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IMPROVING THE STABILITY OF LIMITING ZEROS IN DISCRETE-TIME MODEL FOR NONCOLLOCATED MASS-DAMPER-SPRING MULTIVARIABLE SYSTEMS VIA GENERALIZED SAMPLE HOLD FUNCTION

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Abstract. *It is well-known that unstable zeros limit the achievable control performance. This paper is concerned with the stability of sampling zeros of discrete-time models for mass-damper-spring (MDS) systems with noncollocated sensors and actuators in the case of the generalized sample hold function (GSHF). The asymptotic behaviors of the sampling zeros, for a discrete-time model of a noncollocated MDS system with a GSHF, are analyzed for a sufficiently small sampling period T and the linear approximate expressions are also obtained. In addition, the linear approximate expressions with respect to the sampling period T lead to a sufficient condition for the sampling zeros of the discrete-time model to lie inside the unit circle for a sufficiently small sampling period T. It has been shown that the GSHF can locate the sampling zeros of discrete-time models for noncollocated MDS systems inside the stable region and improve stability properties when the zero-order hold (ZOH) cannot. Moreover, the examples are shown to demonstrate a significant improvement of control performance compared with the use of a ZOH.*
Keywords: Noncollocated mass-damper-spring, Sampling zeros, Stability, Generalized sample hold function

1. **Introduction.** The sampling process is a key element while obtaining sampled-data models to represent continuous-time control systems. The poles of sampled-data models are known to depend only on the sampling period and the poles of the underlying continuous-time system. The transformations of zeros, however, are much more complicated. For example, when a zero-order hold (ZOH) is used to generate the continuous-time system input, the sampled-data models have sampling zeros, which converge to specific locations as the sampling period tends to zero. Furthermore, it has been shown that, for a continuous-time system with at least one of the relative degrees higher than two, the resulting sampled-data models have the nonminimum phase (NMP) zeros for a sufficiently small sampling period T. These NMP zeros limit the control performance that can be achieved, and many techniques based on zero cancellation for control system design are hard to be applied [1, 2, 3].

Even in the case of a ZOH, the stability of zeros is not necessarily preserved in the sampling process except in a very special case: $T \to 0$. Therefore, the limiting zeros, which are the zeros of a discrete-time system in the limiting case when the sampling interval T

tends to zero, have attracted considerable attention, and the efforts have devoted to the analysis of limiting zeros in the earlier research studies [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Because such sampling zeros with ZOH may lie outside the unit circle for small sampling periods, it is common to have NMP discrete-time models, even if the underlying continuous-time system is minimum-phase (MP). These facts sparked interest in other holds such as a first-order hold (FOH) and a fractional-order hold (FROH). Hagiwara et al. [14] have carried out a comparative study and demonstrated that an FOH provides no advantage over ZOH as far as the stability of zeros of the resulting discrete-time systems is concerned. The FROH [15, 16, 17, 18, 19, 20, 21, 22] does yield better discretization zeros, and these known results have definitely shown that the limiting zeros of FROH can be placed inside the unit circle in some cases while ZOH fails to do so, but only within a limited margin, mainly because it has just one tuning parameter, which does not allow to place the discretization zeros as one wishes.

Though discrete system zeros with FROH have more advantage than those of ZOH from the viewpoint of stability, the unstable discretized zeros may still appear in the discrete-time models owing to the existence of sampling zeros of instability. For example, when we sample continuous-time systems having relative degree greater than two, unstable sampling zeros may be also generated by FROH even though the continuous-time system is of MP. To avoid this unstable sampling zeros or improve their stability, further ideas have been introduced such as multirate sampling control and digital control with the generalized sampled-data hold function (GSHF) [23, 24, 25, 26, 27, 28, 29, 30, 31]. Though some evils such as poor intersample behavior in the case of a GSHF cannot be avoided, GSHF can be used to solve many more ambitious control problems for linear system as long as it is formulated exclusively intersample terms. Moreover, in the linear systems, it is well known that GSHF can be also used to shift the zeros of sampled-data models for continuous-time systems because intersample ripples can be suppressed by using a linear-quadratic optimization [24] or alleviated efficiently by minimizing the variation of the control input [26].

