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LINEAR EXTENDED STATE OBSERVER BASED BACK-STEPPING CONTROL FOR UNCERTAIN SISO NONLINEAR SYSTEMS

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Abstract. *In this paper, a control scheme combining linear extended state observer (LESO) technique and back-stepping method is proposed for a class of single-input and single-output (SISO) nonlinear uncertain system with known linear dynamics, unknown nonlinear functions, internal uncertainties and external disturbance. And the system states are not available for measurement. An improved LESO was presented. Compared to the extended state in typical LESO structure, the one in this improved LESO does not contain the known linear terms. This handling approach can decrease control energy. This improved LESO can estimate simultaneously both the systems states and the total disturbances. Through the compensation function of the extended state, the original system can be changed into the linear system. Then we utilized the back-stepping method, which is different from nonlinear PID used during the typical ESO-based control design process, stabilizes the resulting linear system. Rigorous stability analysis shows that the system output asymptotically converges to zero. Finally, the numerical simulation results are given to illustrate the effectiveness of the proposed controller.*
Keywords: System control, Back-stepping control, Extended state observer, Nonlinear systems, Uniformly ultimately bounded

1. Introduction.

In the past few decades, the various adaptive robust control techniques with the high-precision performance have been rapidly developed for the nonlinear uncertain control systems. At present, the back-stepping methods have been widely applied to many control fields, such as spacecraft systems [1], mobile robots systems [2], and general systems [3]. The basic back-stepping control is usually realizable only under condition that the linear and nonlinear system dynamics are known. However, it is well known that the most of real system dynamics are uncertain, even stochastic [4-6]. So it is difficult to directly apply the basic back-stepping control to the uncertain plant with unknown function.

In addition, the system state variables may be partially unavailable in some case. At present, the various estimation techniques based control schemes have been developed to solve the kind of problems (see [7-13] and the references therein).

In recent years, the extended state observer (ESO), as a kind of important estimating technique, is of special interest. The ESO is designed first in [14] for a general n-dimensional SISO nonlinear system. The convergence of nonlinear ESO for SISO system is available only very recently [15]. In order to conveniently tune the parameter gains, [16] proposed the LESO for SISO system. The convergence of LESO for SISO system is investigated firstly in [17] and subsequently in [18]. According to existing results of ESO, the extended state must include all system dynamics. However, some linear dynamics terms contained in the extended state benefit sometimes to the system stability. So in the typical ESO's structure, the practice of this handling extended state probably leads to the energy loss of the control system. Thus it is important how to dispatch the linear terms from the extended state in the typical ESO's structure.

This paper studied the composite LESO-based control problem for a class of SISO nonlinear uncertain systems. The system model includes known linear dynamics, unknown nonlinear dynamics, model uncertainties and external disturbance. And the system states are not available for feedback. An improved LESO structure was developed. The extended state in this improved LESO's structure does not contain the known linear system dynamics compared to one in the typical ESO structure. Using the extended state, i.e., the estimation to the total disturbance compensates the plant model to complete the disturbance attenuation and rejection. As a result, the original system is changed into a linear system. In existing based ESO-based control results, the nonlinear PID controller was usually selected to stabilize the resulting system while we use the back-stepping control law here.

This paper is organized as follows. The problem formulation is presented in Section 2. The design and convergence analysis of LESO is included in Section 3. In Section 4, the back-stepping control law is designed. The simulation results to show the effectiveness of the proposed control algorithm are included in Section 5. Finally, the paper is concluded in Section 6.

2. Problem Statement.

Consider the following uncertain SISO nonlinear system:

$$\begin{cases} x^{(n)} = (b_1 + \Delta b_1)\,x + (b_2 + \Delta b_2)\,x' + \ldots + (b_n + \Delta b_n)\,x^{(n-1)} \\ \qquad\quad + f(x, \ldots, x^{(n-2)}, x^{(n-1)}, t) + (g + \Delta g)\,u + w(t) \\ \quad y = x \end{cases} \tag{1}$$

where $b_1, b_2, \ldots, b_n$ and g are known real constants, $u \in R$ and $y \in R$ are the input and the output, respectively, the unknown function $f(.)$ denotes the complicated nonlinear dynamics, the functions $\Delta b_1, \Delta b_2, \ldots, \Delta b_n$ and Δg are model uncertainties, and $w(t)$ is external disturbance. Denote that $W(t) = \Delta b_1 x + \Delta b_2 x' + \ldots + \Delta b_n x^{(n-1)} + f(.) + \Delta g u + w$. $W(t)$ is viewed as the total disturbances of the systems (1).

