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FUZZY INTEGERS AND METHODS OF CONSTRUCTING THEM TO REPRESENT UNCERTAIN OR IMPRECISE INTEGER INFORMATION

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Abstract. *In this paper the concept of fuzzy integers which can be used to represent uncertain or imprecise integer quantity is proposed, the properties of fuzzy integers are investigated, and a representation theorem of fuzzy integers is obtained. Based on the representation theorem, the closeness of operation for fuzzy integers is discussed, it is shown that the usual addition still preserves the closeness of operation for fuzzy integers, but the usual multiplication and scalar multiplication do not preserve the closeness of the operation for fuzzy integers, and a new multiplication and a new scalar multiplication which preserve the closeness of operation for fuzzy integers are defined. Then for the sake of convenience and more rationality in application, a special kind of fuzzy integers which is called trapezoid-type fuzzy integer is introduced, and a method constructing trapezoid-type fuzzy integers to represent uncertain or imprecise integer quantities is proposed. At last, two practical examples are given to show the method of constructing fuzzy integers to represent uncertain or imprecise integer quantities.*
Keywords: Fuzzy number, Discrete fuzzy number, Fuzzy integer, Trapezoid-type fuzzy integer, Uncertain or imprecise integer quantity

1. Introduction.

Since 1965, professor Zadeh put forward the concept of fuzzy set in [23], and more and more researchers devoted themselves into the theories of fuzzy set and their applications [10, 15]. In 1972, Chang and Zadeh proposed the concept of fuzzy number to study the properties of probability functions in [6]. With the development of theories and applications of fuzzy numbers [7, 8, 9, 16, 18, 19], this concept becomes more and more important.

It is known that the collection of all real numbers possesses continuous attributes, but the collections of the numbers with some special properties like rational numbers or integers do not. The quantity which may be any one of the collection of numbers with continuous attributes is said to be a continuous type quantity, but the quantity which can only be one of the collections of some numbers without continuous attributes is said to be a discrete type quantity. Crisp real number can be used to represent certain and precise continuous type quantity, and crisp discrete number (which can only be one of the collections of some numbers without continuous attributes) can be used to represent certain and precise discrete type quantity. Likewise, fuzzy (continuous) number can be used to represent uncertain or imprecise continuous type quantity, and fuzzy discrete number can be used to represent uncertain or imprecise discrete type quantity.

In engineering or real world, the problems dealt with by us relate not only to uncertain or imprecise continuous type quantities, but also to uncertain or imprecise discrete type quantities such as "a group of people", "a bus of people", "a flock of sheep" and "the grade of the public life in a kind of cities (like medium sized city)". Therefore, studying discrete fuzzy numbers and using discrete fuzzy number theories to deal with uncertain or imprecise discrete type quantities are also important works. In 2001, Voxman defined (1-dimensional) discrete fuzzy numbers and obtained some results about them in [17], which as one kind of special fuzzy sets, have some application backgrounds. In 2005, Wang et al. gave a kind of representation of a discrete fuzzy number using r-level sets in [21]. Then Casasnovas and Riera studied some characters of discrete fuzzy numbers in [1, 2, 3, 4, 5]. In 2012, Riera and Torrens introduced aggregation functions defined on the set of all discrete fuzzy numbers whose support is a subset of consecutive natural numbers, and they are applied to the aggregation of subjective evaluations in [12]. In 2013, Riera and Torrens studied the residual implications on the set of discrete fuzzy numbers in [13]. Xie et al. discussed addition, multiplication and scalar multiplication operations of 2-dimensional discrete fuzzy number in [22]. In 2014, Riera and Torrens introduced aggregation functions on the set of discrete fuzzy numbers whose support is a set of consecutive natural numbers from a couple of discrete aggregation functions, and they can increase the flexibility of the elicitation of qualitative information based on linguistic terms in [14]. Massanet et al. set up a new linguistic computational model based on discrete fuzzy numbers for computing with words in [11]. Recently, we defined a special kind of discrete fuzzy numbers which is called trapezoid fuzzy integers, investigate the operations of trapezoid fuzzy integers, and gave a practical example to show their application in [20].

In fact, many of the uncertain or imprecise discrete type quantities dealt with by us in engineering or real world are all uncertain or imprecise integer type quantities, for example, previously described "a group of people", "a bus of people", "a flock of sheep" and "the grade of the public life in a kind of cities (like medium sized city)" are all uncertain or imprecise integer type quantities. For another example, in [11, 12, 14], the uncertain or imprecise discrete type quantities dealt with in subjective evaluations are also uncertain or imprecise integer type quantities. So studying fuzzy integer numbers and using fuzzy integer numbers to represent uncertain or imprecise integer type quantities (so that we can use the fuzzy integer number theories to solve engineering problems or practical problems related to uncertain or imprecise discrete type quantities) are important and meaningful. Although we studied trapezoid fuzzy integers, the concept of trapezoid fuzzy integers has a big limitation, and they are only suitable to represent the uncertain or imprecise integer type quantities which are Two-side type (see Figure 1), but not suitable to represent the uncertain or imprecise integer type quantities which are Right-side type (see Figure 2) or Left-side type (see Figure 3), or possess some characteristics (see Example 4.1 and Remark 4.1).

In this paper, we study a special kind of discrete fuzzy numbers which possess wider range and are more suitable (compared with trapezoid fuzzy integers) for representing uncertain or imprecise integer quantities, and study their properties and operations and the methods constructing this kind of special discrete fuzzy number. The specific arrangements of this paper are as follows. In Section 2, we briefly review some basic notions, definitions and results about discrete fuzzy numbers. In Section 3, we give the definition of fuzzy integers which is a special kind of discrete fuzzy numbers and can be used to represent such uncertain or imprecise integer information, and study their representation in cut set form. Then using the representation theorem obtained by us, we discuss the properties and the operations (include addition, multiplication and scalar multiplication) and the closeness of the operations and the rules of the operations. In Section 4, for the sake of convenience in application, we introduce a special kind of fuzzy integers which is called trapezoid-type fuzzy integer, and study the problem about how to construct trapezoid-type fuzzy integers to represent uncertain or imprecise integer quantities. At last, we make a conclusion in Section 5.