Over the last three decades, zeros of mass-damper-spring (MDS) systems have been analyzed by some researchers [32, 33, 34, 35, 36, 37]. It has shown that the zeros are closely related to poles and always lie in the left-half plane in the case of the structure with collocated sensors and actuators. In the very motivating work by Ishitobi and Liang [37], the asymptotic properties of limiting zeros of discrete-time models for MDS systems with collocated sensors and actuators have been presented in order to derive further a stability condition of the limiting zeros for sufficiently small period. Moreover, their results are successfully applied to test the stability of zeros for collocated matrix second-order systems. However, the actual MDS systems usually have the noncollocated sensors and actuators in many engineering applications [35]. So far, the asymptotic behaviors of the limiting zeros of discrete-time models for MDS systems with noncollocated sensors and actuators still remain blank, and it is still interesting to find out their asymptotic properties though the impact of the stability of zeros in discrete-time models may not have been strong in control engineering applications. Hence, it is natural to raise the question of how the results of the MDS systems with collocated sensors and actuators can be extended to the case of collocated sensors and actuators.

In this paper, we present a discrete-time model in the case of a GSHF for the MDS systems with noncollocated sensors and actuators, and further analyze the improved asymptotic properties of limiting zeros for MDS discrete-time models proposed. An insightful interpretation of the obtained discrete-time model can be made in terms of sampling zeros, which have no continuous-time counterpart. We also give an explicit characterization of these sampling zeros and show that the approximate expressions of the limiting zeros for the noncollocated MDS discrete-time systems are derived as power series functions with respect to a sampling period. Moreover, we obtain new stability criteria as applications. Finally, two interesting examples are given to validate the main results.

2. **Preliminaries.** In general, consider an n-order, m-input and m-output system described by a state-space form as

$$S_C : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where a state vector $x(t) \in R^n$, an input vector $u(t) \in R^m$ and an output vector $y(t) \in R^m$.

Next, an $\aleph$-mode, m-input and m-output noncollocated multivariable MDS system is expressed as

$$\begin{cases} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Vu(t) \\ y(t) = W^T q(t) \end{cases} \tag{2}$$

where $q \in R^{\aleph}$ is the vector of generalized coordinates, $u \in R^m$ that of applied actuator inputs, $y \in R^m$ that of sensor outputs. Suppose that the mass, damping and stiffness matrices of the system satisfy $M = M^T > O$, $D = D^T \geq O$ and $K = K^T \geq O$, respectively, while the control influence matrices V and W are of full column rank. Further, we assume that

$$rank[D, V] = rank[K, V] = \aleph \tag{3}$$

In the following, it is possible to rewrite the system description (2) to the state-space form by taking the state variable as $x(t) = \begin{bmatrix} q^T(t) & \dot{q}^T(t) \end{bmatrix}^T$ and the form of (1). Namely, it is possible that

$$\begin{cases} A = \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} O \\ M^{-1}V \end{bmatrix} \\ C = \begin{bmatrix} W^T & O \end{bmatrix} \end{cases} \tag{4}$$

where $n = 2\aleph$ is the dimension of the state-space system (4). It is assumed in the rest of the paper that the system (4) is completely controllable and completely observable. Further, the transfer function $G_C(s)$ of linear MDS multivariable systems (4) is represented as:

$$G_C(s) = C(sI - A)^{-1}B = W^T(Ms^2 + Ds + K)^{-1}V \tag{5}$$

We are interested in the discrete-time model for the continuous-time MDS multivariable system (4) with GSHF. However, it is difficult to make a GSHF in practice because it is generally composed of exponential and sinusoidal functions. Thus, we consider a piecewise constant GSHF (PC GSHF) defined by piecewise constant impulse responses [24, 25, 26, 27, 30]

$$h(t) = \begin{cases} \alpha_1, & t \in \Big[0, \dfrac{T}{N}\Big), \\ \alpha_2, & t \in \Big[\dfrac{T}{N}, \dfrac{2T}{N}\Big), \\ \cdots & \cdots \\ \alpha_N, & t \in \Big[\dfrac{(N-1)T}{N}, T\Big). \end{cases} \tag{6}$$