Concerning the system (1) we introduce the following assumptions.

A1) The unknown function $W(t)$ is differentiable, and its derivative function is Lebesgue measurable in t.

A2) Denote that $\frac{dW(t)}{dt} = h(t)$. The norm of the unknown function $h(t)$ is bounded by a known function $\rho(t) : R^+ \to R^+$, that is, for all $t \in R^+$, $\|h(t)\| \le \rho(t)$. Here the function $\rho(t) : R^+ \to R^+$ is bounded by the constant $\rho > 0$, that is, $\rho(t) \le \rho$ for all $t \in R^+$.

A3) The known function $\rho(t) : R^+ \to R^+$ is a Lebesgue measurable function in t. And given a compact interval $[a \quad b] \subset R^+$, the known function $\rho(t) : R^+ \to R^+$ is Lebesgue integral.

Equation (1) can be rewritten in the state space as

$$\begin{cases} \dot{x}_1 = x_2, \ldots, \dot{x}_{n-1} = x_n, \quad \dot{x}_n = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n + gu + W \\ y = x_1 \end{cases} \tag{2}$$

The nonlinear system (2) is assumed to be controllable, the input u is bounded, and the input gain g is assumed to be non-zero. Hence, without loss of generality, we assume that

$g > 0$. In this paper, the control objective lies in designing the LESO-based back-stepping control law to ensure that the system output is convergent to zero in the finite time.

3. LESO Design and Convergence Analysis. Define $x_{n+1}(t) = W(t)$ as the extended state in (2). The corresponding extended system of (2) is given as follows:

$$\begin{cases} \dot{x}_1 = x_2, \ldots, \dot{x}_{n-1} = x_n, \quad \dot{x}_n = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n + x_{n+1} + gu \\ \dot{x}_{n+1} = h \end{cases}$$

where $x = (x_1\ x_2\ \cdots\ x_{n+1})^T \in R^{n+1}$ represents the state of the extended system. The matrix form of the above extended system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_n & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} h \tag{3}$$

The LESO of the system (3) is proposed as follows

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \vdots \\ \dot{\hat{x}}_n \\ \dot{\hat{x}}_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_n & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_n \\ \hat{x}_{n+1} \end{bmatrix} + \begin{bmatrix} \varepsilon^{n-1} a_1 \\ \varepsilon^{n-2} a_2 \\ \varepsilon^{n-3} a_3 \\ \vdots \\ a_n \\ \frac{1}{\varepsilon} a_{n+1} \end{bmatrix} \left(\frac{x_1 - \hat{x}_1}{\varepsilon^n} \right) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g \\ 0 \end{bmatrix} u \tag{4}$$

where $\hat{x} = (\hat{x}_1\ \hat{x}_2\ \cdots\ \hat{x}_n\ \hat{x}_{n+1})^T \in R^{n+1}$ is the state vector of LESO (4), and the constant $\varepsilon > 0$ can be designed later. Denote that $\Phi = \begin{bmatrix} -a_1 & 1 & \cdots & 0 & 0 \\ -a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_n + \varepsilon^n b_1 & \varepsilon^{n-1} b_2 & \cdots & \varepsilon b_n & 1 \\ -a_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix}$. The parameter gains $a_1, a_2, \ldots, a_n, a_{n+1}$ and ε are required to satisfy

$$\Phi^T P + P\Phi = -2Q \tag{5}$$

for some symmetric positive definite matrixes P and Q.

Remark 3.1. *Note that the parameter matrix of LESO (4) is different from one given in reference [16]. In addition, we selected the total disturbance* $W(t) = \Delta b_1 x + \Delta b_2 x' + \ldots + \Delta b_n x^{(n-1)} + f(.) + \Delta g u + w$ *as the extended state, that is* $x_{n+1}(t) = W(t)$. *So the extended state does not include the known linear dynamics term* $b_1 x + b_2 x' + \ldots + b_n x^{(n-1)}$. *This tackling technique is major different from one of the typical ESO given in [16].*

Introduce the observing errors: $Z = (z_1\ z_2\ \cdots\ z_{n+1})^T$ with $z_i = x_i - \hat{x}_i$, $i = 1, 2, \ldots, n+1$. Subtracting (4) from (3) leads to the error dynamics system as follows