2. **Basic Definitions and Notations.**

Let R be the real number set, and I be the integer set. And let $K(R)$ denote the collection of non-empty compact subsets of R. The addition, scalar multiplication and multiplication on the space $K(R)$ are respectively defined as $A+B=\{a+b \mid a\in A, b\in B\}$, $\lambda A=\{\lambda a \mid a\in A\}$ and $AB=\{ab \mid a\in A, b\in B\}$ for any $A,B\in K(R)$, $\lambda\in R$.

A fuzzy subset (for short, a fuzzy set) of R is a function $u: R\to[0,1]$. For each such fuzzy set u, we denote by $[u]^r=\{x\in R^n : u(x)\geq r\}$ for any $r\in(0,1]$, its r-level set. By supp$u$ we denote the support of u, i.e., the set $\{x\in R^n : u(x)>0\}$. By $[u]^0$ we denote the closure of the supp$u$, i.e., $[u]^0=\overline{\{x\in R : u(x)>0\}}$.

For any fuzzy sets u, v and real number k, we define the addition and the multiplication of u and v, and the scalar multiplication of k and u by the following:

$$(u+v)(x)=\sup_{y+z=x}\min\{u(y),v(z)\};$$

$$(uv)(x)=\sup_{yz=x}\min\{u(y),v(z)\};$$

$$(ku)(x)=\begin{cases} u(x/k) & \text{if } k\neq 0\\ \hat{0} & \text{if } k=0\end{cases},$$

where

$$\hat{0}(x)=\begin{cases} 1 & \text{if } x=0\\ 0 & \text{if } x\neq 0\end{cases}.$$

Definition 2.1. [17] *A fuzzy set $u: R\to[0,1]$ is called a discrete fuzzy number if the support of u is finite, i.e., there exist $x_1,x_2,\ldots,x_n\in R$ with $x_1<x_2<\ldots<x_n$ such that $[u]^0=\{x_1,x_2,\ldots,x_n\}$ (the finiteness of the support implies $[u]^0=\mathrm{supp}u$), and there exist natural numbers s, t with $1\leq s\leq t\leq n$ such that*
(1) $u(x_i)=1$ for any natural number i with $s\leq i\leq t$;
(2) $u(x_i)\leq u(x_j)$ for any natural numbers i, j with $1\leq i\leq j\leq s$, $u(x_i)\geq u(x_j)$ for any natural numbers i, j with $t\leq i\leq j\leq n$.

We denote the collection of all discrete fuzzy numbers by F_D.

3. **Fuzzy Integer.**

For any $s_1,s_2\in I$ with $s_1\leq s_2$, we denote

$$\langle s_1,s_2\rangle=\{x\in I : s_1\leq x\leq s_2\}$$

and call it a closed integer interval.

Definition 3.1. *A fuzzy set $u: R\to[0,1]$ is called a fuzzy integer if its support is a closed integer interval (denoted as $\langle \underline{u}(0),\overline{u}(0)\rangle$), and satisfies*
(1) u is normal, i.e., there exists an $\hat{x}\in\langle \underline{u}(0),\overline{u}(0)\rangle$ such that $u(\hat{x})=1$;
(2) $u(x_i)\leq u(x_j)$ for any $x_i,x_j\in\langle \underline{u}(0),\hat{x}\rangle$ with $x_i\leq x_j$;
(3) $u(x_i)\geq u(x_j)$ for any $x_i,x_j\in\langle \hat{x},\overline{u}(0)\rangle$ with $x_i\leq x_j$.

And we denote the collection of all fuzzy integers by F_I.

Proposition 3.1. *Let $u\in F_I$. Then*
(1) $\mathrm{supp}u=[u]^0$;
(2) $u\in F_D$.

Proof: Since $u \in F_I$, supp$u$ is a closed integer interval, so supp$u$ is a finite set. It implies supp$u$ is a closed integer interval, so we see supp$u = [u]^0$ by the definition of $[u]^0$, i.e., Conclusion (1) holds.

From $u \in F_I$, we see $[u]^0$ is a closed integer interval by Conclusion (1), so there exist $x_1, x_2, \ldots, x_n \in I \subset R$ with $x_1 < x_2 < \ldots < x_n$ such that $[u]^0 = \{x_1, x_2, \ldots, x_n\}$ (where it is obvious that $x_1 = \underline{u}(0)$ and $x_n = \overline{u}(0)$). By the Condition (1) of Definition 3.1, we see there exists a natural number i_0 with $0 \le i_0 \le n$ such that $u(x_{i_0}) = 1$ (i.e., $x_{i_0} = \hat{x}$), so $[u]^1 \ne \phi$ ($[u]^1$ is not empty). Denoting $s = \min\{i : x_i \in [u]^1\}$, $t = \max\{i : x_i \in [u]^1\}$, we have that $1 \le s \le i_0 \le t \le n$ and $u(x_i) = 1$ for any natural number i with $s \le i \le t$. In addition, if natural numbers i, j satisfy $1 \le i \le j \le s$, then $1 \le i \le j \le i_0$, i.e., $x_i, x_j \in \langle \underline{u}(0), \hat{x}\rangle$ and $x_i \le x_j$, so we know $u(x_i) \ge u(x_j)$ by the Conclusion (2) of Definition 3.1. Similarly, we can show that if natural numbers i, j satisfy $t \le i \le j \le n$, then $u(x_i) \ge u(x_j)$. Therefore, u satisfies the all conclusions of the definition of discrete fuzzy numbers, so $u \in F_D$.

