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SOME HESITANT INTUITIONISTIC FUZZY LINGUISTIC DISTANCE MEASURES

Yongfeng Pang and Wei Yang

Department of Mathematics
School of Science
Xi'an University of Architecture and Technology
No. 13, Yanta Road, Xi'an 710055, P. R. China
yangweipyf@163.com

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Abstract. *In this paper, we develop hesitant intuitionistic fuzzy linguistic sets by extending the hesitant fuzzy sets to accommodate linguistic arguments and intuitionistic fuzzy values. Then we present several generalized weighted distance measures for hesitant intuitionistic fuzzy linguistic information including the generalized hesitant intuitionistic fuzzy linguistic weighted distance, the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance and the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance. Based on the proposed distances and TOPSIS, we develop a new multiple attribute decision making method. Finally, a practical example concerning the metro project risk assessment is given to illustrate feasibility and practical advantages of the proposed approach.*
Keywords: Hesitant intuitionistic fuzzy linguistic set, Distance measure, Similarity measure, TOPSIS

1. Introduction.

Fuzziness and uncertainty exist extensively in decision making process [1, 2]. Hesitant fuzzy sets (HFSs) are the useful tool to model fuzzy and uncertain information, which are first introduced by Torra [3, 4]. The membership of HFSs is the union of several memberships of fuzzy sets. The HFSs have attracted broad attention due to their flexibility and been studied and applied extensively. Torra [3] discussed the relationship between intuitionistic fuzzy set and HFSs. Some hesitant fuzzy aggregation operators have been developed. The generalized hesitant fuzzy weighted averaging operator and the generalized hesitant fuzzy weighted geometric operator have been developed by Xia and Xu [5]. The hesitant fuzzy geometric Bonferroni mean operator and the hesitant fuzzy Choquet geometric Bonferroni mean operator have been given by Zhu et al. [6]. A wide range of hesitant fuzzy power aggregation operators have been proposed by Zhang [7]. Some hesitant fuzzy prioritized aggregation operators have been proposed by Wei [8]. Some distance measures and correlation measures have been developed. Xu and Xia [9] developed a number of hesitant ordered weighted distance measures. The weighted correlation coefficient of dual hesitant fuzzy sets has been presented by Ye [10]. A generalized hesitant fuzzy synergetic weighted distance measure is presented by Peng et al. [11]. Distance measures are very important in various fields such as decision making, pattern recognition, and machine learning. Moreover, the distance measures are the basis of many well-known multiple attribute decision making methods including TOPSIS, ELECTRE, VIKOR. In this paper, we develop several new distance measures for the new extended hesitant fuzzy set.

The hesitant fuzzy set has been generalized to accommodate interval values [12], triangular fuzzy value [13], linguistic argument [12-14]. In some decision making problems, decision makers would like to evaluate with linguistic arguments, which can reflect the fuzzy nature of human thinking. Rodríguez et al. [14] developed the concept of a hesitant fuzzy linguistic term set, in which several possible linguistic values are used to assess an indicator. Compared with the other linguistic decision models, the hesitant fuzzy linguistic term sets are more flexible and convenient to reflect the decision makers preferences. The hesitant fuzzy linguistic decision model can deal with the situation that experts hesitate between several possible linguistic terms to assess an element, which cannot reflect membership of each linguistic value satisfying the attribute. In fact, the experts may think memberships of linguistic terms are different. The intuitionistic fuzzy set [17] characterized by a membership degree and a non-membership degree is a useful tool to model the uncertainty and vagueness. The intuitionistic fuzzy set can effectively model the memberships of linguistic arguments. Hence, the linguistic arguments combining the intuitionistic fuzzy values can be used when evaluating. Yang et al. [18] developed hesitant intuitionistic linguistic fuzzy set. For example, in order to evaluate performance of a brand of air conditioner, an expert may think it belongs to the degree of 'good' being $(0.6, 0.2)$, the degree of 'slightly good' being $(0.7, 0.3)$ and the degree of 'fair' being $(0.5, 0.4)$, which can be represented as hesitant intuitionistic fuzzy linguistic element as $\left\{\left(s_7, (0.6, 0.2)\right), \left(s_6, (0.7, 0.3)\right), \left(s_5, (0.5, 0.4)\right)\right\}$, where $\left(s_i, (\mu_i, \nu_i)\right)$ is the intuitionistic fuzzy linguistic element. However, in aggregation process, we cannot make sure linguistic terms belonging to the given linguistic term set. In order to overcome this difficulty, we first extend hesitant intuitionistic fuzzy set to make the linguistic term belong to a continuous set in this paper. Since distance measures are so important, we have not found any studies focusing on hesitant linguistic intuitionistic fuzzy information. We define several distance and similarity measures between two collections of hesitant intuitionistic fuzzy linguistic elements. Based on the proposed distance measures and TOPSIS, we propose a new multiple attribute decision making method to accommodate hesitant intuitionistic fuzzy linguistic information.