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \\ \dot{z}_{n+1} \end{bmatrix} = \begin{bmatrix} -\varepsilon^{-1} a_1 & 1 & \cdots & 0 & 0 \\ -\varepsilon^{-2} a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon^{-n} a_n + b_1 & b_2 & \cdots & b_n & 1 \\ -\varepsilon^{-(n+1)} a_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ z_{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h \end{bmatrix} \tag{6}$$

Theorem 3.1. *Consider the error system (6) satisfying Assumptions A1)-A3). Then the following results hold.*

(I) Given any $(Z_0, t_0) \in R^{n+1} \times R^+$, *there exists a continuous solution:* $Z(t) : [t_0 \quad t_1] \to R^{n+1}$ *with* $Z(t_0) = Z_0$.

(II) The solution of Equation (6) $Z(t) : [t_0 \quad t_1] \to R^{n+1}$, $Z(t_0) = Z_0$, *is uniformly ultimately bounded. Specifically, we have* $\|Z(t)\| \le \kappa(\varepsilon)$ *for* $t \ge t_f(\varepsilon)$, *where* $t_f(\varepsilon)$ *is a finite time and* $\kappa(\varepsilon) = \frac{2\lambda_{\max}(P)\|P\|\rho}{\lambda_{\min}(Q)\lambda_{\min}(P)}\varepsilon$. *(Appendix details the proof of Theorem 3.1)*

So the LESO (4) can estimate both states and extended state of the system (3), that is,

$$\hat{x}_1 \to x_1,\ \hat{x}_2 \to x_2,\ \dots,\ \hat{x}_n \to x_n,\ \hat{x}_{n+1} \to x_{n+1} = W(t). \tag{7}$$

Remark 3.2. *It follows from Theorem 3.1 that the estimation error enters the closed ball* $\{Z : \|Z\| \le \kappa(\varepsilon)\}$ *after a finite time* $t_f(\varepsilon)$. *It is easy to verify that the radius of the above closed ball can be adjusted by the design parameter* ε, *and because of* $\lim_{\varepsilon \to 0^+} \kappa(\varepsilon) = 0$, *the estimation error* Z *converges to the origin as* ε *goes to zero. It should be pointed out that the proposed LESO can be feasible in case of the extended state including all linear dynamics terms.*

Remark 3.3. *To eliminate the peaking phenomenon coming from the high gain, we introduce saturation on the signal* $\hat{x} = (\hat{x}_1\ \hat{x}_2 \cdots \hat{x}_n\ \hat{x}_{n+1})^T$ *such that* $\hat{x}^s = (\hat{x}_1^s\ \hat{x}_2^s \cdots \hat{x}_n^s\ \hat{x}_{n+1}^s)^T$, *where* $\hat{x}^s = \left(d_1 \mathrm{sat}\left(\frac{\hat{x}_1}{S_1}\right)\ d_2 \mathrm{sat}\left(\frac{\hat{x}_2}{S_2}\right)\ \cdots\ d_n \mathrm{sat}\left(\frac{\hat{x}_n}{S_n}\right)\ d_{n+1} \mathrm{sat}\left(\frac{\hat{x}_{n+1}}{S_{n+1}}\right)\right)^T$ *with* $S_i > d_i$, $i = 1, 2, \dots, n, n+1$. *Here* d_i *denote the upper boundedness of* $\hat{x}_i$, *that is,* $|\hat{x}_i| \le d_i$, $i = 1, 2, \dots, n, n+1$.

4. **Back-stepping Control Design.**

In this section, we develop the back-stepping control scheme for the case where all system states and total disturbance are available using LESO approach.

Step 1: At this step, the first equation in Equation (2) is considered, i.e., $\dot{x}_1 = x_2$. The virtual control law x_2^* is designed as $x_2^* = -k_1\chi_1$, where $k_1 > 0$, $\chi_1 = x_1$.

Step 2: Consider $\dot{x}_2 = x_3$. Letting $\chi_2 = x_2 - x_2^*$, we have $\dot{\chi}_2 = x_3 - \frac{\partial x_2^*}{\partial x_1}x_2$. Choose a virtual control law x_3^* as $x_3^* = -\chi_1 - k_2\chi_2 + \frac{\partial x_2^*}{\partial x_1}x_2$, where $k_2 > 0$.