Theorem 3.1. *Let $u \in F_I$. Then the following statements (1)-(3) hold:*
(1) $[u]^r$ is a closed integer interval for any $r \in [0,1]$;
(2) $[u]^{r_2} \subset [u]^{r_1}$, for any $r_1, r_2 \in [0,1]$ with $0 \le r_1 \le r_2 \le 1$;
(3) $\cap_{n=1}^{\infty}[u]^{r_n} = [u]^r$ for any positive non-decreasing sequence $\{r_n\}_{n=1}^{\infty}$ with $lim_{n\to\infty} r_n = r \in (0,1]$.

And conversely, if $\{A_r \subset I : \ r \in [0,1]\}$ satisfies the following conditions (i)-(iii):
(i) A_r is a closed integer interval for any $r \in [0,1]$;
(ii) $A_{r_2} \subset A_{r_1}$, for any $r_1, r_2 \in [0,1]$ with $0 \le r_1 \le r_2 \le 1$;
(iii) $\cap_{n=1}^{\infty} A_{r_n} = A_r$ for any positive non-decreasing sequence $\{r_n\}_{n=1}^{\infty}$ with $lim_{n\to\infty} r_n = r \in (0,1]$,
then there exists a unique $u \in F_I$ such that $[u]^r = A_r$, for any $r \in [0,1]$.

Proof: The proof of the first part of the theorem: Let $u \in F_I$. By the definition of $[u]^r$, $r \in [0,1]$, Conclusion (2) of the theorem obviously holds. Then, for $r \in (0,1]$, we see $[u]^r \subset [u]^0 = \langle \underline{u}(0), \overline{u}(0)\rangle$. In addition, from the normality and $[u]^1 \subset [u]^r$, we know $[u]^r \ne \phi$. Denoting $\underline{x} = \min[u]^r$ and $\overline{x} = \max[u]^r$, we see $\underline{x}, \overline{x} \in I$ and $[u]^r \subset \langle \underline{x}, \overline{x}\rangle$ since $[u]^r$ is an empty finite integer set. Let $\hat{x}$ be a normal point of u, then $\underline{x} \le \hat{x} \le \overline{x}$. Therefore, for any $x \in \langle \underline{x}, \overline{x}\rangle$, we see x is an integer with $\underline{x} \le x \le \hat{x}$ or $\hat{x} \le x \le \overline{x}$. As $\underline{x} \le x \le \hat{x}$, we know $u(x) \ge u(\underline{x}) \ge r$ from Condition (2) of Definition 3.1, so $x \in [u]^r$. Likewise, as $\hat{x} \le x \le \overline{x}$, we can also see $x \in [u]^r$ from Condition (3) of Definition 3.1. Thus, we obtain $\langle \underline{x}, \overline{x}\rangle \subset [u]^r$, so $[u]^r = \langle \underline{x}, \overline{x}\rangle$. Therefore, we see for any $r \in [0,1]$, $[u]^r$ is always a closed integer interval since $[u]^0$ is also a closed integer interval by Definition 3.1, so Conclusion (1) of the theorem holds. Let $\{r_n\}_{n=1}^{\infty}$ be a positive non-decreasing sequence with $lim_{n\to\infty} r_n = r \in (0,1]$. For any $n = 1, 2, \ldots$, from $r_n \le r$, we know $[u]^r \subset [u]^{r_n}$ by Conclusion (2) of the theorem, so we have $[u]^r \subset \cap_{n=1}^{\infty}[u]^{r_n}$. Let $x \in \cap_{n=1}^{\infty}[u]^{r_n}$. Then $x \in [u]^{r_n}$, i.e., $u(x) \ge r_n$ holds for any $n = 1, 2, \ldots$, it implies $u(x) \ge r$, so $x \in [u]^r$. Thus, we obtain $\cap_{n=1}^{\infty}[u]^{r_n} = [u]^r$, i.e., Conclusion (3) of the theorem also holds. The proof of the first part of the theorem is completed.

The proof of the second part of the theorem: Let $A_r \subset I$, $r \in [0,1]$ satisfy Conditions (i)-(iii) of the theorem. Writing

$$u(x) = \begin{cases} \sup\{r \in [0,1] : x \in A_r\} & \text{if } x \in A_0 \\ 0 & \text{if } x \overline{\in} A_0 \end{cases}$$

in the following we prove that $u \in F_I$ and satisfy $[u]^r = A_r$, for any $r \in [0,1]$.

Firstly, we show $[u]^r = A_r$ for any $r \in [0,1]$. Let $r_0 \in [0,1]$. If $x \in A_{r_0}$, then $r_0 \in \{r \in [0,1] : x \in A_r\}$. Hence, $u(x) = \sup\{r \in [0,1] : x \in A_r\} \ge r_0$, i.e., $x \in [u]^{r_0}$.

Therefore, we have $A_{r_0} \subset [u]^{r_0}$ for any $r_0 \in [0,1]$. Conversely, if $r_0 \in [0,1]$ and $x \in [u]^{r_0}$, then $u(x) \geq r_0$ as $r_0 \neq 0$, and $u(x) > r_0$ as $r_0 = 0$, i.e., $\sup\{r \in [0,1] : x \in A_r\} \geq r_0$ as $r_0 \neq 0$, and $\sup\{r \in [0,1] : x \in A_r\} > r_0$ as $r_0 = 0$. If $\sup\{r \in [0,1] : x \in A_r\} > r_0$ ($r_0 \in [0,1]$), by the definition of supremum, we see that there exists $\grave{r} \in (0,1]$ with $x \in A_{\grave{r}}$ such that $\grave{r} > r_0$, so by Condition (ii) of the theorem, we know $x \in A_{\grave{r}} \subset A_{r_0}$. If $\sup\{r \in [0,1] : x \in A_r\} = r_0$ ($r_0 \in (0,1]$), then there exists a positive non-decreasing sequence $\{r_n\}_{n=1}^{\infty}$ with $x \in A_{r_n}$ ($n = 1, 2, \ldots$) such that $lim_{n\to\infty} r_n = r_0 \in (0,1]$, so we have that $x \in \cap_{n=1}^{\infty} A_{r_n} = A_{r_0}$ by Condition (iii) of the theorem. Thus, we also obtain $[u]^{r_0} \subset A_{r_0}$ for any $r_0 \in [0,1]$, so $[u]^r = A_r$ holds for any $r \in [0,1]$.