Clearly, PC GSHF keeps a regular partition in time of sampling interval $[0, T)$ as in the case of the ZOH (see Figure 1). When multiplicity output of PC GSHF showed in Figure

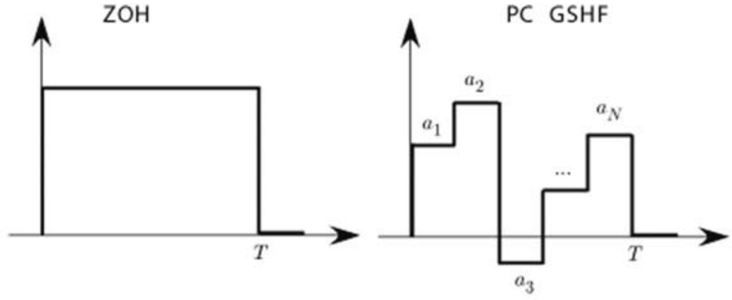

FIGURE 1. Pulse response of a ZOH and a PC GSHF

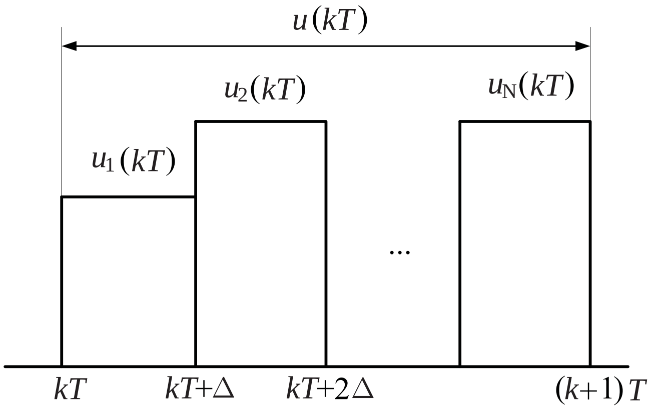

FIGURE 2. Multiplicity output of a PC GSHF

2 is considered, each sampling period T is equally divided into N subperiods of length $\Delta = \dfrac{T}{N}$ and the control input over the subinterval $[kT, \Delta]$ is described by

$$u(kT + \Delta) = u_j(kT), \quad \frac{(j-1)T}{N} \leq \Delta < \frac{jT}{N} \tag{7}$$

From (6) and (7), it can be rewritten as

$$u_j(kT) = \alpha_j u(kT), \quad j = 1, \ldots, N \tag{8}$$

where α_j is a real constant.

Remark 2.1. *PC GSHF can arbitrarily approximate any GSHF by taking N large, and can be readily implemented. The discrete-time system with ZOH, in general, may be considered as a particular case of PC GSHF when $N = 1$ or $\alpha_1 = \cdots = \alpha_N$.*

Let S_D be an MDS discrete-time model of a series connection of a GSHF, the continuous-time system with MDS and a sampler with a sampling period T. We consider the relationship between limiting zeros of the MDS discrete-time multivariable model and that of the origin MDS multivariable system (4) or (5), and the corresponding MDS discrete-time multivariable model (9) with PC GSHF is described in the following.

$$S_D : \begin{cases} \boldsymbol{x}\left((k+1)T\right) = \Phi \boldsymbol{x}(kT) + \Gamma_G \boldsymbol{u}(kT) \\ \boldsymbol{y}(kT) = \boldsymbol{C}\boldsymbol{x}(kT) \end{cases} \tag{9}$$

$$k = 0, 1, 2, \ldots$$

where

$$\Phi = e^{AT}, \quad \Gamma_G = \sum_{j=1}^{N} \Gamma_j \alpha_j, \quad \Gamma_j = \int_{(1-\frac{j}{N})T}^{(1-\frac{j-1}{N})T} e^{At} \boldsymbol{B} dt$$
$$j = 1, 2, \ldots, N$$