Step i: Consider $\dot{x}_i = x_{i+1}$. Letting $\chi_i = x_i - x_i^*$, we have $\dot{\chi}_i = x_{i+1} - \sum_{j=1}^{i-1} \frac{\partial x_i^*}{\partial x_j}x_{j+1}$. Choose a virtual control law x_{i+1}^* as $x_{i+1}^* = -\chi_{i-1} - k_i\chi_i + \sum_{j=1}^{i-1} \frac{\partial x_i^*}{\partial x_j}x_{j+1}$, where $k_i > 0$.

Step n: The final control law is designed in this step. Consider $\dot{x}_n = b_1x_1 + b_2x_2 + \dots + b_nx_n + W + u$.

Letting $\chi_n = x_n - x_n^*$, we have $\dot{\chi}_n = b_1x_1 + b_2x_2 + \dots + b_nx_n + W + gu - \sum_{j=1}^{n-1} \frac{\partial x_n^*}{\partial x_j}x_{j+1}$. The final control law is designed as

$$u = 1/g\left(-\hat{\chi}_{n-1} - k_n\hat{\chi}_n - b_1\hat{x}_1 - b_2\hat{x}_2 - \dots - b_n\hat{x}_n - \hat{x}_{n+1} + \sum_{j=1}^{n-1} \frac{\partial \hat{x}_n^*}{\partial \hat{x}_j}\hat{x}_{j+1}\right) \tag{8}$$

where $k_n > 0$, $\hat{\chi}_1 = \hat{x}_1$, $\hat{\chi}_2 = \hat{x}_2 - \hat{x}_2^*, \dots, \hat{\chi}_i = \hat{x}_i - \hat{x}_i^*, \dots, \hat{\chi}_{n-1} = \hat{x}_{n-1} - \hat{x}_{n-1}^*$, $\hat{\chi}_n = \hat{x}_n - \hat{x}_n^*$, $\hat{x}_2^* = -k_1\hat{\chi}_1$, $\hat{x}_3^* = -\hat{\chi}_1 - k_2\hat{\chi}_2 + \frac{\partial \hat{x}_2^*}{\partial \hat{x}_1}\hat{x}_2, \dots, \hat{x}_{i+1}^* = -\hat{\chi}_{i-1} - k_i\hat{\chi}_i + \sum_{j=1}^{i-1} \frac{\partial \hat{x}_i^*}{\partial \hat{x}_j}\hat{x}_{j+1}, \dots, \hat{x}_n^* = -\hat{\chi}_{n-2} - k_{n-1}\hat{\chi}_{n-1} + \sum_{j=1}^{n-2} \frac{\partial \hat{x}_{n-1}^*}{\partial \hat{x}_j}\hat{x}_{j+1}$.

Remark 4.1. *The composite controller (8) has been designed based on the back-stepping method and the LESO technique. The leading difference from the conventional back-stepping method is that the disturbance estimations are introduced in the final control law to compensate the total disturbances at nth step.*

Remark 4.2. *The back-stepping control scheme (8) is a function of the LESO's outputs* $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{x}_{n+1}$. *Since the inputs of LESO (4) are* x_1 *(or* y*) and* u. *So the proposed back-stepping control scheme only requires the output* y *of the system (1) to be measureable.*

The above design procedure of the back-stepping control can be summarized in the following theorem.

Theorem 4.1. *Considering the uncertain nonlinear system (1). The LESO is designed as (4)-(5). Under the proposed back-stepping control scheme (8), the output* y *asymptotically converges to zero.*

Proof: The closed-loop can be described as

$$\begin{cases} \dot{\chi}_1 = \chi_2 - k_1\chi_1, \dot{\chi}_2 = \chi_3 - k_2\chi_2 - \chi_1, \dot{\chi}_3 = \chi_4 - k_3\chi_3 - \chi_2, \ldots, \\ \dot{\chi}_{n-1} = \chi_n - k_{n-1}\chi_{n-1} - \chi_{n-2} \\ \dot{\chi}_n = b_1x_1 + b_2x_2 + \ldots + b_nx_n + W - \sum\limits_{j=1}^{n-1} \frac{\partial x_n^*}{\partial x_j} x_{j+1} - \hat{\chi}_{n-1} - k_n\hat{\chi}_n - b_1\hat{x}_1 \\ \qquad -b_2\hat{x}_2 - \ldots - b_n\hat{x}_n - \hat{x}_{n+1} + \sum\limits_{j=1}^{n-1} \frac{\partial \hat{x}_n^*}{\partial \hat{x}_j}\hat{x}_{j+1} \end{cases} \tag{9}$$