The rest of the paper is organized as follows. In Section 2, we first review some basic concepts on the hesitant intuitionistic fuzzy linguistic term set. In Section 3, we develop several generalized distance measures including the generalized hesitant intuitionistic fuzzy linguistic weighted distance, the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance and generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance. We propose a multiple attribute decision making method based on the new distance measure and TOPSIS. In Section 4, a practical example is given to illustrate feasibility and practical advantages of new method. The conclusions are given in the last section.

2. **Problem Statement and Preliminaries.**

An HFS is defined in terms of a function that returns a set of membership values of each element in the domain.

Definition 2.1. [3] *Letting X be a reference set, an HFS A on X is a function h that returns a subset of values in $[0, 1]$ when it is applied to X:*

$$A = \{< x, h_A(x) > \mid x \in X\}, \tag{1}$$

where $h_A(x)$ is a set of some different values in $[0, 1]$, representing the possible membership degrees of the element $x \in X$ to A. $h_A(x)$ is called a hesitant fuzzy element (HFE).

Suppose that $S = \{s_i \mid i = 0, \dots, g\}$ is a finite and totally ordered discrete term set, where s_i represents a possible value for a linguistic variable. For example, a set of nine terms S [19] can be expressed as $S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, s_7 = \text{good}, s_8 = \text{very good}, s_9 =$

extremely good}. In order to preserve all the information, the discrete linguistic term sets S can be extended to a continuous one $\bar{S} = \{s_\alpha \mid s_0 \leq s_\alpha \leq s_g, \alpha \in [0, g]\}$.

Definition 2.2. [14] *Let X be a reference set and $\bar{S}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLS) $\bar{A}$ on X is an ordered finite subset of consecutive linguistic terms of $\bar{S}$*

$$\bar{A} = \left\{< x_i, \bar{h}_{\bar{A}}(x_i) > \mid x_i \in X, i = 1, 2, \ldots, n\right\}, \tag{2}$$

where $\bar{h}_{\bar{A}}(x_i) : X \to \bar{S}$ denotes all the possible linguistic evaluation values of element $x_i \in X$. For convenience, we call $\bar{h}_{\bar{A}}(x_i)$ a hesitant fuzzy linguistic element (HFLE), which can be represented as $\bar{h}_{\bar{A}}(x_i) = \{s_i \mid s_i \in \bar{h}_{\bar{A}}(x_i)\}$, where s_i is a linguistic argument.

Definition 2.3. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a reference set and $\bar{S}$ be a linguistic term set. A hesitant intuitionistic fuzzy linguistic term set (HIFLS) $\tilde{A}$ on X is defined as*

$$\tilde{A} = \left\{< x_i, \tilde{h}_{\tilde{A}}(x_i) > \mid x_i \in X, i = 1, 2, \ldots, n\right\}, \tag{3}$$

where $\tilde{h}_{\tilde{A}}(x_i) : X \to H$ denotes all possible intuitionistic fuzzy linguistic evaluation values of element $x_i \in X$. For convenience, we call $\tilde{h}_{\tilde{A}}(x_i)$ a hesitant intuitionistic fuzzy linguistic element (HIFLE), which can be represented as $\tilde{h}_{\tilde{A}}(x_i) = \left\{\left(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)\right) \mid \left(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)\right) \in \tilde{h}_{\tilde{A}}(x_i)\right\}$, where $\tilde{s}_{\theta(i)} \in \bar{S}$ is a linguistic argument and (α_i, β_i) is the intuitionistic fuzzy value. $\left(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)\right)$ is intuitionistic fuzzy linguistic element (IFLE).

Definition 2.4. *Let $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}$ be HIFLEs, $\lambda > 0$. Some operations on these sets are defined as follows*

(1) $\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)) \in \tilde{h}_1, (\tilde{s}_{\theta(j)}, (\alpha_j, \beta_j)) \in \tilde{h}_2} \left\{\left(s_{\theta(i)+\theta(j)}, (\alpha_i + \alpha_j - \alpha_i\alpha_j,\ \beta_i\beta_j)\right)\right\}$,

(2) $\tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)) \in \tilde{h}_1, (\tilde{s}_{\theta(j)}, (\alpha_j, \beta_j)) \in \tilde{h}_2} \left\{\left(s_{\theta(i)\theta(j)}, (\alpha_i\alpha_j, \beta_i + \beta_j - \beta_i\beta_j)\right)\right\}$,

(3) $\lambda\tilde{h} = \bigcup_{(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)) \in \tilde{h}} \left\{\left(s_{\lambda\theta(i)}, (1 - (1 - \alpha_i)^\lambda, \beta_i^\lambda)\right)\right\}$,

(4) $(\tilde{h})^\lambda = \bigcup_{(\tilde{s}_{\theta(i)}, (\alpha_i, \beta_i)) \in \tilde{h}} \left\{\left(s_{(\theta(i))^\lambda}, (\alpha_i^\lambda, 1 - (1 - \beta_i)^\lambda)\right)\right\}$.

