB\right|\Big[|F| + \left|\Delta\hat{f}(\underline{x}, \underline{\theta}^*)\right| + d_{\max}\Big] + \underline{e}^TPBbu_r$$

with P verifying the LMI form: $A^TP + PA < 0$.

Then

$$\dot{V} \le \left|\underline{e}^TPB\right|\Big[|F| + \left|\Delta\hat{f}(\underline{x}, \underline{\theta}^*)\right| + d_{\max}\Big] + \underline{e}^TPBbu_r \tag{33}$$

If $\left|e^TPB\right| > \varepsilon$, $sat(e^TPB) = sgn(e^TPB)$.

Substituting u_r, defined in (21), into (33), then we have $\dot{V} \le 0$.

If $\left|e^TPB\right| \le \varepsilon$, $sat(e^TPB) = \frac{(e^TPB)}{\varepsilon}$, then $u_r = -\frac{1}{b}\frac{(e^TPB)}{\varepsilon}\Big[|F| + \left|\Delta\hat{f}(\underline{x}, \underline{\theta}^*)\right| + d_{\max}\Big]$.

And

$$\dot{V} \le \left|e^TPB\right|\Big[|F| + \left|\Delta\hat{f}(\underline{x}, \underline{\theta}^*)\right| + d_{\max}\Big] - \frac{(e^TPB)^2}{\varepsilon}\Big[|F| + \left|\Delta\hat{f}(\underline{x}, \underline{\theta}^*)\right| + d_{\max}\Big] \tag{34}$$

Therefore: $\dot{V} \le 0$.

4. **Simulations Results.** A large number of fuzzy control applications can be considered in the industrial processes and more specifically in the fields of automobiles and Renewable Energies. In this section, to show the performance of the presented controller we consider a classical test for uncertain nonlinear system. We test the type-2 adaptive fuzzy controller on the tracking control of a Duffing forced oscillation system described by the following equation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.1x_2 - x_1^3 + bu(t) + d(t) \\ y = x_1 \end{cases}$$

The objective of control is to maintain the system tracking the desired angle trajectory:

$$y_r(t) = \frac{\pi}{3}\sin(0.01t)$$

The coefficients of the Hurwitz polynomial are set as: $c_0 = 1$ and $c_1 = 7$.

The initial values of $x_1(0) = 1$ and $x_2(0) = 1$.

Let the adaptive gain $\gamma = 0.5$, $M_f = 1$, and $\varepsilon = 0.1$.

The external disturbance is represented by: $d(t) = 12 * \cos(t)$.

The solution of the LMI (30), using the Matlab Toolbox, is: $P = \begin{bmatrix} 1.2580 & 0.1769 \\ 0.1769 & 0.1068 \end{bmatrix}$.

The upper function F is $|x_1^3|$.

The membership functions for system state x_1 are represented in Figure 3. There are 3 rules to approximate the system function: $\Delta f(\underline{x}, t) = -x_1^3$.

Figure 4 demonstrates the tracking performance in presence of disturbances. The corresponding fuzzy control signal is given in Figure 5. The tracking error is illustrated in Figure 6.

5. **Conclusion.** In this paper, we presented an optimal adaptive type-2 fuzzy control for a class of nonlinear systems based on the Lyapunov synthesis approach and LMI. We introduced the type-2 FLS to approximate the unknown nonlinear term.

The main advantage of the proposed controller, interval type-2 FLC, is that it does not need any knowledge about the nonlinear term.