It obtains from (7) that $\hat{\chi}_1 \to \chi_1, \hat{\chi}_2 \to \chi_2, \ldots, \hat{\chi}_n \to \chi_n, \hat{x}_1^* \to x_1^*, \hat{x}_2^* \to x_2^*, \ldots, \hat{x}_n^* \to x_n^*$. Ignoring the estimation errors in $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ and $\hat{x}_{n+1}$ leads to $\dot{\chi}_n = b_1x_1 + b_2x_2 + \ldots + b_nx_n + W - \sum\limits_{j=1}^{n-1} \frac{\partial x_n^*}{\partial x_j} x_{j+1} - \hat{\chi}_{n-1} - k_n\hat{\chi}_n - b_1\hat{x}_1 - b_2\hat{x}_2 - \ldots - b_n\hat{x}_n - \hat{x}_{n+1} + \sum\limits_{j=1}^{n-1} \frac{\partial \hat{x}_n^*}{\partial \hat{x}_j}\hat{x}_{j+1} = -\chi_{n-1} - k_n\chi_n$. So the closed-loop system (9) can be rewritten as

$$\begin{cases} \dot{\chi}_1 = \chi_2 - k_1\chi_1, \dot{\chi}_2 = \chi_3 - k_2\chi_2 - \chi_1, \dot{\chi}_3 = \chi_4 - k_3\chi_3 - \chi_2, \ldots, \\ \dot{\chi}_{n-1} = \chi_n - k_{n-1}\chi_{n-1} - \chi_{n-2}, \dot{\chi}_n = -k_n\chi_n - \chi_{n-1} \end{cases} \tag{10}$$

Denote that $\chi = [\chi_1 \ \chi_2 \ \ldots \ \chi_n]^T$. Consider the Lyapunov function candidate $V = \frac{1}{2}\chi^T\chi$. The time derivative of V along the solutions of (10) is $\dot{V} = -k_1\chi_1^2 - k_2\chi_2^2 - \ldots - k_n\chi_n^2$. Thus we can know that the closed-loop system (10) is asymptotically stable. Furthermore, the output y asymptotically converges to zero. The proof of the theorem is complete.

5. **Simulation Study.** In this section, the simulation results are given to illustrate the effectiveness of the proposed control techniques. We consider a second-order system modeled by $\ddot{x} = (-2 + \Delta_1(t))x + (3 + \Delta_2(t))\dot{x} + f(\dot{x}, x, t) + (1 + \Delta g)u + w$. Its state-space form is described as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -2 + \Delta b_1 & 3 + \Delta b_2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ f(x_1, x_2, t) + w + \Delta gu \end{bmatrix} \tag{11}$$

where the model uncertainties $\Delta b_1 = 0.02\sin(\pi t)$, $\Delta b_2 = 0.02\cos(\pi t)$, $\Delta \text{g} = 0.01\sin(\pi t)$, the system nonlinear dynamics function $f = -3x_1x_2^2$, the external disturbance $w(t) = 2\sin(0.1\pi t) + 3\sin(0.2\sqrt{t+1})$. So the total disturbance $W = \Delta_1x_1 + \Delta_2x_2 + \Delta gu + f + w = 0.02\sin(\pi t)x_1 + 0.02\cos(\pi t)x_2 + 0.01\sin(\pi t)u - 3x_1x_2^2 + 2\sin(0.5\pi t) + 3\sin(0.2\sqrt{t+1})$. To proceed the design of LESO and back-stepping controller, the design parameters are chosen as $\varepsilon = 0.01$, $a_1 = 2.8252$, $a_2 = 4.4085$, $a_3 = 0.9129$, $k_1 = 20$, $k_2 = 36$. The initial states are arbitrarily chosen as $x_1 = -0.045$, $x_2 = 0.06$. The LESO and the back-stepping controller are, respectively, given by $\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$ $+ \begin{bmatrix} 282.52 \\ 44085 \\ 912900 \end{bmatrix} (x_1 - \hat{x}_1) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$ and

$$u(t) = -\chi_1(t) - 36\chi_2(t) - 2\hat{x}_1(t) - 3\hat{x}_2(t) - \hat{x}_3(t) + \frac{\partial x_2^*(t)}{\partial \hat{x}_1(t)} \hat{x}_2(t) \tag{12}$$

with $\chi_1(t) = \hat{x}_1(t)$, $x_2^*(t) = -20\chi_1(t)$, $\chi_2(t) = \hat{x}_2(t) - x_2^*(t)$, $\frac{\partial x_2^*(t)}{\partial \hat{x}_1(t)} = -20$.

Figures 1 and 2 give the simulation results which are obtained by applying the controller (12) to the system (11). In Figure 1, the plot (a) shows the system output profile, and the plot (b) shows the control law profile, respectively. From the plot (a) in Figure 1, we can see that the system output is convergent to zero with small error except some spikes caused by the high gain in LESO. In addition, the plot (b) in Figure 1 shows that the control magnitude is also small. Figure 2 shows the tracking performance of LESO. From Figure 2, we can see that the outputs of LESO can closely track the system states and total disturbances with very small errors. So the replacements of system states and total disturbances with their estimations are valid during the design of control law.

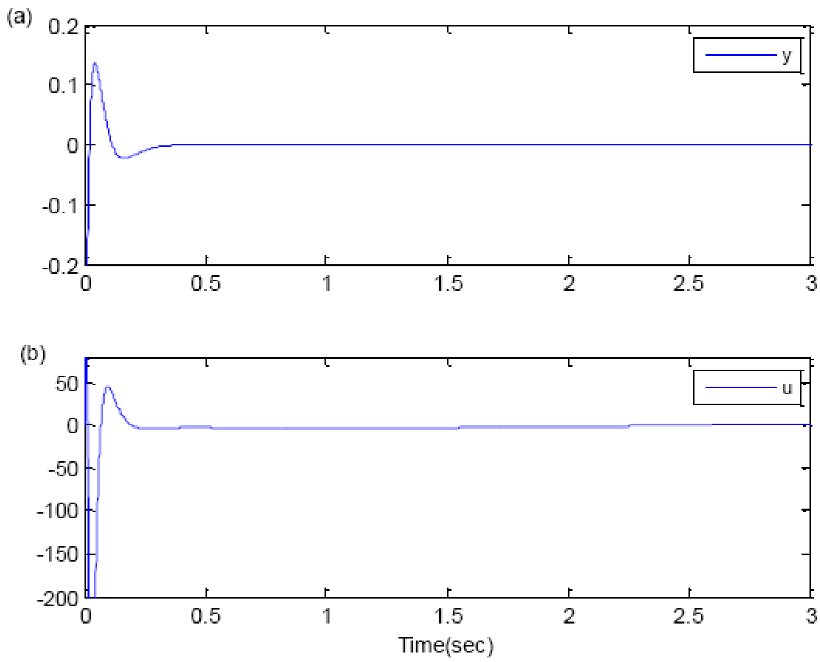

FIGURE 1. Trajectories of output y as well as control law u

6. **Conclusions.** In this paper, a composite LESO-based backstepping control scheme has been presented for a class of SISO uncertain systems. Compared to the typical LESO, the extended state in this improved LESO structure does not contain the known linear terms. This improved LESO can estimate simultaneously both the systems states and the total disturbances. The compensation function of the extended state can change the original system into the linear system. The proposed back-stepping method, which is different from the nonlinear PID used during the typical ESO-based control design process,