Further, the pulse transfer function matrix $G_D(z)$ for (9) is described by

$$G_D(z) = \boldsymbol{C}(zI - \Phi)^{-1}\Gamma_G \tag{10}$$

In addition, zeros of multivariable systems are defined in several ways. Multivariable zeros can be termed system zeros, invariant zeros, transmission zeros and so on. In spite of many differences and ambiguities, all those definitions of multivariable zeros refer or claim to be extensions to those for single-input single-output (SISO) systems. Then, the definition of invariant zeros, transmission zeros and system zeros for the system all coincides and some of the properties of zeros in SISO systems are inherited in the discretization process [38]. Thus, these zeros are simply called the zeros throughout this article. These zeros are the complex roots, including multiplicities, of $det\Gamma(s)$, where $\Gamma(s)$ denotes the pencil or the system matrix of S_C defined by

$$\Gamma(s) = \begin{bmatrix} A - sI_n & B \\ \boldsymbol{C} & O_m \end{bmatrix} \tag{11}$$

Similarly, the zeros of S_D possess the same properties as S_C does for a sufficiently small T [39], and are calculated using the system matrix $det\Gamma_T(z) = 0$ of S_D, where

$$\Gamma(s) = \begin{bmatrix} \Phi - zI_n & \Gamma_G \\ \boldsymbol{C} & O_m \end{bmatrix} \tag{12}$$

A zero of the continuous-time system is said to be stable or unstable if it has $R_e(s) < 0$ or $R_e(s) > 0$, and a zero of the discrete-time model is said to be stable or unstable if it has $|z| < 1$ or $|z| > 1$.

In this paper, the word 'an approximate zero of order M' is used as follows. If it holds for small sampling periods T that

$$z(T) = \overline{z}(T) + O(T^{M+1}) \tag{13}$$

where $z(T)$ denotes an exact value of a zero of a discrete-time model which depends on T, then a function $\overline{z}(T)$ is called an approximate zero of order M for $z(T)$.

3. **Main Results.** In this section, we first define the characterization of linear MDS multivariable system, such as the degrees of infinite elementary divisors and the number of continuous-time zeros. On the basis of the above analysis, we give the linear approximate asymptotic expressions of limiting zeros in the case of a PC GSHF with respect to the sampling period T for linear MDS multivariable discrete-time model. Finally, the new stability criteria of sampling zeros can be obtained by analyzing the asymptotic expansions.

From (4), we have straightforward by calculating

$$\begin{aligned} CB &= \begin{bmatrix} W^T & O \end{bmatrix} \begin{bmatrix} O \\ M^{-1}V \end{bmatrix} = O \\ CAB &= \begin{bmatrix} W^T & O \end{bmatrix} \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} O \\ M^{-1}V \end{bmatrix} \\ &= \begin{bmatrix} O & W^T \end{bmatrix} \begin{bmatrix} O \\ M^{-1}V \end{bmatrix} \end{aligned}$$

$$= W^T M^{-1} V \tag{14}$$

From Equation (14), the matrix M is positive definite, and the matrices V and W are both column full rank, so the following results hold: when it holds $|CAB| \neq 0$, the degrees of infinite elementary divisors of linear MDS multivariable system (4) or (5) are equal to three; when it holds $|CAB| = 0$, the degrees of infinite elementary divisors of linear MDS multivariable system (4) or (5) are greater than or equal to four. Without loss of generality, we consider that the degrees of infinite elementary divisors are equal to three in the proof of Theorem 3.1 in this section. In addition, it can be found in the MDS discrete-time model with PC GSHF that the system S_D (9) has $2\aleph - m$ zeros for almost all sampling period while the continuous-time multivariable system (4) has $2\aleph - 2m$ zeros. The zeros of S_D are classified into two categories, i.e., those that correspond to the zeros of the continuous-time system, and those that do not have any continuous-time counterparts. The former $2\aleph - 2m$ called intrinsic zeros and the latter m zeros are called sampling zeros.

Theorem 3.1. *Let r_i $(i = 1, \cdots, 2\aleph - 2m)$ be the zeros of the linear MDS multivariable system (2). Then, the corresponding intrinsic zeros z_i $(i = 1, \cdots, 2\aleph - 2m)$ are represented as*

$$z_i = 1 + r_i T + O(T^2) \tag{15}$$

and for the sampling zeros $z_{2\aleph-2m+i}$ $(i = 1, \cdots, m)$ generated in the sampling process, there is a relation

$$z_{2\aleph-2m+i} = \left(1 - \frac{2c_N^1(\alpha)}{c_N^2(\alpha)}\right) - \lambda_i T + O\left[(T^2)\right] \tag{16}$$

where λ_i $(i = 1, \cdots, m)$ denote the eigenvalues of the matrix $\dfrac{(3c_N^2(\alpha) - 2c_N^3(\alpha))c_N^1(\alpha)}{3[c_N^2(\alpha)]^2}\tilde{A}_{11}$, and $\tilde{A}_{11} = (-W^T M^{-1} D M^{-1} V)(W^T M^{-1} V)^{-1}$.