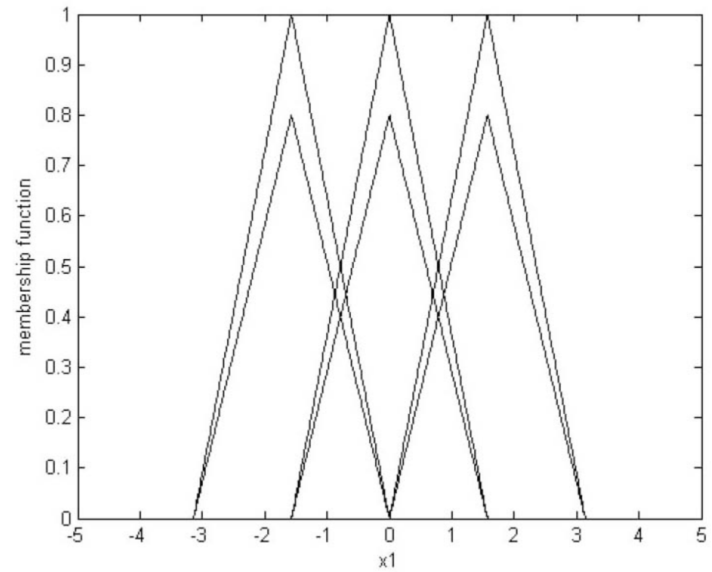

FIGURE 3. Type-2 membership functions used for the adaptive fuzzy controller

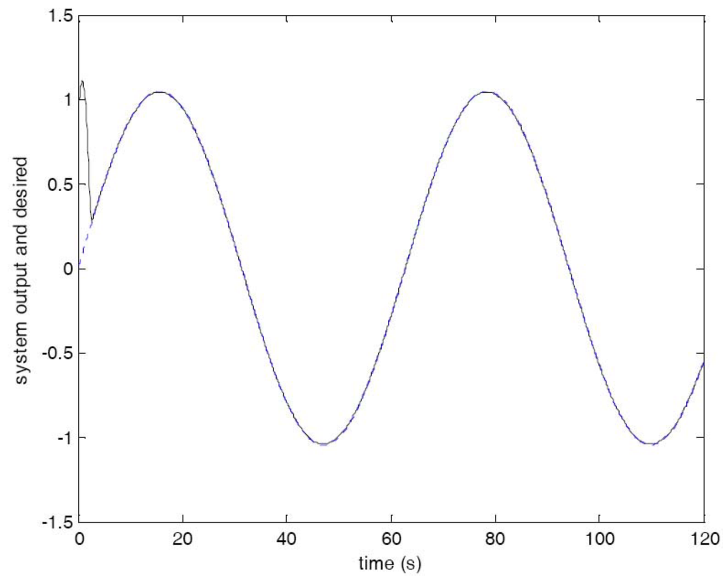

FIGURE 4. The state x_1 and its desired value x_d

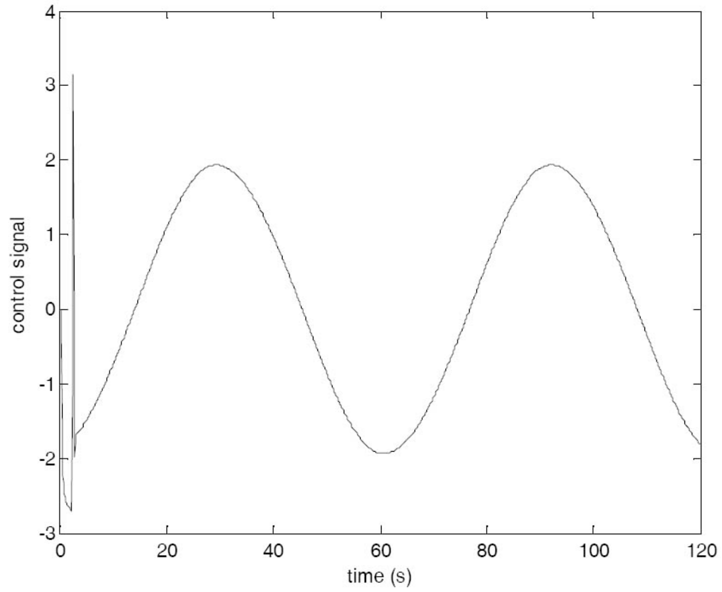

FIGURE 5. Control signal

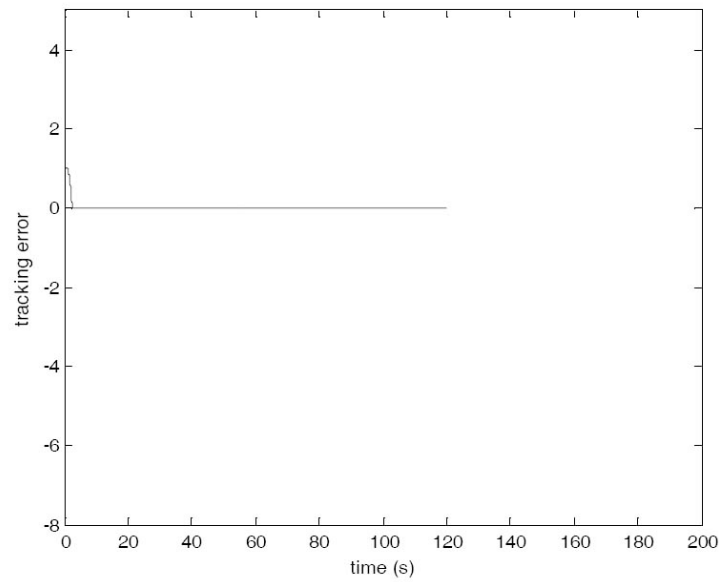

FIGURE 6. The tracking error

The auxiliary part of control is implemented to improve the system performance by suppressing the influence of external disturbance and removing the fuzzy approximation error. The synthesis of this signal relies on LMI and Lyapunov approach.

The simulation results have shown the effectiveness of the optimal adaptive type-2 fuzzy controller for a class of nonlinear systems in achieving the desired performance in the presence of external disturbances and fuzzy approximation error.

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