Secondly, we show $u \in F_I$. From the definition of u and $\text{supp}u = [u]^0 = A_0$ and Condition (i) of the theorem, we see that u is a fuzzy set of R, and its support $\text{supp}u$ is a closed integer interval (denoted as $\langle \underline{u}(0), \overline{u}(0) \rangle$). The normality of u can be seen from the non-empty of $[u]^1$ (by $[u]^1 = A_1$ and Condition (i) of the theorem), so Condition (1) of Definition 3.1 holds. Let $\hat{x} \in \langle \underline{u}(0), \overline{u}(0) \rangle$ such that $u(\hat{x}) = 1$, and $x_i, x_j \in \langle \underline{u}(0), \hat{x} \rangle$ with $x_i \leq x_j$. We prove that $u(x_i) \leq u(x_j)$, i.e., Condition (2) of Definition 3.1 holds by reductio in the following. If $u(x_i) \leq u(x_j)$ does not hold, then $u(x_i) > u(x_j)$, so we see $[u]^{u(x_i)} \subset [u]^{u(x_j)}$ and $x_j \overline{\in} [u]^{u(x_i)}$ by Condition (ii) of the theorem and the definition of $u(x_i)$-level set of u. It implies $x_i \geq \min[u]^{u(x_i)} > x_j$ which contradicts to $x_i \leq x_j$, so $u(x_i) \leq u(x_j)$, i.e., Condition (2) of Definition 3.1 holds. Similarly, we can show Condition (3) of Definition 3.1 also holds, so by the definition (Definition 3.1) of fuzzy integers, we have $u \in F_I$.

At last, the uniqueness of u is obvious, so the proof of the theorem is completed.

Remark 3.1. *Theorem 3.1 tells us that for any $u \in F_I$ with $r \in [0,1]$, $[u]^r$ is always a closed integer interval. We denote the closed integer interval by $\langle \underline{u}(r), \overline{u}(r) \rangle$, i.e., $[u]^r = \langle \underline{u}(r), \overline{u}(r) \rangle$.*

Theorem 3.2. *Let $u \in F_D$. Then $u \in F_I \iff [u]^r$ is a closed integer interval for any $r \in [0,1]$.*

Proof: By the definition (Definition 2.1) of discrete fuzzy numbers and Theorem 3.1, the theorem can be directly shown.

Lemma 3.1. *Let $\langle s_1, t_1 \rangle$, $\langle s_2, t_2 \rangle$ be closed integer intervals. Then*

$$\langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle = \langle s_1 + s_2, t_1 + t_2 \rangle$$

Proof: Let $x \in \langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle$. Then there exist $y \in \langle s_1, t_1 \rangle$ and $z \in \langle s_2, t_2 \rangle$ such that $x = y + z$. From $s_1 \leq y \leq t_1$, $s_2 \leq z \leq t_2$ and $y, z \in I$, we see $s_1 + s_2 \leq x \leq t_1 + t_2$ and $x \in I$, so $x \in \langle s_1 + s_2, t_1 + t_2 \rangle$. Thus, $\langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle \subset \langle s_1 + s_2, t_1 + t_2 \rangle$ holds. Conversely, let $x \in \langle s_1 + s_2, t_1 + t_2 \rangle$, then $s_1 + s_2 \leq x \leq t_1 + t_2$ and $x \in I$. Denoting $y = x - s_1$, then $s_2 \leq y \leq t_1 + t_2 - s_1 \leq t_2$ and $y \in I$, i.e., $y \in \langle s_2, t_2 \rangle$, so from $x = y + s_1$ and $s_1 \in \langle s_1, t_1 \rangle$, we have $x \in \langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle$. Therefore, $\langle s_1 + s_2, t_1 + t_2 \rangle \subset \langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle$ also holds, so we have $\langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle = \langle s_1 + s_2, t_1 + t_2 \rangle$.

Remark 3.2. *For any closed integer intervals $\langle s_1, t_1 \rangle$, $\langle s_2, t_2 \rangle$, and any $m \in I$, inclusion relations*

$$\langle s_1, t_1 \rangle \langle s_2, t_2 \rangle \subset \langle \min\{s_1 s_2, s_1 t_2, t_1 s_2, t_2 t_2\}, \max\{s_1 s_2, s_1 t_2, t_1 s_2, t_2 t_2\} \rangle$$

and

$$m \langle s_1, t_1 \rangle \subset \begin{cases} \langle m s_1, m t_1 \rangle & \text{if } m \geq 0 \\ \langle m t_1, m s_1 \rangle & \text{if } m < 0 \end{cases}$$

hold, but in turn, the inclusion relations

$$\langle s_1, t_1 \rangle \langle s_2, t_2 \rangle \supset \langle \min\{s_1 s_2, s_1 t_2, t_1 s_2, t_2 t_2\}, \max\{s_1 s_2, s_1 t_2, t_1 s_2, t_2 t_2\} \rangle$$

and

$$m\langle s_1, t_1\rangle \supset \begin{cases} \langle ms_1, mt_1\rangle & \text{if } m \geq 0 \\ \langle mt_1, ms_1\rangle & \text{if } m < 0 \end{cases}$$

are not true. So

$$\langle s_1, t_1\rangle\langle s_2, t_2\rangle = \langle \min\{s_1s_2, s_1t_2, t_1s_2, t_2t_2\}, \max\{s_1s_2, s_1t_2, t_1s_2, t_2t_2\}\rangle$$

$$m\langle s_1, t_1\rangle = \begin{cases} \langle ms_1, mt_1\rangle & \text{if } m \geq 0 \\ \langle mt_1, ms_1\rangle & \text{if } m < 0 \end{cases}$$

may not be true. Of course, as $k \in R$,

$$k\langle s_1, t_1\rangle = \begin{cases} \langle ks_1, kt_1\rangle & \text{if } k \geq 0 \\ \langle kt_1, ks_1\rangle & \text{if } k < 0 \end{cases}$$

more may not be true.