Proof: First, there exist non-singular matrices $P = \begin{bmatrix} P_{11} & P_{12} \\ O & P_{22} \end{bmatrix}$ and $Q = \begin{bmatrix} P_{11}^{-1} & O \\ Q_{21} & Q_{22} \end{bmatrix}$ which yield

$$\hat{\Gamma}(s) = P\Gamma(s)Q = \begin{bmatrix} \hat{A} - sI_n & \hat{B} \\ \hat{C} & O_m \end{bmatrix} \tag{17}$$

where P_{11}, P_{12}, P_{22}, Q_{21}, Q_{22} are matrices of dimension $n \times n$, $n \times m$, $m \times m$, $m \times n$, $m \times m$, respectively, and

$$\begin{aligned} \hat{A} &= \begin{bmatrix} O_m & I_m & O_{m\times(n-2m)} \\ O_m & O_m & O_{m\times(n-2m)} \\ O_{(n-2m)\times m} & O_{(n-2m)\times m} & A_f \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} O_m \\ I_m \\ O_{(n-2m)\times m} \end{bmatrix} \\ \hat{C} &= \begin{bmatrix} I_m & O_m & O_{m\times(n-2m)} \end{bmatrix} \end{aligned} \tag{18}$$

The matrix manipulation by P and Q does not change the values of the zeros and the eigenvalues of A_f coincide with the zeros of S_C.

For a discrete-time MDS system (9) in the case of a GSHF, there exist matrices $\tilde{P} = block - diag(P_{11}, P_{22})$ and $\tilde{Q} = block - diag(P_{11}^{-1}, Q_{22})$ which transform $\Gamma_T(z)$ to

$$\tilde{\Gamma}_T(z) = \tilde{P}\Gamma_T(z)\tilde{Q} = \begin{bmatrix} \tilde{\Phi} - zI_n & \tilde{\Gamma}_G \\ \hat{C} & O_m \end{bmatrix} \tag{19}$$

where $\tilde{\Phi} = e^{\tilde{A}T}$, $\tilde{A} = P_{11}AP_{11}^{-1}$, $\tilde{\Gamma}_G = \sum_{j=1}^{N} \alpha_j \tilde{\Gamma}_j$, $\tilde{\Gamma}_j = \int_{(1-\frac{j}{N})T}^{(1-\frac{j-1}{N})T} e^{\tilde{A}\tau}\hat{B}d\tau$. Similarly, the zeros of $\tilde{\Gamma}_T(z)$ are not changed by the matrix manipulation with $\tilde{P}$ and $\tilde{Q}$. Then we can express the matrix $\tilde{A}$ by use of block matrices such as

$$\begin{array}{cc} & \begin{array}{ccc} m & m & n-2m \end{array} \\ \tilde{A} = & \left[\begin{array}{ccc} \tilde{A}_{11} & I_m & O \\ \tilde{A}_{21} & O_m & \tilde{A}_{23} \\ \tilde{A}_{31} & O & A_f \end{array}\right] \end{array} \begin{array}{c} m \\ m \\ n-2m \end{array} \tag{20}$$

where $\tilde{A}_{ij}$ are some constant matrices.

Next, define the block matrices $\tilde{U} = block-diag(V^{-1}, U)$, $\tilde{V} = block-diag(V, T^{-1}I_m)$, $V = block-diag(I_{m-k}, TI_k, I_{n-m})$, $U = block-diag(I_{m-k}, T^{-1}I_k)$ and let

$$\bar{\Gamma}_T(z) = \tilde{U}\tilde{\Gamma}_T(z)\tilde{V} = \left[\begin{array}{cc} \bar{\Phi} - zI_n & \bar{\Gamma}_G \\ \hat{C} & O_m \end{array}\right] \tag{21}$$

where $\bar{\Phi} = e^{\bar{A}}$, $\bar{A} = V^{-1}\tilde{A}VT$, $\bar{\Gamma}_G = \sum_{j=1}^{N} \alpha_j \bar{\Gamma}_j$, $\bar{\Gamma}_j = \int_{(1-\frac{j}{N})T}^{(1-\frac{j-1}{N})T} e^{\bar{A}\tau}\hat{B}d\tau$.