Definition 2.5. *Let $\tilde{a}_i = \left(s_{\theta(i)}, (\alpha_i, \beta_i)\right)$ be an IFLE, then the score function $s(\tilde{a}_i)$ of IFLE $\tilde{a}_i$ can be defined as $s(\tilde{a}_i) = \frac{\theta(i)}{g}(\alpha_i - \beta_i)$, and the accuracy function $h(\tilde{a}_i)$ can be defined as $h(\tilde{a}_i) = \frac{\theta(i)}{g}(\alpha_i + \beta_i)$, where g is the number of linguistic variables in linguistic term set.*

Based on the score function and accuracy function of IFLE, we can rank the IFLEs. Letting $\tilde{a}_i$ and $\tilde{a}_j$ be two IFLEs, if $s(\tilde{a}_i) > s(\tilde{a}_j)$, then $\tilde{a}_i > \tilde{a}_j$; if $s(\tilde{a}_i) = s(\tilde{a}_j)$ and if $h(\tilde{a}_i) > h(\tilde{a}_j)$, then $\tilde{a}_i > \tilde{a}_j$; if $s(\tilde{a}_i) = s(\tilde{a}_j)$ and $h(\tilde{a}_i) = h(\tilde{a}_j)$, then $\tilde{a}_i = \tilde{a}_j$.

Definition 2.6. *Let $\tilde{h} = \left\{\left(s_{\theta(i)}, (\alpha_i, \beta_i)\right) \mid \left(s_{\theta(i)}, (\alpha_i, \beta_i)\right) \in \tilde{h}_{\tilde{A}}(x_i)\right\}$ be HIFLE, the score function $S(\tilde{h})$ can be defined as $S(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum \frac{\theta(i)}{g}(\alpha_i - \beta_i)$, and the accuracy function $A(\tilde{h})$ can be defined as $A(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum \frac{\theta(i)}{g}(\alpha_i + \beta_i)$, where $l_{\tilde{h}}$ is the number of IFLEs in $\tilde{h}$ and g is the number of linguistic arguments in linguistic term set S.*

Based on the score function and accuracy function, we present the following method to compare HIFLEs. Let $\tilde{h}_1$ and $\tilde{h}_2$ be two HIFLEs. If $S(\tilde{h}_1) < S(\tilde{h}_2)$, then $\tilde{h}_1 < \tilde{h}_2$; if $S(\tilde{h}_1) = S(\tilde{h}_2)$ and $A(\tilde{h}_1) < A(\tilde{h}_2)$, then $\tilde{h}_1 < \tilde{h}_2$; if $S(\tilde{h}_1) = S(\tilde{h}_2)$ and $A(\tilde{h}_1) = A(\tilde{h}_2)$, then $\tilde{h}_1 = \tilde{h}_2$.

Let $\tilde{h}_1$, $\tilde{h}_2$ be two HIFLEs and $l = \max\{l_{\tilde{h}_1}, l_{\tilde{h}_2}\}$, where $l_{\tilde{h}_1}$, $l_{\tilde{h}_2}$ are the numbers of IFLEs in $\tilde{h}_1$ and $\tilde{h}_2$, respectively. If $l_{\tilde{h}_1} \neq l_{\tilde{h}_2}$, the shorter one should be extended until they have the same numbers of IFLEs in order to define the distance between the two HIFLEs more accurately. The decision maker can add IFLEs according to his/her risk attitude. If the decision maker is risk-seeking, the largest IFLE can be added; if the decision maker is risk-averse, the smallest IFLE can be added; if the decision maker is risk-neutral, the average IFLE can be added. In the real decision making, the decision makers can extend the HIFLEs according to their own risk attitudes and real needs.

Definition 2.7. *Letting $\tilde{h}_1$, $\tilde{h}_2$ be HIFLEs, then the distance measure between $\tilde{h}_1$ and $\tilde{h}_2$ is defined as $d(\tilde{h}_1,\tilde{h}_2)$, which satisfies the following properties:* (1) $0 \le d(\tilde{h}_1,\tilde{h}_2) \le 1$, (2) $d(\tilde{h}_1,\tilde{h}_2) = 0$ *if and only if* $\tilde{h}_1 = \tilde{h}_2$, (3) $d(\tilde{h}_1,\tilde{h}_2) = d(\tilde{h}_2,\tilde{h}_1)$.