Theorem 3.3. *If* $u, v \in F_I, k \in R$, *then for any* $r \in [0, 1]$,
(1) $[u + v]^r = [u]^r + [v]^r$;
(2) $[ku]^r = k[u]^r$;
(3) $[uv]^r = [u]^r[v]^r$.

Proof: Let $u, v \in F_I, k \in I$. We first show that $[u+v]^r = [u]^r + [v]^r$, for any $r \in [0, 1]$. For a fixed $r \in [0, 1]$, if $x \in [u]^r + [v]^r$, then there exist $y \in [u]^r$ and $z \in [v]^r$ such that $x = y + z$. Hence, we deduce that $(u+v)(x) = \sup_{s+t=x} \min(u(s), v(t)) \geq \min(u(y), v(z)) \geq r$, i.e., $x \in [u + v]^r$. This leads to $[u + v]^r \supset [u]^r + [v]^r$ for any $r \in [0, 1]$. Conversely, for any fixed $r \in [0, 1]$, if $x \in [u + v]^r$, then we have $(u + v)(x) \geq r$ as $r \neq 0$, and $(u + v)(x) > r$ as $r = 0$, i.e., $\sup_{s+t=x} \min(u(s), v(t)) \geq r$ as $r \neq 0$, and $\sup_{s+t=x} \min(u(s), v(t)) > r$ as $r = 0$. If $\sup_{s+t=x} \min(u(s), v(t)) > r \in [0, 1]$, then there exist $s_r, t_r \in R$ with $s_r + t_r = x$ such that $\min\{u(s_r), v(t_r)\} > r$, so we have $s_r + t_r = x$ and $u(s_r) > r$ and $v(t_r) > r$, i.e., $x = s_r+t_r$ and $s_r \in [u]^r$ and $t_r \in [v]^r$, it implies $x \in [u]^r+[v]^r$. If $\sup_{s+t=x} \min(u(s), v(t)) = r \in (0, 1]$, then there exist positive non-decreasing sequence $\{\min(u(s_n), v(t_n))\}_{n=1}^{\infty}$ with $s_n + t_n = x$ such that $\lim_{n\to\infty} \min\{u(s_n), v(t_n)\} = r$. From $s_n \in [u]^0$ and $t_n \in [v]^0$ $(n = 1, 2, \ldots)$, we see that $\{s_n : n = 1, 2, \ldots\}$ and $\{t_n : n = 1, 2, \ldots\}$ are all finite, so $\{\min(u(s_n), v(t_n)) : n = 1, 2, \ldots\}$ is also finite. It implies that there exists natural number n_0 such that $\min(u(s_{n_0}), v(t_{n_0})) = \sup\{\min(u(s_n), v(t_n)) : n = 1, 2, \ldots\} = r$, so $u(s_{n_0}) \geq r$ and $v(t_{n_0}) \geq r$, i.e., $s_{n_0} \in [u]^r$ and $t_{n_0} \in [v]^r$, and then $x = s_{n_0} + t_{n_0} \in [u]^r + [v]^r$. Thus, $[u + v]^r \subset [u]^r + [v]^r$ for any $r \in [0, 1]$ also holds, so $[u + v]^r = [u]^r + [v]^r$ for any $r \in [0, 1]$, i.e., Conclusion (1) of the theorem is true.

We next show that $[ku]^r = k[u]^r$ for any $k \in R$ and $r \in [0, 1]$. By using the definitions of ku and $k[u]^r$, we can see that $[0u]^r = 0[u]^r$ for any $r \in [0, 1]$, i.e., for $k = 0$, $[ku]^r = k[u]^r$ holds for any $r \in [0, 1]$. Let $k \neq 0$. Then, for a fixed $r \in [0, 1]$, if $x \in k[u]^r$, then there exists $y \in [u]^r$ such that $x = ky$. This leads to $(ku)(x) = u(x/k) = u(ky/k) = u(y) \geq r$, i.e., $x \in [ku]^r$. Therefore, we obtain $[ku]^r \supset k[u]^r$ for any $k \neq 0$ and $r \in [0, 1]$. Conversely, for any fixed $r \in [0, 1]$, if $x \in [ku]^r$, then $u(x/k) = (ku)(x) \geq r$, so $x/k \in [u]^r$, i.e., $x \in k[u]^r$. Therefore, we also obtain $[ku]^r \subset k[u]^r$ for any $k \neq 0$ and $r \in [0, 1]$.

By using similar proof of Conclusion (1) of the theorem, we can also prove that $[uv]^r = [u]^r[v]^r$ for any $r \in [0, 1]$. Thus, the proof is completed.

By Theorems 3.1 and 3.3 and Lemma 3.1, we can obtain the following result:

Theorem 3.4. *If* $u, v \in F_I$, *then* $u + v \in F_I$, *and*

$$\langle \underline{(u + v)}(r), \overline{(u + v)}(r)\rangle = \langle \underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\rangle$$

for any $r \in [0, 1]$.

Remark 3.3. *Theorem 3.4 tells us that the usual addition "+" preserves the closeness of the operation for fuzzy integers. However, by Theorem 3.3 and Remark 3.2, we can see that the usual scalar multiplication and multiplication do not preserve the closeness of the operation for fuzzy integers.*

For the sake of the application of fuzzy integers, we define a new scalar multiplication and multiplication operations which preserve the closeness of the operation. For this reason, we give the following denotation and result:

For any $x \in R$, we use $\lfloor x \rceil$ to indicate the integer which is obtained by arithmetic rounding to x.

Theorem 3.5. *Let $u, v \in F_I, k \in R$, and for each $r \in [0,1]$*

$$A_r = \begin{cases} \langle \lfloor m\underline{u}(r) \rceil, \lfloor m\overline{u}(r) \rceil \rangle & \text{if } k \geq 0 \\ \langle \lfloor m\overline{u}(r) \rceil, \lfloor m\underline{u}(r) \rceil \rangle & \text{if } k < 0 \end{cases}$$

$$B_r = \langle \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}\rangle$$

Then families of sets $\{A_r : r \in [0,1]\}$ and $\{B_r : r \in [0,1]\}$ both satisfy Conditions (i)-(iii) of Theorem 3.1.

Proof: The theorem can be easily shown, so we omit the proof.

By Theorems 3.1 and 3.5, we can give the following definition:

Definition 3.2. *Let $u, v \in F_I, k \in R$. We define scalar multiplication $k \circ u$ (of k and u) and multiplication $u \circ v$ (u and v) as the fuzzy integers decided by $\{A_r : r \in [0,1]\}$ and $\{B_r : r \in [0,1]\}$ in Theorem 3.5, respectively.*

Theorem 3.6. *Let $u, v \in F_I, k \in R$. Then $k \circ u, u \circ v \in F_I$, and*
(1) $\underline{k \circ u}(r) = \lfloor k\underline{u}(r) \rceil$ and $\overline{k \circ u}(r) = \lfloor k\overline{u}(r) \rceil$ as $k \geq 0$,
$\underline{k \circ u}(r) = \lfloor k\overline{u}(r) \rceil$ and $\overline{k \circ u}(r) = \lfloor k\underline{u}(r) \rceil$ as $k < 0$;
(2) $\underline{u \circ v}(r) = \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$,
$\overline{u \circ v}(r) = \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$.

Proof: The theorem can be directly obtained by Theorem 3.5 and Definition 3.2.

Theorem 3.7. *Let $u, v \in F_I, k \in R$. Then*
(1) $ku = k \circ u$ as $ku \in F_I$;
(2) $uv = u \circ v$ as $uv \in F_I$.

Proof: The theorem can be easily shown, so we omit the proof.

4. **Method to Construct Fuzzy Integer.**

4.1. **Trapezoid-type fuzzy integer.**

Theorem 4.1. *Let s_0, s_1, t_1 and $t_0 \in I$ with $s_0 \leq s_1 \leq t_1 \leq t_0 \in I$. If the fuzzy set $u : R \to [0,1]$ is defined as*

$$u(x) = \begin{cases} 1 & \text{if } x \in \langle s_1, t_1 \rangle \\ \frac{x - s_0}{s_1 - s_0} & \text{if } x \in \langle \underline{m}, s_1 \rangle \\ \frac{t_0 - x}{t_0 - t_1} & \text{if } x \in \langle t_1, \overline{m} \rangle \\ 0 & \text{if } x \overline{\in} \langle \underline{m}, \overline{m} \rangle \end{cases}$$

where $\underline{m}, \overline{m} \in I$ with $s_0 \leq \underline{m} \leq s_1$ and $t_1 \leq \overline{m} \leq t_0$, then $u \in F_I$.

Proof: The theorem can be easily shown, so we omit the proof.

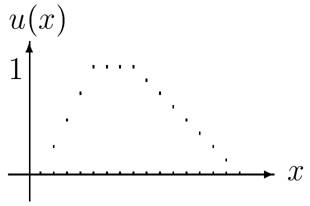
Definition 4.1. *Let s_0, s_1, t_1 and $t_0 \in I$ with $s_0 \le s_1 \le t_1 \le t_0 \in I$. If the fuzzy set $u : R \to [0, 1]$ is defined as*

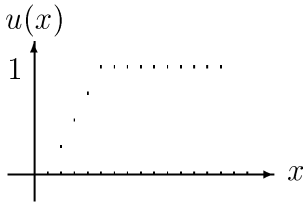
$$u(x) = \begin{cases} 1 & \text{if} \;\; x \in \langle s_1, t_1 \rangle \\ \frac{x - s_0}{s_1 - s_0} & \text{if} \;\; x \in \langle \underline{m}, s_1 \rangle \\ \frac{t_0 - x}{t_0 - t_1} & \text{if} \;\; x \in \langle t_1, \overline{m} \rangle \\ 0 & \text{if} \;\; x \,\overline{\in}\, \langle \underline{m}, \overline{m} \rangle \end{cases}$$

then we call u a trapezoid-type fuzzy integer, and denote it as $u = F_I(s_0|_{\underline{m}}, s_1, t_1, t_0|_{\overline{m}})$. Specially, if $s_1 = t_1$ (denoted n), then we call the trapezoid-type fuzzy integer u a triangle-type fuzzy integer, and denote it as $u = F_I(s_0|_{\underline{m}}, n, t_0|_{\overline{m}})$, where $\underline{m}, \overline{m} \in I$ with $s_0 \le \underline{m} \le s_1$ and $t_1 \le \overline{m} \le t_0$.

And we denote the collection of all trapezoid-type fuzzy integers by $Tra - F_I$, and the collection of all triangle-type fuzzy integers by $Tri - F_I$.

4.2. **Constructing methods.** In this section, we establish the method constructing fuzzy integers to represent an object characterized by a group of uncertain or imprecise integers information. For the sake of convenience in stating, we first introduce the following concepts.