Obviously, the roots of $\Gamma_T(z) = 0$ coincide with the roots of $\bar{\Gamma}_T(z) = 0$. Therefore, the limiting zeros of the linear MDS discrete-time system (9) in the case of a GSHF are obtained from Equation (21).

For a sufficiently small T, we consider the approximate expansion of $\bar{A}$ with respect to T up to the first-order term, and the matrix $\bar{A}$ can be expressed as

$$\bar{A} = \left[\begin{array}{ccc} \tilde{A}_{11}T & I_m & O \\ O_m & O_m & \tilde{A}_{23}T \\ O & O & A_fT \end{array}\right] + O\left[(T^2)\right] \tag{22}$$

The product of (22) leads to

$$\bar{A}^2 = \left[\begin{array}{ccc} O_m & \tilde{A}_{11}T & \tilde{A}_{23}T \\ O_m & O_m & O \\ O & O & O \end{array}\right] + O\left[(T^2)\right], \quad \bar{A}^3 = O\left[(T^2)\right] \tag{23}$$

Then, by simple straightforward calculation, we have

$$\begin{aligned} \bar{\Phi} &= I_n + \bar{A} + \frac{1}{2}\bar{A}^2 + O\left[(T^2)\right] \\ &= \left[\begin{array}{ccc} I_m + \tilde{A}_{11}T & I_m + \frac{1}{2}\tilde{A}_{11}T & \frac{1}{2}\tilde{A}_{23}T \\ O_m & I_m & \tilde{A}_{23}T \\ O & O & I_{n-2m} + A_fT \end{array}\right] + O\left[(T^2)\right] \end{aligned} \tag{24}$$

$$\begin{aligned} \bar{\Gamma}_G &= \left(c_N^1(\alpha)I_n + \frac{1}{2}c_N^2(\alpha)\bar{A} + \frac{1}{6}c_N^3(\alpha)\bar{A}^2\right)\hat{B} + O\left[(T^2)\right] \\ &= \left[\begin{array}{c} \frac{1}{2}c_N^2(\alpha)I_m + \frac{1}{6}c_N^3(\alpha)\tilde{A}_{11}T \\ c_N^1(\alpha)I_m \\ O \end{array}\right] + O\left[(T^2)\right] \end{aligned} \tag{25}$$

On the basis of the above analysis, substituting (24) and (25) into (21), it immediately obtains:

$$\left|\bar{\Gamma}_T(z)\right| = Q_0Q_1Q_2 \tag{26}$$

where Q_0 is a constant, and Q_1 and Q_2 represent respectively the asymptotic properties of sampling zeros and intrinsic zeros in the following:

$$Q_1 = \left(1 - \frac{2c_N^1(\alpha)}{c_N^2(\alpha)} - z\right) I_m - \frac{(3c_N^2(\alpha) - 2c_N^3(\alpha))c_N^1(\alpha)}{3[c_N^2(\alpha)]^2}\tilde{A}_{11}T + O\left[(T^2)\right] \quad (27)$$

$$Q_2 = (1-z)I_{n-2m} + A_f T + O\left[(T^2)\right] \quad (28)$$

where $\tilde{A}_{11} = (CA^2B)(CAB)^{-1}$. Hence, $(2\aleph - 2m)$ zeros z_i for $i = 1, \cdots, 2\aleph - 2m$ of S_D (9) can be expressed as $z_i = 1 + r_i T + O(T^2)$; furthermore, the remaining m zeros (sampling zeros) $z_{2\aleph-2m+i}$ for $i = 1, \cdots, m$ of S_D (9) have the form of $z_{2\aleph-2m+i} = \left(1 - \frac{2c_N^1(\alpha)}{c_N^2(\alpha)}\right) - \lambda_i T + O(T^2)$ for $i = 1, \cdots, m$, where λ_i $(i = 1, \cdots, m)$ denote the eigenvalues of the matrix $\frac{(3c_N^2(\alpha) - 2c_N^3(\alpha))c_N^1(\alpha)}{3[c_N^2(\alpha)]^2}\tilde{A}_{11}$.

As a result, the proof is complete.

Remark 3.1. *A proof of Theorem 3.1 is given for the case of the degrees of the infinite elementary divisors being three for simplicity of description. Further, it is straightforward to extend the proof to the general case such as the degrees of the infinite elementary divisors being four or five* $\cdots$.

Remark 3.2. *It is found from Theorem 3.1 that the intrinsic zeros of the linear MDS discrete-time system* (9) *converge to* $z = 1$*, and the sampling zeros are expressed approximately by the parameters* α_1*,* α_2*,* α_3 *and eigenvalues of the matrices* $\frac{(3c_N^2(\alpha) - 2c_N^3(\alpha))c_N^1(\alpha)}{3[c_N^2(\alpha)]^2}\tilde{A}_{11}$.

Remark 3.3. *In the proof of Theorem 3.1, for more details on this procedure, such as the structure of the matrices* P_{11} *and* P_{11}^{-1}*, it can see the review by Suda* [40] *on limiting zeros of linear time-invariant systems.*

Remark 3.4. *In Theorem 3.1, all of the zeros are stable, i.e., located strictly inside the unit circle when the equation* $\frac{\alpha_1 + \alpha_2 + \alpha_3}{5\alpha_1 + 3\alpha_2 + \alpha_3}$ *is greater than 0 for a sufficiently small sampling period* T*. Theorem 3.1 means that the limiting zeros of linear MDS discrete-time system* (9) *can be assigned inside the unit circle by choosing design parameters* α_i *of PC GSHF. Therefore, the PC GSHF with choosing the suitable parameters* α_i *can produce* $G_D(z)$ *with all stable zeros for a wider class of continuous-time plant* $G_C(s)$ *than ZOH.*

4. **Numerical Example.** Consider a 3-mode, 2-input, 2-output MDS system as Figure 3 with noncollocated actuators and sensors described by (2), where

$$D = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}, K = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, W^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$M = diag(1, 1, 1) \quad (29)$$

The system (2) with the parameters (29) has two zeros and the degrees of infinite elementary divisors are all three since it has readily seen that $CB = 0$ and $CAB = W^T M^{-1} V$ is nonsingular. The values of the two zeros are -0.7085 and -11.2915 and they are stable. Hence, the intrinsic zeros of the corresponding discrete-time system lie inside the unit circle for a sufficiently small sampling period. From [37], in the case of a ZOH, one of the sampling zeros is located inside the unit circle for a sufficiently small sampling period and the other outside. Now we consider the stability of limiting zeros

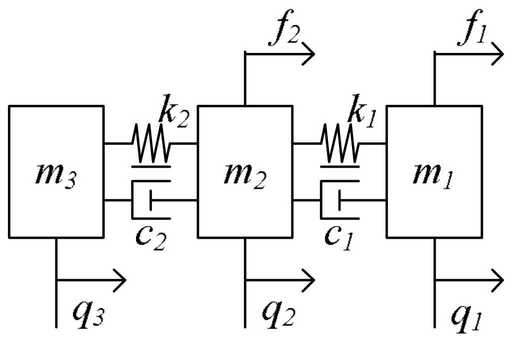

FIGURE 3. A 3-mode, 2-input, 2-output mass-damper-spring system

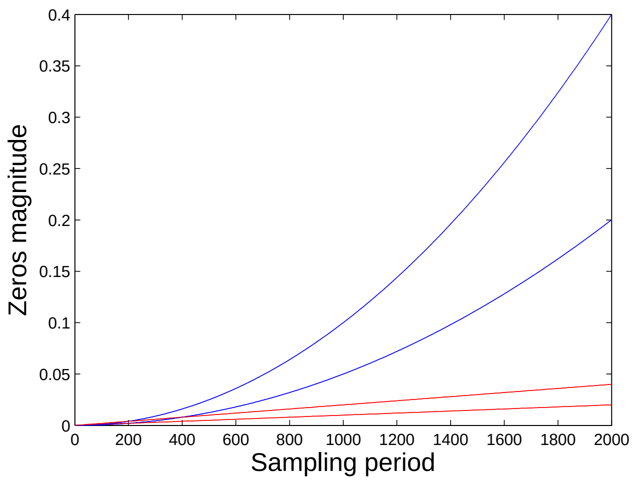

FIGURE 4. The magnitudes of zeros of the discrete-time system with PC GSHF for $T = 0.001$