3. Distance Measures between HIFLEs.

3.1. Distance measures between HIFLEs in discrete case.

Let $\mathbb{H}_1 = \{\tilde{h}_1^{(1)}, \tilde{h}_2^{(1)}, \dots, \tilde{h}_n^{(1)}\}$ and $\mathbb{H}_2 = \{\tilde{h}_1^{(2)}, \tilde{h}_2^{(2)}, \dots, \tilde{h}_n^{(2)}\}$ be two sets of HIFLEs and the associated weight vector be $w = (w_1, w_2, \dots, w_n)$ with $w_i \ge 0$ and $\sum_{i=1}^n w_i = 1$ and $\lambda > 0$. The HIFLEs have been extended until they have the same number of IFLEs. l is the number of IFLEs in $\tilde{h}_i^{(k)}$ and g is the number of linguistic variables in linguistic term set.

The generalized hesitant intuitionistic fuzzy linguistic weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1,\mathbb{H}_2) = \left(\sum_{j=1}^{n}\frac{w_j}{3l}\sum_{i=1}^{l}\left(\left|\theta^{(1)}_{(\sigma(i)j)} - \theta^{(2)}_{(\sigma(i)j)}\right|^{\lambda}/g^{\lambda} + \left|\mu^{(1)}_{\sigma(i)j} - \mu^{(2)}_{\sigma(i)j}\right|^{\lambda} + \left|\nu^{(1)}_{\sigma(i)j} - \nu^{(2)}_{\sigma(i)j}\right|^{\lambda}\right)\right)^{1/\lambda}, \tag{4}$$

where $\left(s_{\theta^{(k)}_{(\sigma(i)j)}}, \left(\mu^{(k)}_{\sigma(i)j}, \nu^{(k)}_{\sigma(i)j}\right)\right) \in \tilde{h}_j^{(k)}$, $\left(s_{\theta^{(k)}_{(\sigma(i)j)}}, \left(\mu^{(k)}_{\sigma(i)j}, \nu^{(k)}_{\sigma(i)j}\right)\right) \ge \left(s_{\theta^{(k)}_{(\sigma(i+1)j)}}, \left(\mu^{(k)}_{\sigma(i+1)j}, \nu^{(k)}_{\sigma(i+1)j}\right)\right)$, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, n$, $k = 1, 2$.

If $\lambda = 1$, the generalized hesitant intuitionistic fuzzy linguistic weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic weighted Hamming distance measure as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2) = \sum_{j=1}^{n}\frac{w_j}{3l}\sum_{i=1}^{l}\left(\left|\theta^{(1)}_{(\sigma(i)j)} - \theta^{(2)}_{(\sigma(i)j)}\right|/g + \left|\mu^{(1)}_{\sigma(i)j} - \mu^{(2)}_{\sigma(i)j}\right| + \left|\nu^{(1)}_{\sigma(i)j} - \nu^{(2)}_{\sigma(i)j}\right|\right). \tag{5}$$

If $\lambda = 2$, the generalized hesitant intuitionistic fuzzy linguistic weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic weighted Euclidean distance measure as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2) = \left(\sum_{j=1}^{n}\frac{w_j}{3l}\sum_{i=1}^{l}\left(\left|\theta^{(1)}_{(\sigma(i)j)} - \theta^{(2)}_{(\sigma(i)j)}\right|^{2}/g^{2} + \left|\mu^{(1)}_{\sigma(i)j} - \mu^{(2)}_{\sigma(i)j}\right|^{2} + \left|\nu^{(1)}_{\sigma(i)j} - \nu^{(2)}_{\sigma(i)j}\right|^{2}\right)\right)^{1/2}. \tag{6}$$

The generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1,\mathbb{H}_2) = \left(\sum_{j=1}^{n} w_j\Big(\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)} - \theta^{(2)}_{(\sigma(i)j)}\right|^{\lambda}/g^{\lambda}\right.\right.$$

$$+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|^{\lambda}+\left|\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\right|^{\lambda}\Big\}\Big)\Bigg)^{1/\lambda}. \quad (7)$$

If $\lambda = 1$, the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic weighted Hamming-Housdorff distance measure as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2)=\sum_{j=1}^{n} w_j\left(\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\right|/g+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|+\left|\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\right|\right\}\right). \quad (8)$$

If $\lambda = 2$, the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic weighted Euclidean-Housdorff distance measure as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2)=\left(\sum_{j=1}^{n} w_j\left(\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\right|^2/g^2+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|^2+\left|\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\right|^2\right\}\right)\right)^{1/2}. \quad (9)$$

The generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\sum_{j=1}^{n}\frac{w_j}{2}\Bigg(\frac{1}{3l}\sum_{i=1}^{l}\Big(\Big(\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\Big)^{\lambda}/g^{\lambda}+\Big(\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\Big)^{\lambda}$$
$$+\Big(\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\Big)^{\lambda}\Big)+\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\right|^{\lambda}/g^{\lambda}$$
$$+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|^{\lambda}+\left|\nu^{(1)}_{\sigma(i)}-\nu^{(2)}_{\sigma(i)}\right|^{\lambda}\Big\}\Bigg)\Bigg)^{1/\lambda}. \quad (10)$$