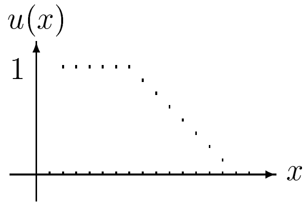


FIGURE 1. Two-side type FIGURE 2. Right-side type FIGURE 3. Left-side type

For an uncertain or imprecise integer information u,

(1) if its maximal membership degree point(s) is (are) in the middle part (see Figure 1), then we said it to be of Two-side type;

(2) if its maximal membership degree point(s) is (are) only in the right (see Figure 2), then we said it to be of Right-side type;

(3) if its maximal membership degree point(s) is (are) only in the left (see Figure 3), then we said it to be of Left-side type;

Consider an object (denoted by O) which is characterized by an imprecise or uncertain integer information. And suppose the following data set about the object is from m sources (such as observations or samples) in an imprecise or uncertain environment:

$$x_1, x_2, \ldots, x_m, \;\; (x_i \in I, i = 1, 2, \ldots, m)$$

The problem to solve is how to construct a fuzzy integer (from the set of data: $x_1, x_2, \ldots, x_m$) to represent the object O which is characterized by imprecise or uncertain integer quantity.

The constructing method: First: We work out the means μ of the object O from $x_1, x_2, \ldots, x_m$, $x_i \in I$, $i = 1, 2, \ldots, m$:

$$\mu = \frac{1}{m} \sum_{i=1}^{m} x_i \tag{1}$$

Second: We work out the left separation degrees $L\sigma$ and right separation degrees $R\sigma$ of the object O from $x_1, x_2, \ldots, x_m$, $x_i \in I$, $i = 1, 2, \ldots, m$, respectively:

$$\begin{array}{l} L\sigma = \frac{1}{N_L}\sum_{x_i<\mu}(\mu - x_i) \\ R\sigma = \frac{1}{N_R}\sum_{x_i>\mu}(x_i - \mu) \end{array} \tag{2}$$

where N_L and N_R are the number of the character values which satisfy $x_i \le \mu$ in $x_1, x_2, \ldots, x_m$, $x_i \in I$, $i = 1, 2, \ldots, m$ and the number of the character values which satisfy $x_i \ge \mu$ in $x_1, x_2, \ldots, x_m$, $x_i \in I$, $i = 1, 2, \ldots, m$, respectively.

Third: Make a domain $\langle \alpha, \beta \rangle$ (such that all the possible character values of the object O are in it) of the character value of the object O according to the practical case, and denote

$$\begin{array}{c} \underline{\mu - \lambda L\sigma} = \max\{n \in I : \ \ n \le \mu - \lambda L\sigma\} \\ \overline{\mu + \lambda R\sigma} = \min\{n \in I : \ \ n \ge \mu + \lambda R\sigma\} \\ (\underline{\max}) = \max\{\mu - \lambda L\sigma, \alpha\} \\ (\overline{\min}) = \min\{\mu + \lambda R\sigma, \beta\} \end{array}$$

where λ is a parameter, that may be chosen in interval $[2, 4]$ according to practical case.

(1) When the object quantity is of Two-side type, we construct a trapezoid-type fuzzy integer u as

$$u = F_I\left(\underline{\mu - \lambda L\sigma}|_{(\underline{\max})}, \underline{N}, \overline{N}, \overline{\mu + \lambda R\sigma}|_{(\overline{\min})}\right) \tag{3}$$

where $\underline{N} = \max\{n \in I : \ \ n \le \mu\}$ and $\overline{N} = \min\{n \in I : \ \ n \ge \mu\}$.

(2) When the object quantity is of Right-side type, we construct a trapezoid-type fuzzy integer u as

$$u = F_I(\underline{\mu - \lambda L\sigma}|_{(\underline{\max})}, \underline{N}, \beta, \beta|_\beta) \tag{4}$$

where $\underline{N} = \max\{n \in I : \ \ n \le \mu\}$.

(3) When the object quantity is of Left-side type, we construct a trapezoid-type fuzzy integer u as

$$u = F_I\left(\alpha|_\alpha, \alpha, \overline{N}, \overline{\mu + \lambda R\sigma})_{(\overline{\min})}\right) \tag{5}$$

where $\overline{N} = \min\{n \in I : \ \ n \ge \mu\}$.

Then we can use the trapezoid-type fuzzy integer u to express the object O.

In the following, we give practical examples to show the method constructing a fuzzy integer to represent an uncertain or imprecise integer quantity.

Example 4.1. *One day, there are* 5 *bus from City A to City B. Then the number (i.e., "*5 *bus people") of people from City A to City B by the* 5 *buses is an uncertain integer quantity, so we can use a fuzzy integer to represent the uncertain integer quantity "*5 *bus people".*

According to the provisions of the transport company, not only over-passenger is not allowed, but also the passengers less than half of the bus capacity are not allowed, so one bus from City A to City B only can be taken by 19-38 people. For buses from City A to City B, suppose the following set of data comes from the previous statistics (50 Samples) of the number of passengers of one bus.

$$37, 19, 35, 37, 26, 36, 32, 38, 37, 34, 36, 36, 31, 25, 32, 38, 36, 36, 34, 34, 19, 33, 25, 29, 36$$

$$28, 20, 36, 30, 27, 23, 21, 24, 34, 36, 31, 38, 25, 35, 37, 22, 29, 37, 38, 35, 34, 32, 28, 38, 33$$

*The problem to solve is how to construct a fuzzy integer (from the set of data) to represent the uncertain integer quantity "*5 *big bus people".*

First: By Formula (1), we work out the means μ *of the set of data:* $\mu = \frac{1}{m}\sum_{i=1}^{m} x_i = 31.64$.

Second: By Formula (2), we work out the left separation degrees $L\sigma$ and right separation degrees $R\sigma$ of the set of data: $L\sigma = \frac{1}{N_L}\sum_{x_i<\mu}(\mu - x_i) = 6.09$ *and* $R\sigma = \frac{1}{N_R}\sum_{x_i>\mu}(x_i - \mu) = 3.84$.

Third: Taking $\lambda = 3$ *and a domain* $\langle \alpha, \beta\rangle = \langle 19, 38\rangle$ *of the uncertain integer quantity "1 big bus people" according to the practical case (the provisions of the transport company), we have* $\mu - \lambda L\sigma = 13.37$ *and* $\mu + \lambda R\sigma = 43.16$, *so*

$$\underline{\mu - \lambda L\sigma} = \max\{n \in I : \ n \le \mu - \lambda L\sigma\} = 13$$

$$\overline{\mu + \lambda R\sigma} = \min\{n \in I : \ n \ge \mu + \lambda R\sigma\} = 44$$

$$(\underline{\max}) = \max\{\mu - \lambda L\sigma, \alpha\} = 19$$

$$(\overline{\min}) = \min\{\mu + \lambda R\sigma, \beta\} = 36$$

And due to the Two-side type attribute of the uncertain integer quantity "1 big bus people", we construct a trapezoid-type fuzzy integer u *as* $u = F_I(13|_{19}, 31, 32, 44|_{36})$ *by Formula (3).*

Then we can use the fuzzy integer $5 \circ u = 5 \circ F_I(13|_{19}, 31, 32, 44|_{36})$ *to express the uncertain integer quantity "5 big bus people".*

Remark 4.1. *It is obvious that* $u = F_I(13|_{19}, 31, 32, 44|_{36})$ *constructed in Example 4.1 is not trapezoid fuzzy integer which is introduced in* [20]. *From the constructions of* $u = F_I(13|_{19}, 31, 32, 44|_{36})$ *and trapezoid fuzzy integers, we see that it is rational to use trapezoid-type fuzzy integer* u *representing "1 big bus people", but it becomes unwise to use a trapezoid fuzzy integer representing "1 big bus people".*

Example 4.2. *Let* $L = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *represent* $\Omega = \{EB, VB, B, MB, F, MG, G, VG, EG\}$, *where the letters refer ordinarily to the linguistic terms: Extremely Bad, Very Bad, Bad, More or Less Bad, Fair, More or Less Good, Good, Very Good and Extremely Good. An expert panel consisting of 100 experts evaluate a person (denoted by* P*) for his or her working ability. Everyone in the expert panel is asked to choose only one number in* L *which is considered by the expert to be best matching with the working ability of person* P. *Suppose the following data set be from the evaluation results of the 100 experts:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 3 & 5 & 9 & 20 & 63 & 0 \end{pmatrix}$$

where n_i *in*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 & n_9 \end{pmatrix}$$

is the total number of experts which choose i *in* L. *We can consider to construct a fuzzy integer to represent the working ability of person* P.

We can work out the value

$$\mu = \Sigma_{i=1}^{9} \frac{n_i}{100} i = 0.03 \times 4 + 0.05 \times 5 + 0.09 \times 6 + 0.2 \times 7 + 0.63 \times 8 = 7.35$$

Taking it into account that 7 *is the nearest integer to* 7.35 *and* $4 = \min\{i \in L | n_i \ne 0\}$ *and* $8 = \max\{i \in L | n_i \ne 0\}$, *then we should construct* $u = F_I(3, 7, 9)$ *to represent the working ability of person* P *if the trapezoid fuzzy integers introduced in* [20] *are only considered. However, it has a defect since* $u(8) = F_I(3, 7, 9)(8) = 0.5$ *is not consistent with* $n_8 = 63$. *If we construct the fuzzy integer* $u = F_I(3|_4, 7, 10|_8)$ *to represent the working ability of person* P, *then the defect is overcome since* $u(8) = F_I(3|_4, 7, 10|_8)(8) = 0.6667$ *is basically consistent with* $n_8 = 63$.

Remark 4.2. *Of course, using fuzzy integers to represent uncertain or imprecise integer quantities like Example 4.1 and 4.2 is not our destination. We introduce, study and construct fuzzy integers in order to turn the problem of processing uncertain or imprecise integer quantities into the problem of processing fuzzy integers by using fuzzy integer space theory. Thus, fuzzy integers can be applied in some engineering or real world.*

5. **Conclusion.** In this paper, we firstly gave the definition of fuzzy integers (Definition 3.1), showed that they are discrete fuzzy numbers (Proposition 3.1), and obtained a representation theorem of fuzzy integers in cut-sets form (Theorem 3.1). Then based on the representation theorem of fuzzy integers, we showed that a discrete fuzzy number is a fuzzy integer if and only if its cut-sets are all closed integer intervals (Theorem 3.2), obtained the operation rule in cut-sets form about usual addition and multiplication and scalar multiplication (Theorem 3.3), showed that the usual addition still preserves the closeness of operation for fuzzy integers (Theorem 3.4), but the usual multiplication and scalar multiplication do not preserve the closeness of the operation for fuzzy integers (Remark 3.3), and defined a new multiplication and a new scalar multiplication which preserve the closeness of operation for fuzzy integers (Definition 3.2). And then, for the sake of convenience in application, we introduced a special kind of fuzzy integers which is called trapezoid-type fuzzy integer (Definition 4.1), and proposed a method constructing trapezoid-type fuzzy integers to represent uncertain or imprecise integer quantities. At last, we gave practical example to show the method constructing fuzzy integer to represent an uncertain or imprecise integer quantity.

In the future work, we can study the problems of establishing suitable measures in fuzzy integer space to identify, classify and rank imprecise or uncertain integer information, and put obtained results into the applications in engineering field, such as industrial alarm system in uncertain environment.

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