for a discrete-time MDS system in the case of a GSHF. The magnitudes of zeros of the corresponding discrete-time model with PC GSHF are shown in Figure 4 for the sampling period $T = 0.001$. All the zeros stay inside the unit circle for the corresponding stability condition in Remark 3.4. The improvement of stability can be achieved by means of a suitable choice of the parameter α_j $(j = 1, 2, 3)$. Same as to PC GSHF can make an inverse stable discrete-time system when ZOH cannot.

The second MDS multivariable system with noncollocated actuators and sensors (2) is assumed to have the matrices

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

Then in this case, one of the degrees of infinite elementary divisors is greater than or equal to four since it holds that $CB = 0$ and $|CAB| = |W^T M^{-1} V| = 0$. Moreover, the system (2) with the parameters (30) has a stable zero -0.6667. When the MDS system (2) is discretized by a ZOH, the corresponding discrete-time system has one unstable sampling zero -3.73 and another stable sampling zero -0.268 for $T = 0.001$s [37]. Now, we consider the limiting zeros of sampled-data models for the MDS multivariable systems in the case of a PC GSHF. For system (2) with the parameters (30), loci of the absolute values of limiting zeros of the corresponding discrete-time systems by using a PC GSHF are shown in Figure 5. Furthermore, the good performance of the controlled system can

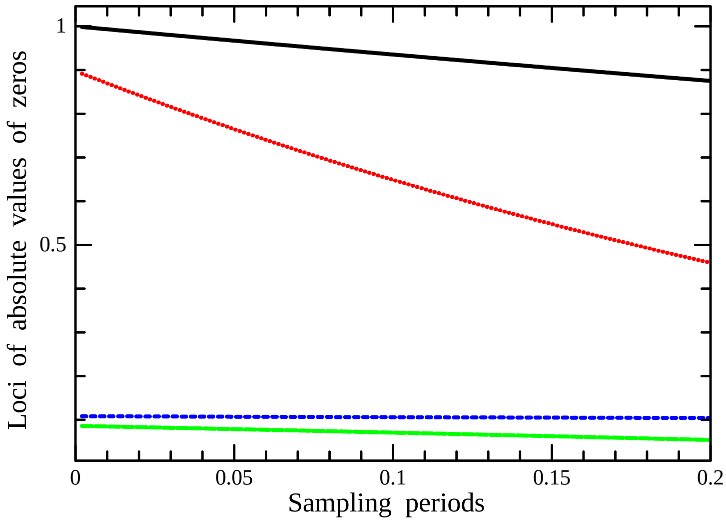

Figure 5. Loci of absolute values of limiting zeros of the discrete-time system with PC GSHF

be achieved in the case of the PC GSHF because the stability of the zeros of discrete-time system is significantly improved.

5. Conclusions.

This paper investigates the asymptotic behavior of the limiting zeros of MDS discrete-time multivariable systems with noncollocated sensors and actuators in the case of a PC GSHF. In this case, the corresponding discrete-time multivariable systems have m intrinsic zeros that approach the point $z = 1$ and $2\aleph - 2m$ sampling zeros which can be expressed approximately by the parameters α_1, α_2, α_3 and eigenvalues of the matrices $\dfrac{(3c_N^2(\alpha) - 2c_N^3(\alpha))c_N^1(\alpha)}{3[c_N^2(\alpha)]^2}\tilde{A}_{11}$. Moreover, the stability of the zeros of the discrete-time MDS multivariable systems for a sufficiently small sampling period is derived. As a result of this study, it has been shown that the zeros of the MDS discrete-time multivariable systems with PC GSHF can be located inside the stability region when the ZOH fails to do so, by choosing the suitable value of the design parameter α_j.

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