If $\lambda = 1$, then the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic hybrid weighted Hamming distance as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2)=\sum_{j=1}^{n}\frac{w_j}{2}\Bigg(\frac{1}{3l}\sum_{i=1}^{l}\Big(\Big(\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\Big)/g+\Big(\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\Big)+\Big(\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\Big)\Big)$$
$$+\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\right|/g+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|+\left|\nu^{(1)}_{\sigma(i)}-\nu^{(2)}_{\sigma(i)}\right|\Big\}\Bigg). \quad (11)$$

If $\lambda = 2$, then the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the hesitant intuitionistic fuzzy linguistic hybrid weighted Euclidean distance as follows:

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\sum_{j=1}^{n}\frac{w_j}{2}\Bigg(\frac{1}{3l}\sum_{i=1}^{l}\Big(\Big(\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\Big)^{2}/g^{2}+\Big(\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\Big)^{2}$$
$$+\Big(\nu^{(1)}_{\sigma(i)j}-\nu^{(2)}_{\sigma(i)j}\Big)^{2}\Big)+\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}-\theta^{(2)}_{(\sigma(i)j)}\right|^{2}/g^{2}$$
$$+\left|\mu^{(1)}_{\sigma(i)j}-\mu^{(2)}_{\sigma(i)j}\right|^{2}+\left|\nu^{(1)}_{\sigma(i)}-\nu^{(2)}_{\sigma(i)}\right|^{2}\Big\}\Bigg)\Bigg)^{1/2}. \quad (12)$$

3.2. **Distance measures between HIFLEs in continuous case.** Let $x \in [a, b]$, $w(x)$ be the weight of x with $w(x) \in [0, 1]$ and $\int_a^b w(x)dx = 1$. Let $\mathbb{H}_1 = \{\tilde{h}_1^{(1)}(x), \tilde{h}_2^{(1)}(x), \ldots, \tilde{h}_n^{(1)}(x)\}$ and $\mathbb{H}_2 = \{\tilde{h}_1^{(2)}(x), \tilde{h}_2^{(2)}(x), \ldots, \tilde{h}_n^{(2)}(x)\}$ be two sets of HIFLEs. We define some generalized hesitant intuitionistic fuzzy linguistic weighted distance measures in continuous case as follows.

The generalized continuous hesitant intuitionistic fuzzy linguistic weighted distance measure between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1, \mathbb{H}_2) = \left(\int_a^b \frac{w(x)}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right|^\lambda / g^\lambda + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right|^\lambda + \left|\nu_{\sigma(i)j}^{(1)}(x) - \nu_{\sigma(i)j}^{(2)}(x)\right|^\lambda\right) dx\right)^{1/\lambda}. \tag{13}$$

If $\lambda = 1$, the generalized continuous hesitant intuitionistic fuzzy linguistic weighted distance measure between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the continuous hesitant intuitionistic fuzzy linguistic weighted distance measure as follows

$$d(\mathbb{H}_1, \mathbb{H}_2) = \int_a^b \frac{w(x)}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right| / g + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right| + \left|\nu_{\sigma(i)j}^{(1)}(x) - \nu_{\sigma(i)j}^{(2)}(x)\right|\right) dx. \tag{14}$$

If $\lambda = 2$, the generalized continuous hesitant intuitionistic fuzzy linguistic weighted distance measure between $\mathbb{H}_1$ and $\mathbb{H}_2$ becomes the continuous hesitant intuitionistic fuzzy linguistic weighted Euclidean distance measure as follows

$$d(\mathbb{H}_1, \mathbb{H}_2) = \left(\int_a^b \frac{w(x)}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right|^2 / g^2 + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right|^2 + \left|\nu_{\sigma(i)j}^{(1)}(x) - \nu_{\sigma(i)j}^{(2)}(x)\right|^2\right) dx\right)^{1/2}. \tag{15}$$

If $w(x) = \frac{1}{b-a}$, $\forall x \in [a, b]$, then the above Equations (13)-(15) reduce to the following generalized continuous hesitant intuitionistic fuzzy linguistic distance measure, the continuous hesitant intuitionistic fuzzy linguistic distance measure and the continuous hesitant intuitionistic fuzzy linguistic Euclidean distance measure, respectively.

$$d(\mathbb{H}_1, \mathbb{H}_2) = \left(\frac{1}{b-a} \int_a^b \frac{1}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right|^\lambda / g^\lambda + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right|^\lambda + \left|\nu_{\sigma(i)j}^{(1)}(x) - \nu_{\sigma(i)j}^{(2)}(x)\right|^\lambda\right) dx\right)^{1/\lambda}. \tag{16}$$

$$d(\mathbb{H}_1, \mathbb{H}_2) = \frac{1}{b-a} \int_a^b \frac{1}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right| / g + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right| + \left|\nu_{\sigma(i)j}^{(1)}(x) - \nu_{\sigma(i)j}^{(2)}(x)\right|\right) dx. \tag{17}$$

$$d(\mathbb{H}_1, \mathbb{H}_2) = \left(\frac{1}{b-a} \int_a^b \frac{1}{3l} \sum_{i=1}^{l} \left(\left|\theta_{(\sigma(i)j)}^{(1)}(x) - \theta_{(\sigma(i)j)}^{(2)}(x)\right|^2 / g^2 + \left|\mu_{\sigma(i)j}^{(1)}(x) - \mu_{\sigma(i)j}^{(2)}(x)\right|^2\right.$$

$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^2\Big)dx\Bigg)^{1/2}. \tag{18}$$

The generalized continuous hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance measure between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\int_a^b\frac{w(x)}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^\lambda/g^\lambda+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^\lambda$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^\lambda\Big\}\Big)dx\Bigg)^{1/\lambda}. \tag{19}$$

If $\lambda=1$, the above distance measure becomes the continuous hesitant intuitionistic fuzzy linguistic weighted Hamming-Hausdorff distance measure as follows

$$d(\mathbb{H}_1,\mathbb{H}_2)=\int_a^b\frac{w(x)}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|/g+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|\Big\}\Big)dx. \tag{20}$$

If $\lambda=2$, the generalized continuous hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance measure becomes the continuous hesitant intuitionistic fuzzy Euclidean-Hausdorff distance measure as follows

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\int_a^b\frac{w(x)}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^2/g^2+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^2$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^2\Big\}\Big)dx\Bigg)^{1/2}. \tag{21}$$

If $w(x)=\frac{1}{b-a}$, $\forall x\in[a,b]$, then the above Equations (19)-(21) reduce to the following generalized continuous hesitant intuitionistic fuzzy linguistic Hausdorff distance measure, the continuous hesitant intuitionistic fuzzy linguistic Hamming-Hausdorff distance measure and the continuous hesitant intuitionistic fuzzy Euclidean-Hausdorff distance measure, respectively

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\frac{1}{b-a}\int_a^b\frac{1}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^\lambda/g^\lambda+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^\lambda$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^\lambda\Big\}\Big)dx\Bigg)^{1/\lambda}. \tag{22}$$

$$d(\mathbb{H}_1,\mathbb{H}_2)=\frac{1}{b-a}\int_a^b\frac{1}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|/g+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|\Big\}\Big)dx. \tag{23}$$

$$d(\mathbb{H}_1,\mathbb{H}_2)=\Bigg(\frac{1}{b-a}\int_a^b\frac{1}{3l}\Big(\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^2/g^2+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^2$$
$$+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^2\Big\}\Big)dx\Bigg)^{1/2}. \tag{24}$$

The generalized continuous hesitant intuitionistic fuzzy linguistic hybrid weighted distance between $\mathbb{H}_1$ and $\mathbb{H}_2$ can be defined as

$$d(\mathbb{H}_1,\mathbb{H}_2)=\left(\int_a^b\frac{w(x)}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)^{\lambda}/g^{\lambda}+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)^{\lambda}\right.\right.\right.$$
$$\left.+\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)^{\lambda}\right)+\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^{\lambda}/g^{\lambda}\right.$$
$$\left.\left.\left.+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^{\lambda}+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^{\lambda}\right\}\right)dx\right)^{1/\lambda}. \tag{25}$$

If $\lambda=1$, the above distance measure becomes the continuous hesitant intuitionistic fuzzy linguistic hybrid Hamming weighted distance as follows

$$d(\mathbb{H}_1,\mathbb{H}_2)=\int_a^b\frac{w(x)}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)/g+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)\right.\right.$$
$$\left.+\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)\right)+\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|/g\right.$$
$$\left.\left.+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|\right\}\right)dx. \tag{26}$$

If $\lambda=2$, the continuous hesitant intuitionistic fuzzy linguistic hybrid weighted distance becomes the continuous hesitant intuitionistic fuzzy linguistic hybrid Euclidean weighted distance as follows

$$d(\mathbb{H}_1,\mathbb{H}_2)=\left(\int_a^b\frac{w(x)}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)^{2}/g^{2}+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)^{2}\right.\right.\right.$$
$$\left.+\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)^{2}\right)+\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^{2}/g^{2}\right.$$
$$\left.\left.\left.+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^{2}+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^{2}\right\}\right)dx\right)^{1/2}. \tag{27}$$

If $w(x)=\frac{1}{b-a}$, $\forall x\in[a,b]$, then the above Equations (25)-(27) reduce to the following generalized continuous hesitant intuitionistic fuzzy linguistic hybrid distance, continuous hesitant intuitionistic fuzzy linguistic hybrid Hamming distance and continuous hesitant intuitionistic fuzzy linguistic hybrid distance, respectively

$$d(\mathbb{H}_1,\mathbb{H}_2)=\left(\frac{1}{b-a}\int_a^b\frac{1}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)^{\lambda}/g^{\lambda}+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)^{\lambda}\right.\right.\right.$$
$$\left.+\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)^{\lambda}\right)+\frac{1}{3}\max_i\left\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^{\lambda}/g^{\lambda}\right.$$
$$\left.\left.\left.+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^{\lambda}+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^{\lambda}\right\}\right)dx\right)^{1/\lambda}. \tag{28}$$

$$d(\mathbb{H}_1,\mathbb{H}_2)=\frac{1}{b-a}\int_a^b\frac{1}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)/g+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)\right.\right.$$
$$+\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)\Big)+\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|/g$$

$$+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|\Big\}\Bigg)dx. \tag{29}$$

$$d(\mathbb{H}_1,\mathbb{H}_2)=\left(\frac{1}{b-a}\int_a^b\frac{1}{2}\left(\frac{1}{3l}\sum_{i=1}^{l}\left(\left(\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right)^2/g^2+\left(\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right)^2\right.\right.\right.$$
$$+\left.\left(\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right)^2\right)+\frac{1}{3}\max_i\Big\{\left|\theta^{(1)}_{(\sigma(i)j)}(x)-\theta^{(2)}_{(\sigma(i)j)}(x)\right|^2/g^2$$
$$\left.\left.+\left|\mu^{(1)}_{\sigma(i)j}(x)-\mu^{(2)}_{\sigma(i)j}(x)\right|^2+\left|\nu^{(1)}_{\sigma(i)j}(x)-\nu^{(2)}_{\sigma(i)j}(x)\right|^2\Big\}\right)dx\right)^{1/2}. \tag{30}$$

Algorithm

Step 1. Since different HIFLEs have different numbers of IFLEs, we extend the HIFLEs by adding the minimum IFLE in each attribute evaluation value until they have the same number of IFLEs for the same attribute.

Step 2. Determine the hesitant intuitionistic fuzzy linguistic positive ideal solution (HIFLPIS) H^+ and the hesitant intuitionistic fuzzy linguistic negative ideal solution (HIFLNIS) H^- as follows

$$\text{HIFLPIS}:\ H^+=\left(\tilde{h}_1^+,\tilde{h}_2^+,\dots,\tilde{h}_n^+\right)=\left(\max_i\tilde{h}_{i1},\max_i\tilde{h}_{i2},\dots,\max_i\tilde{h}_{in}\right), \tag{31}$$

$$\text{HIFLNIS}:\ H^-=\left(\tilde{h}_1^-,\tilde{h}_2^-,\dots,\tilde{h}_n^-\right)=\left(\min_i\tilde{h}_{i1},\min_i\tilde{h}_{i2},\dots,\min_i\tilde{h}_{in}\right). \tag{32}$$

Step 3. Calculate the generalized hesitant intuitionistic fuzzy linguistic weighted distance between each alternative hesitant intuitionistic fuzzy linguistic evaluation values H_i $(i=1,2,\dots,m)$ and the HIFLPIS H^+, HIFLNIS H^- as follows

$$d_i^+=\left(\sum_{j=1}^{n}\frac{w_j}{3l}\sum_{i=1}^{l}\left(\left|\theta_{(\sigma(i)j)}-\theta^+_{(\sigma(i)j)}\right|^\lambda/g^\lambda+\left|\mu_{\sigma(i)j}-\mu^+_{\sigma(i)j}\right|^\lambda+\left|\nu_{\sigma(i)j}-\nu^+_{\sigma(i)j}\right|^\lambda\right)\right)^{1/\lambda}, \tag{33}$$

$$d_i^-=\left(\sum_{j=1}^{n}\frac{w_j}{3l}\sum_{i=1}^{l}\left(\left|\theta_{(\sigma(i)j)}-\theta^-_{(\sigma(i)j)}\right|^\lambda/g^\lambda+\left|\mu_{\sigma(i)j}-\mu^-_{\sigma(i)j}\right|^\lambda+\left|\nu_{\sigma(i)j}-\nu^-_{\sigma(i)j}\right|^\lambda\right)\right)^{1/\lambda}. \tag{34}$$

Step 4. Calculate the relative closeness CC_i $(i=1,2,\dots,m)$ of each alternative A_i $(i=1,2,\dots,m)$ as follows:

$$CC_i=\frac{d_i^-}{d_i^-+d_i^+},\ i=1,2,\dots,m. \tag{35}$$

Step 5. Rank the alternatives according to the ranking of the relative closeness values. The larger the alternative's relative closeness value is, the better the alternative is.

4. **Numerical Example.** A real example of the metro project risk assessment is adopted in this section. Various types of risk exist in construction process of metro project. If accidents happen during construction, the project may suffer from serious losses and social effects. Suppose that there is project needing to be constructed. Multiple experts from different fields are invited to evaluate the project risk. After preliminary screening, five alternatives A_i $(i=1,2,\dots,5)$ are left for further evaluation. Four attributes, C_1 – policy risk, C_2 – environmental risk, C_3 – technical risk, C_4 – financing risk, are taken into consideration in selecting the alternative. The weight vector of the attributes is $w=(0.30,0.20,0.15,0.35)$. The experts evaluate the alternatives with IFLEs and the

Table 1. Decision matrix $\widetilde{D}$

	C_1	C_2
A_1	$\{(s_3,(0.5,0.2))\}$	$\{(s_7,(0.7,0.1)),(s_8,(0.8,0.2))\}$
A_2	$\{(s_6,(0.5,0.4)),(s_7,(0.5,0.5))\}$	$\{(s_2,(0.6,0.2))\}$
A_3	$\{(s_4,(0.6,0.1))\}$	$\{(s_5,(0.7,0.3))\}$
A_4	$\{(s_7,(0.8,0.2)),(s_8,(0.6,0.3))\}$	$\{(s_4,(0.5,0.2))\}$
A_5	$\{(s_3,(0.6,0.3))\}$	$\{(s_6,(0.6,0.2)),(s_7,(0.8,0.1)),(s_8,(0.7,0.3))\}$
	C_3	C_4
A_1	$\{(s_6,(0.6,0.3))\}$	$\{(s_4,(0.7,0.2)),(s_6,(0.5,0.4))\}$
A_2	$\{(s_7,(0.7,0.1)),(s_8,(0.6,0.2))\}$	$\{(s_5,(0.6,0.2))\}$
A_3	$\{(s_2,(0.5,0.4)),(s_3,(0.6,0.3)),(s_4,(0.5,0.2))\}$	$\{(s_8,(0.6,0.1))\}$
A_4	$\{(s_6,(0.6,0.2)),(s_5,(0.7,0.3))\}$	$\{(s_3,(0.6,0.2))\}$
A_5	$\{(s_2,(0.7,0.2))\}$	$\{(s_6,(0.8,0.1))\}$

Table 2. The extended decision matrix $\widetilde{D'}$

	C_1	C_2
A_1	$\{(s_3,(0.5,0.2)),(s_3,(0.5,0.2))\}$	$\{(s_7,(0.7,0.1)),(s_8,(0.8,0.2)),(s_2,(0.6,0.2))\}$
A_2	$\{(s_6,(0.5,0.4)),(s_7,(0.5,0.5))\}$	$\{(s_2,(0.6,0.2)),(s_2,(0.6,0.2)),(s_2,(0.6,0.2))\}$
A_3	$\{(s_4,(0.6,0.1)),(s_3,(0.5,0.2))\}$	$\{(s_5,(0.7,0.3)),(s_2,(0.6,0.2)),(s_2,(0.6,0.2))\}$
A_4	$\{(s_7,(0.8,0.2)),(s_8,(0.6,0.3))\}$	$\{(s_4,(0.5,0.2)),(s_2,(0.6,0.2)),(s_2,(0.6,0.2))\}$
A_5	$\{(s_3,(0.6,0.3)),(s_3,(0.5,0.2))\}$	$\{(s_6,(0.6,0.2)),(s_7,(0.8,0.1)),(s_8,(0.7,0.3))\}$
	C_3	C_4
A_1	$\{(s_6,(0.6,0.3)),(s_2,(0.5,0.4)),(s_2,(0.5,0.4))\}$	$\{(s_4,(0.7,0.2)),(s_6,(0.5,0.4))\}$
A_2	$\{(s_7,(0.7,0.1)),(s_8,(0.6,0.2)),(s_2,(0.5,0.4))\}$	$\{(s_5,(0.6,0.2)),(s_3,(0.6,0.2))\}$
A_3	$\{(s_2,(0.5,0.4)),(s_3,(0.6,0.3)),(s_4,(0.5,0.2))\}$	$\{(s_8,(0.6,0.1)),(s_3,(0.6,0.2))\}$
A_4	$\{(s_6,(0.6,0.2)),(s_5,(0.7,0.3)),(s_2,(0.5,0.4))\}$	$\{(s_3,(0.6,0.2)),(s_3,(0.6,0.2))\}$
A_5	$\{(s_2,(0.7,0.2)),(s_2,(0.5,0.4)),(s_2,(0.5,0.4))\}$	$\{(s_6,(0.8,0.1)),(s_3,(0.6,0.2))\}$

hesitant intuitionistic fuzzy linguistic decision matrix $\widetilde{D}$ can be got as in Table 1. We use the new algorithm to rank the alternatives.

Step 1. