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NONLINEAR FILTER DESIGN FOR A CLASS OF POLYNOMIAL DISCRETE-TIME SYSTEMS: AN INTEGRATOR APPROACH

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Abstract. *This paper presents the filter design to estimate the state of polynomial discrete-time systems. The problem of designing filter for polynomial discrete-time systems is difficult because it is generally a nonconvex problem. More precisely, it is a nonconvex problem due to the fact that the relationship between the Lyapunov function and the filter gain is not jointly convex. The problem is related to the issue of bilinear matrix inequalities in linear systems, where the decision variables are not jointly convex, that is, there is a cross product of the decision variable. Hence, the problem cannot be solved by a semidefinite programming (SDP). Therefore, in this paper an integrator is proposed to be incorporated into the filter structure. By doing this, a convex solution can be guaranteed. The filter gains are then computed using SOS-SDP techniques. The solution has been established without any assumptions about nonlinear terms of the error dynamics.*

Keywords: Integrator, Polynomial discrete-time systems, Filter design, Sum of squares

1. Introduction.

Nonlinear systems play a vital role in the control systems engineering point of view due to the fact that in practice all plants are nonlinear in nature. The nonlinear systems have received a great deal of attention and many contributions have been made to this literature [1, 2, 3, 4]. However, the proposed approaches remain restrictive to particular classes of nonlinear models, and there is no general method for the analysis or synthesis of general nonlinear systems. That is the reason why nonlinear systems must be continued in research. Specifically, in this paper, we consider the polynomial system because this system constitutes an important class of nonlinear systems which has the advantage to describe the dynamical behavior of a large set of processes and its ability to approach any model of nonlinear systems by using polynomial expansion. That is why the polynomial systems attract considerable attention from control researchers to involve themselves in this area, especially on the stability analysis and controller synthesis of polynomial systems [5, 6, 7, 8, 9, 10, 11].

The filter design theory for linear system is first developed by Luenberger [12], who suggests a comprehensive and complete answer to the problem. The author showed how the available system inputs and outputs can be used to construct an estimation of the system state vector. Since then, a lot of works have been done in the framework of filter design for linear systems, see [13, 14, 15, 16, 17, 18].

However, filter design problem in nonlinear systems is much more challenging than its counterpart. Although it is hard to design a filter for nonlinear systems, some useful results can still be found in [19, 20, 21, 22]. In [19], new observer design methods were proposed for nonlinear systems with discrete measurements in the context of solving simultaneous nonlinear equation. The methods are continuous Newton method and Broyden's method. Meanwhile, in [20], the authors proposed a solution in terms of the Hamilton Jacobi Inequalities (HJIs) to solve the problem. However, it is well known that to solve HJIs is very difficult because there is no existing computational tool available to solve them. Moreover, in [21] an observer system is presented where the feedback interconnection of a linear system and a time-varying multivariable sector nonlinearity can be solved by linear matrix inequalities (LMIs). In [22], the paper presents a nonlinear observer design technique based on Lyapunov second method which produces an observer gain matrix that stabilizes the error dynamics for a class of nonlinear systems. The S-procedure method has been considered to solve this observer design problem.

In regard to the filter design for polynomial continuous-time systems, the result can be found in [23, 24, 25]. The authors in [23] addressed the filter design by using Lyapunov-type stochastic stability. Meanwhile, in [24, 25] H_∞ filter design for a class of continuous-time systems with sector-bounded nonlinearities had been presented. The S-procedure approach has been used to solve the sector-bounded nonlinearities and the convex optimization algorithm was utilized to obtain the solution. The sum of squares (SOS) decomposition method is utilized in both papers. The SOS decomposition method is first established by Parillo, see [5]. By using this method, the algorithmic analysis of nonlinear systems using Lyapunov method can be performed effectively because the conversion from SOS decomposition to the semidefinite programming (SDP) can be done in MATLAB using SOSTOOLS [26]. However, when talking about polynomial discrete-time systems, only few results are available, see [27, 28]. In [27, 28] the papers computed the filter and observer gains, the invariant sets, matrix sum of squares relaxation and semidefinite programming for discrete-time systems had been applied.

In this paper, the integrator method is proposed to ensure that a convex solution to the problem of filter design can be obtained. The filter design has been derived in terms of polynomial matrix inequalities (PMIs), which are formulated as SOS constraints. Then, to compute the filter gains, SOS techniques have been used to reduce the problems to semidefinite programming (SDP). The integrator method has been proposed in [29, 30, 31, 32] in the framework of controller design for polynomial discrete-time systems. Based on the results, this method is valuable to convexify the nonconvex filter design problem for polynomial discrete-time systems.

The contribution of this paper can be summarised as follows:

- A convex solution to the filter design problem is obtained in a less conservative way than the available approaches by introducing an integrator into the filter structure.
- In comparison to the work done in [27, 28], our proposed method yields a global solution to the filter design problem in polynomial discrete-time systems.

The paper is organized as follows. Section 2 provides system description and preliminaries. The main results are highlighted in Section 3. The next section gives a numerical example. Conclusions are given out in Section 5.

2. **System Description and Preliminaries.** Consider the following unforced polynomial discrete-time system:

$$\begin{aligned} x(k+1) &= A(x(k))x(k) \\ y &= C(x(k))x(k) \end{aligned} \tag{1}$$

where $x(k) \in \Re^{n\times 1}$ is the state vector, and $y \in \Re^{p\times 1}$ is the measurement output. Meanwhile, $A(x(k)) \in \Re^{n\times n}$ and $C(x(k)) \in \Re^{p\times n}$ are known polynomial matrices with appropriate dimension.

A filter to estimate the state $x(k)$ from y is selected to be of the following form:

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x}(k))\hat{x}(k) + L(\hat{x}(k))\big(y - \hat{y}\big) \\ \hat{y} &= C(\hat{x}(k))\hat{x}(k) \end{aligned} \tag{2}$$

where $\hat{x}(k) \in \Re^{n\times 1}$ is the state vector, and $\hat{y} \in \Re^{p\times 1}$ is the measurement output. Meanwhile, $A(\hat{x}(k)) \in \Re^{n\times n}$, $L(\hat{x}(k)) \in \Re^{n\times p}$ and $C(\hat{x}(k)) \in \Re^{p\times n}$ are known polynomial matrices with appropriate dimension. To study the convergence performance of the filter described in (2), we will look at the dynamics of the estimation error defined by $e = \hat{x}(k) - x(k)$. The error dynamic is then given as follows:

$$\begin{aligned} e(k+1) &= \hat{x}(k+1) - x(k+1) \\ &= A(\hat{x}) + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) - A(x)x \\ &= \big[A(\hat{x}) + L(\hat{x})C(\hat{x})\big]e \\ &\quad + \big[A(\hat{x}) - A(x) + L(\hat{x})C(\hat{x}) - L(\hat{x})C(x)\big]x \end{aligned} \tag{3}$$

Now, let $\tilde{e} = [e, x]^T \in \Re^{2n\times 1}$; therefore, the system in (3) can be re-written as follows:

$$\tilde{e}(k+1) = \phi(x, \hat{x})\tilde{e} \tag{4}$$

where,

$$\phi(x, \hat{x}) = \begin{bmatrix} A(\hat{x}) + L(\hat{x})C(\hat{x}) & A(\hat{x}) - A(x) + L(\hat{x})\big(C(\hat{x}) - C(x)\big) \\ 0 & A(x) \end{bmatrix} \tag{5}$$

where the dimension of $\phi(x, \hat{x}) \in \Re^{2n\times 2n}$.

Theorem 2.1. *Consider the system* (1)*, the error dynamic shown in* (4) *is asymptotically stable if there exist polynomial matrices* $L(\hat{x}) \in \Re^{n\times p}$ *and* $P(\tilde{e}) \in \Re^{2n\times 2n}$ *such that the following conditions are satisfied:*

$$P(\tilde{e}) > 0 \tag{6}$$

$$\begin{bmatrix} P(\tilde{e}) & \phi^T(x, \hat{x})P^T(\tilde{e}(k+1)) \\ P(\tilde{e}(k+1))\phi(x, \hat{x}) & P(\tilde{e}(k+1)) \end{bmatrix} > 0 \tag{7}$$

Proof: Let the Lyapunov function be selected as follows:

$$V(\tilde{e}) = (\tilde{e})^T P(\tilde{e})\tilde{e} \tag{8}$$

The difference between $V(\tilde{e}(k+1))$ and $V(\tilde{e}(k))$ along (4) with (2) is given below:

$$\begin{aligned} \Delta(V(\tilde{e})) &= V(\tilde{e}(k+1)) - V(\tilde{e}) \\ &= \tilde{e}^T(k+1)P(\tilde{e}(k+1))\tilde{e}(k+1) - \tilde{e}^T P(\tilde{e})\tilde{e} \\ &= \tilde{e}^T\big[\phi^T(x, \hat{x})P(\tilde{e}(k+1))\phi(x, \hat{x}) - P(\tilde{e})\big]\tilde{e} \end{aligned} \tag{9}$$

Suppose (7) is feasible, then multiplying it to the left by $diag[I, P(\tilde{e}(k+1))]$ and to the right by $diag[I, P^T(\tilde{e}(k+1))]$ and by applying the Schur complement, we have

$$\phi^T(x, \hat{x})P(\tilde{e}(k+1))\phi(x, \hat{x}) - P(\tilde{e}) < 0 \tag{10}$$

Knowing that (10) holds, then $\Delta V(\tilde{e}) < 0$, which implies that the error dynamic (4) with the filter (2) is globally asymptotically stable.

Remark 2.1. *Theorem 2.1 provides a sufficient condition for the existence of filter gains and given in terms of solutions to a set of parameterised PMIs. However, notice that the $P(\tilde{e}(k+1))$ appears in the PMIs; therefore, the inequalities are not jointly convex. This is because the cross products between the decision variables of $P(e)$ and the decision of $L(x)$ exist. If $P(\tilde{e})$ is fixed then (9) is convex in $L(x)$ and if $L(x)$ is fixed then (9) is convex in $P(\tilde{e})$. However, (9) is not jointly convex in $P(e)$ and $L(x)$. This issue is related to the issue of bilinear matrix inequalities in linear systems, where the decision variables are not jointly convex, that is, there is a cross product of the decision variables. One might think to select the Lyapunov matrix to be of $P(e)$ instead of $P(\tilde{e})$. However, such a selection does not help the solution to be convex because the problem remains persistent. Hence, to directly solve the Theorem 2.1 is hard because the PMIs need to be checked for all combination of $P(\tilde{e})$ and $L(\hat{x})$, which results in solving an infinite number of PMIs. In light of the aforementioned problem, in our work, an integrator is proposed to be incorporated into the filter dynamics. In doing so, a convex solution to the filter design problem in polynomial discrete-time systems can be rendered efficiently. The most detail of this integrator method is illustrated in the following section.*

3. Main Results.

In this section, the significance of incorporating an integrator into the filter dynamics will be illustrated.

A following nonlinear filter is proposed:

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{aligned} \tag{11}$$

where, $\hat{x} \in \Re^{n\times 1}$ and $x_f \in \Re^{n\times 1}$ is an augmented state.

Now, error is defined as follows:

$$\bar{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \hat{x} - x \\ x_f \end{bmatrix} \tag{12}$$

where the dimension of $e_1 \in \Re^{n\times 1}$, $e_2 \in \Re^{n\times 1}$ and $\bar{e} \in \Re^{2n\times 1}$.

The error dynamics is then given by

$$\begin{aligned} \bar{x}(k+1) = \begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} &= \begin{bmatrix} \hat{x}(k+1) - x(k+1) \\ x_f(k+1) \end{bmatrix} \\ &= \begin{bmatrix} A(\hat{x})\hat{x} + x_f - A(x)x \\ x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{bmatrix} \\ &= \begin{bmatrix} A(\hat{x})e_1 + e_2 + \big(A(\hat{x}) - A(x)\big)x \\ e_2 + L(\hat{x})C(\hat{x})e_1 + \big(L(\hat{x})C(\hat{x}) - L(\hat{x})C(x)\big)x \end{bmatrix} \end{aligned} \tag{13}$$

Next, let define $\check{e} = [e_1, x, e_2]^T \in \Re^{3n\times 1}$, and hence, the error dynamics described in (13) can be re-written as follows:

$$\check{e}(k+1) = \phi_2(x,\hat{x})\check{e} \tag{14}$$

where,

$$\phi_2(x,\hat{x}) = \begin{bmatrix} A(\hat{x}) & A(\hat{x}) - A(x) & 1 \\ L(\hat{x})C(\hat{x}) & L(\hat{x})C(\hat{x}) - L(\hat{x})C(x) & 1 \\ 0 & A(x) & 0 \end{bmatrix} \tag{15}$$

The dimension of $\phi_2(x,\hat{x}) \in \Re^{3n\times 3n}$.

Theorem 3.1. *Consider the system* (1), *the error dynamics shown in* (14) *is asymptotically stable if there exists a symmetric polynomial matrix* $P(e_1) \in \Re^{3n\times 3n}$, *polynomial matrices* $L(\hat{x}) \in \Re^{n\times p}$ *and* $G(\hat{x}) \in \Re^{3n\times 3n}$ *such that the following conditions are satisfied:*

$$P(e_1) > 0 \tag{16}$$

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x}) + G(\hat{x}) - P(e_1(k+1)) \end{bmatrix} > 0 \tag{17}$$

where,

$$G(\hat{x}) = \begin{bmatrix} g_{11}(\hat{x}) & g_{12}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{15}(\hat{x}) & g_{16}(\hat{x}) \\ g_{21}(\hat{x}) & g_{22}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{25}(\hat{x}) & g_{26}(\hat{x}) \\ g_{31}(\hat{x}) & g_{32}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{35}(\hat{x}) & g_{36}(\hat{x}) \\ g_{41}(\hat{x}) & g_{42}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{45}(\hat{x}) & g_{46}(\hat{x}) \\ g_{51}(\hat{x}) & g_{52}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{55}(\hat{x}) & g_{56}(\hat{x}) \\ g_{61}(\hat{x}) & g_{62}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{65}(\hat{x}) & g_{66}(\hat{x}) \end{bmatrix}$$

$$\text{and let} \quad G_2(\hat{x}) = \begin{bmatrix} g_{13}(\hat{x}) & g_{14}(\hat{x}) \\ g_{23}(\hat{x}) & g_{24}(\hat{x}) \end{bmatrix}. \tag{18}$$

The filter is given by

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{aligned} \tag{19}$$

where,

$$L(\hat{x}) = K(\hat{x})G_2^{-1}(\hat{x}) \tag{20}$$

Proof: Let the Lyapunov function be selected as follows:

$$V(\check{e}) = \check{e}^T P(e_1)\check{e} \tag{21}$$

Then, the difference between $V(\check{e}(k+1))$ and $V(\check{e}(k))$ along (14) with (11) is given below:

$$\begin{aligned} \Delta(V(\check{e})) &= V(\check{e}(k+1)) - V(\check{e}) \\ &= \check{e}^T(k+1)P(e_1(k+1))\check{e}(k+1) - \check{e}^T P(e_1)\check{e} \\ &= \check{e}^T\big[\phi_2^T(x,\hat{x})P(e_1(k+1))\phi_2(x,\hat{x}) - P(\check{e})\big]\check{e} \end{aligned} \tag{22}$$

Suppose (17) is feasible, thus $G^T(\hat{x}) + G(\hat{x}) > P(e_1(k+1)) > 0$. This implies that $G(\hat{x})$ is nonsingular. Since $P(e_1(k+1))$ is positive definite, the inequality

$$\big(P(e_1(k+1)) - G(\hat{x})\big)P^{-1}(e_1(k+1)) \times \big(P(e_1(k+1)) - G(\hat{x})\big)^T > 0 \tag{23}$$

holds. Therefore, establish

$$G(\hat{x}(k))P^{-1}(e_1(k+1))G^T(\hat{x}) \geq G(\hat{x}) + G^T(\hat{x}) - P(e_1(k+1)) \tag{24}$$

This immediately gives

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x})P^{-1}(e_1(k+1))G(\hat{x}) \end{bmatrix} > 0 \tag{25}$$

Next, by multiplying (25) on the right by $diag[I, G^{-1}(\hat{x}(k))]^T$ and on the left by $diag[I, G^{-1}(\hat{x}(k))]$, we get

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x}) \\ \phi_2(x,\hat{x}) & P^{-1}(e_1(k+1)) \end{bmatrix} > 0 \tag{26}$$

Then, by applying the Schur complement into (26), we have

$$\phi_2^T(x,\hat{x})P(e_1(k+1))\phi_2(x,\hat{x}) - P(e_1) < 0 \tag{27}$$

Knowing that (27) holds, then $\Delta V(\check{e}) < 0$, which implies that the error dynamic (14) with the filter (11) is globally asymptotically stable.

Remark 3.1. *One might think how the term $L(\hat{x}) = K(\hat{x})G_2^{-1}(\hat{x})$ can suddenly appear in Theorem 3.1. The fact is that a change-of-variable technique has been applied in the above proof, where $K(\hat{x}) = L(\hat{x})G_2(\hat{x})$. This is explicitly applied in Theorem 3.1. It is also important to note here that to allow the same value of $L(\hat{x})$ can be obtained, the polynomial matrix $G(\hat{x})$ must be enforced to be of a certain structure: see Equation* (18)*. Although, the $G(\hat{x})$ must be of a certain form, the results are still not too conservative because it is independent from the Lyapunov matrix.*

Remark 3.2. *The inequality* (17) *of Theorem 3.1 is convex. This is true because the terms in $P(e_1(k+1))$ are jointly convex. For clarity, refer to the following expansion version of $P(e_1(k+1))$,*

$$P(e_1(k+1)) = P\big[A(\hat{x})\hat{x} + x_f - A(x)x\big] \tag{28}$$

From (28)*, the x_f is an augmented state, hence the $P(e_1(k+1))$. Therefore, the Theorem 3.1 can be possibly solved via SDP.*

Unfortunately, to solve Theorem 3.1 is hard because we need to solve an infinite set of state-dependent PMIs. To relax these conditions, we utilise a SOS decomposition approach [5] and therefore the conditions given in Theorem 3.1 can be converted into SOS conditions and they are given by the following corollary:

Corollary 3.1. *Consider the system* (1)*, the error dynamics shown in* (14) *is asymptotically stable if there exists a symmetric polynomial matrix $P(e_1)$, polynomial matrices $L(\hat{x})$ and $G(\hat{x})$, and positive constant ϵ_1 and ϵ_2 such that the following conditions are satisfied:*

$$v_1^T[P(e_1) - \epsilon_1 I]v_1 \qquad \text{is a SOS} \tag{29}$$

$$v_2^T \begin{bmatrix} P(e_1) - \epsilon_2 I & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x}) + G(\hat{x}) - P(e_1(k+1)) - \epsilon_2 I \end{bmatrix} v_2 \qquad \text{is a SOS} \tag{30}$$

where, v_1 and v_2 are free vectors with appropriate dimensions,

$$G(\hat{x}) = \begin{bmatrix} g_{11}(\hat{x}) & g_{12}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{15}(\hat{x}) & g_{16}(\hat{x}) \\ g_{21}(\hat{x}) & g_{22}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{25}(\hat{x}) & g_{26}(\hat{x}) \\ g_{31}(\hat{x}) & g_{32}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{35}(\hat{x}) & g_{36}(\hat{x}) \\ g_{41}(\hat{x}) & g_{42}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{45}(\hat{x}) & g_{46}(\hat{x}) \\ g_{51}(\hat{x}) & g_{52}(\hat{x}) & g_{13}(\hat{x}) & g_{14}(\hat{x}) & g_{55}(\hat{x}) & g_{56}(\hat{x}) \\ g_{61}(\hat{x}) & g_{62}(\hat{x}) & g_{23}(\hat{x}) & g_{24}(\hat{x}) & g_{65}(\hat{x}) & g_{66}(\hat{x}) \end{bmatrix}$$

$$\text{and let} \quad G_2(\hat{x}) = \begin{bmatrix} g_{13}(\hat{x}) & g_{14}(\hat{x}) \\ g_{23}(\hat{x}) & g_{24}(\hat{x}) \end{bmatrix}. \tag{31}$$

Note that the dimension of $g_{11}(\hat{x})$ to $g_{66}(\hat{x})$ is $n \times n$ and $G_2(\hat{x})$ is $2n \times 2n$. Therefore, the filter is given by

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{aligned} \tag{32}$$

where,

$$L(\hat{x}) = K(\hat{x})G_2^{-1}(\hat{x}) \tag{33}$$

4. **Numerical Example.** The following polynomial system is considered:

$$x(k+1) = \begin{bmatrix} 1 & -0.01 \\ 0.01 + 0.01x_1x_2 & 1 - 0.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$y = [x_1; x_2] \tag{34}$$

Then, by applying Corollary 3.1 where the $P(e_1)$ is set to be of degree of 2, polynomial matrix $G(\hat{x})$ is in degree of 4, and polynomial matrix $L(\hat{x})$ is set to be degree of 6, a feasible solution is obtained. The results of the error between the estimation state and the actual state can be seen in Figures 1 and 2. The initial condition of the actual state is $x(0) = [1 \quad 1]$ and the estimation state is $\hat{x}(0) = [0.5 \quad 0.5]$.

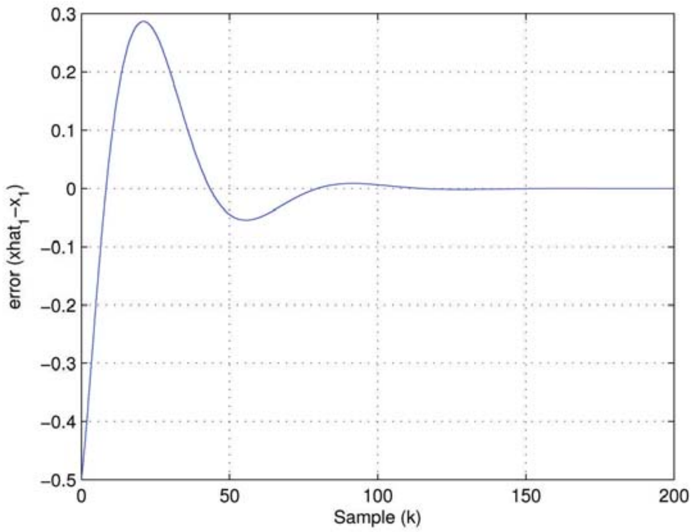

FIGURE 1. Trajectory of the $\hat{x}_1 - x_1$

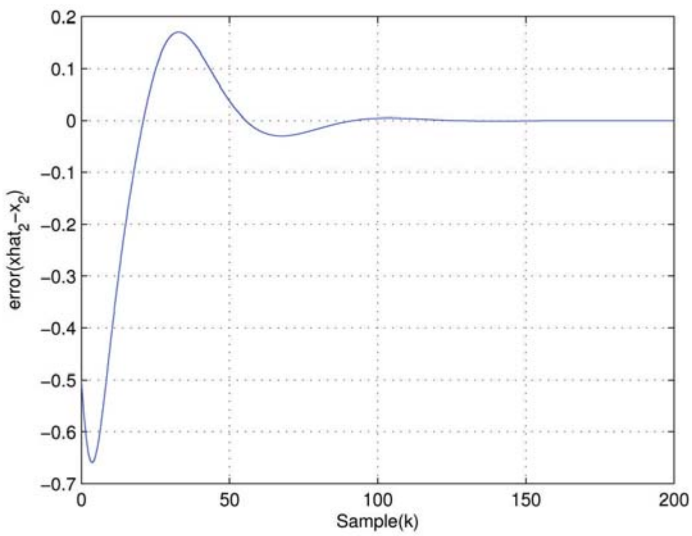

FIGURE 2. Trajectory of the $\hat{x}_2 - x_2$

Remark 4.1. *It should be mentioned here that the error between the actual state and the estimation state is quite large. This is due to the fact that the degree of the polynomial matrices $P(e_1)$, $G(\hat{x})$, and $K(\hat{x})$ cannot be increased further because of the limitation of the memory space of our machine. This significantly affects the feasibility of the solution because with the current set-up, the feasibility of the solution is very low which is* 0.15*. We believe that a better solution might be obtained by increasing the polynomial degree of the mentioned parameters, and consequently yields a better estimation.*

Remark 4.2. *The values of the polynomial matrices $P(e_1)$, $G(\hat{x})$, and $K(\hat{x})$ are omitted here due to large in size.*

5. **Conclusions.** The filter design of polynomial discrete-time systems has been studied in this paper. The non-convex issue of filter design has been encountered by incorporating an integrator into the filter structure. The existence of our proposed filter is given in terms of the solvability of the PMIs, which is formulated as SOS constraints and can be solved by any SOS solver. In this work, SOSTOOLS has been used to solve the SOS-PMIs. The SOS decomposition approach is utilized to solve the polynomial matrix inequalities and convert the nonconvex problem to the convex problem. Unlike the work performed in [27], our proposed methodology provides a solution to the global filter design. However, a current limitation of the proposed approach is the large computational cost. The future work that is important to be considered is to include the H_∞ performance and the uncertainty to the problem.

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