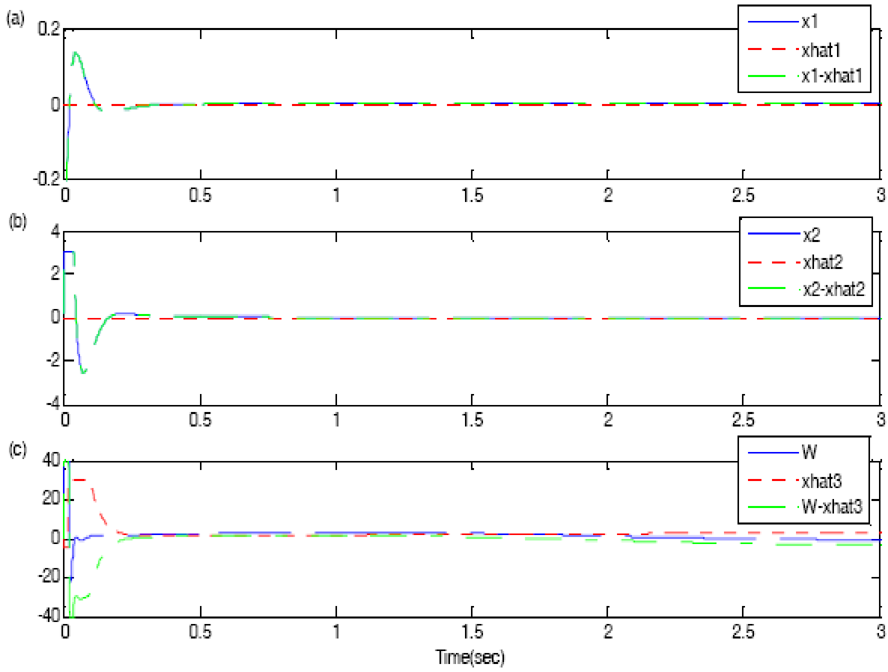

FIGURE 2. Tracking performance of LESO

can effectively stabilize the resulting linear system. The proposed control method is an alternative way to solve the control problem for the SISO nonlinear plant model with uncertainty. The simulation results illustrate the effectiveness of the proposed control scheme.

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Appendix. Proof of Theorem 3.1

Proof: (I) Let $\bar{\Phi} = \begin{bmatrix} -\varepsilon^{-1}a_1 & 1 & \cdots & 0 & 0 \\ -\varepsilon^{-2}a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon^{-n}a_n + b_1 & b_2 & \cdots & b_n & 1 \\ -\varepsilon^{-(n+1)}a_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix}$, $\Psi(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ h(t) \end{bmatrix} \in R^{n+1}$.

Consider the function $H(Z,t) = \bar{\Phi}Z(t) + \Psi(t)$. As a consequence of assumptions A1)-A3), $H(Z,t)$ is a Caratheodory function, that is, it is continuous in Z for all $t \in R$ and Lebesgue measurable for all $Z \in R^{n+1}$ in t. Given a compact set E of R^{n+1} and a compact interval $[a \quad b]$, there exists a Lebesgue integral function $M(t) : [a \quad b] \to R$ such that $\|H(Z,t)\| \le M(t)$ for all $(Z,t) \in E \times [a \quad b]$. Thus, given (Z_0, t_0), E and $[a \quad b]$ such that $Z_0 \in \text{int}E$ and $t_0 \in [a \quad b]$, there exists a continuous solution, $Z(t) : [t_0 \quad t_1] \to R^{n+1}$, $Z(t_0) = Z_0$. (see [19]).

(II) For every $i = 1, 2, \ldots, n+1$, set $\eta_i(t) = \frac{z_i(\varepsilon t)}{\varepsilon^{n+1-i}}$, $\eta = (\eta_1 \ \ldots \ \eta_{n+1})^T$. It follows from (3) and (4) that for every $i \in \{1, 2, \ldots, n-1\}$, η_i satisfies

$$\begin{aligned} \frac{d\eta_i(t)}{dt} &= \frac{d}{dt}\left(\frac{x_i(\varepsilon t) - \hat{x}_i(\varepsilon t)}{\varepsilon^{n+1-i}}\right) \\ &= \frac{x_{i+1}(\varepsilon t) - \hat{x}_{i+1}(\varepsilon t)}{\varepsilon^{n-i}} - a_i\left(\frac{x_1(\varepsilon t) - \hat{x}_1(\varepsilon t)}{\varepsilon^n}\right) \\ &= \eta_{i+1}(t) - a_i\eta_1(t), \end{aligned}$$

for $i = n$,

$$\begin{aligned} \frac{d\eta_n(t)}{dt} &= \frac{d}{dt}\left(\frac{x_n(\varepsilon t) - \hat{x}_n(\varepsilon t)}{\varepsilon}\right) \\ &= \sum_{i=1}^{n} b_i(x_i(\varepsilon t) - \hat{x}_i(\varepsilon t)) + (x_{n+1}(\varepsilon t) - \hat{x}_{n+1}(\varepsilon t)) - a_n\left(\frac{x_1(\varepsilon t) - \hat{x}_1(\varepsilon t)}{\varepsilon^n}\right) \\ &= \sum_{i=1}^{n} b_i\varepsilon^{n+1-i}\eta_i(t) + \eta_{n+1}(t) - a_n\eta_1(t) \end{aligned}$$

and for $i = n+1$, $\frac{d\eta_{n+1}(t)}{dt} = \frac{d}{dt}\left(x_{n+1}(\varepsilon t) - \hat{x}_{n+1}(\varepsilon t)\right) = -a_{n+1}\eta_1(t) + \varepsilon h(t)$. We then put all these equations together into the following differential equations satisfied by $\eta_i(t)$, $i = 1, \ldots, n+1$.

$$\begin{cases} \frac{d\eta_1(t)}{dt} = \eta_2(t) - a_1\eta_1(t), \\ \frac{d\eta_2(t)}{dt} = \eta_3(t) - a_2\eta_1(t), \\ \quad\vdots \\ \frac{d\eta_n(t)}{dt} = \sum_{i=1}^{n} b_i\varepsilon^{n+1-i}\eta_i(t) + \eta_{n+1}(t) - a_n\eta_1(t), \\ \frac{d\eta_{n+1}(t)}{dt} = -a_{n+1}\eta_1(t) + \varepsilon h(t) \end{cases} \tag{*}$$

Choose a Lyapunov candidate function $V = \frac{1}{2}\eta^T P\eta$. Evaluating the time derivative of V on the solutions of (*), we obtain $\dot{V} = \eta^T P\Phi\eta + \varepsilon\eta^T P\Psi(t) = -\eta^T Q\eta + \varepsilon\eta^T P\Psi(t) \le -\lambda_{\min}(Q)\|\eta\|^2 + \varepsilon\rho\|P\|\|\eta\|$.

Performing some manipulations gives

$$\dot{V} \le -\lambda_{\min}(Q)\|\eta\|^2 + \varepsilon\rho\|P\|\|\eta\| \le -2\mu V + k_1\varepsilon\sqrt{V} \tag{**}$$

where $\mu = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$, $k_1 = \frac{\sqrt{2}\|P\|}{\sqrt{\lambda_{\min}(P)}}\rho$. It follows from (**) that $\dot{V} \le -\mu V - \mu\sqrt{V}\left(\sqrt{V} - R_+\right)$, where $R_+ = \frac{k_1\varepsilon}{\mu} > 0$. Hence, as long as $\sqrt{V} > R_+$, that is, $V > R_+^2$, we have $-\mu\sqrt{V}\left(\sqrt{V} - R_+\right) < 0$. Therefore, if $V(t_0) > R_+^2$ and $V(t) > R_+^2$ for $t \ge t_0$, then $\dot{V} \le -\mu V$, which implies that $V(t) \le \exp(-\mu(t - t_0))V(t_0)$. Thus, we can find a finite time $t_f(\varepsilon)$ such that $V(t) \le R_+^2$ for $t \ge t_0 + t_f(\varepsilon)$, where $t_f(\varepsilon) = \frac{1}{\mu}\ln\left(\frac{V(t_0)}{R_+^2}\right)$.

On the other hand, if $V(t_0) \le R_+^2$, then $V(t) \le R_+^2$ for $t \ge t_0$. Therefore, there exists a finite time $t_f(\varepsilon)$ such that $V(t) \le R_+^2$ for $t \ge t_0 + t_f(\varepsilon)$. We also have that $\frac{1}{2}\lambda_{\min}(P)\|\eta\|^2 \le V(t) \le R_+^2$. Using this fact coupled with the definition of R_+ we have for $t \ge t_0 + t_f(\varepsilon)$ that $\|\eta\| \le \sqrt{\frac{2}{\lambda_{\min}(P)}}\frac{k_1\varepsilon}{\mu} = \frac{2\lambda_{\max}(P)\|P\|\rho}{\lambda_{\min}(Q)\lambda_{\min}(P)}\varepsilon$. Furthermore, we have for $t \ge t_0 + t_f(\varepsilon)$ that

$$\begin{aligned} \|Z\| &= \left(z_1^2 + z_2^2 + \ldots + z_{n+1}^2\right)^{\frac{1}{2}} = \left(\left(\varepsilon^n\eta_1\left(\tfrac{t}{\varepsilon}\right)\right)^2 + \left(\varepsilon^{n-1}\eta_2\left(\tfrac{t}{\varepsilon}\right)\right)^2 + \ldots + \left(\eta_{n+1}\left(\tfrac{t}{\varepsilon}\right)\right)^2\right)^{\frac{1}{2}} \\ &\le \left(\left(\eta_1\left(\tfrac{t}{\varepsilon}\right)\right)^2 + \left(\eta_2\left(\tfrac{t}{\varepsilon}\right)\right)^2 + \ldots + \left(\eta_{n+1}\left(\tfrac{t}{\varepsilon}\right)\right)^2\right)^{\frac{1}{2}} \le \frac{2\lambda_{\max}(P)\|P\|\rho}{\lambda_{\min}(Q)\lambda_{\min}(P)}\varepsilon. \end{aligned}$$

The proof of the theorem is complete.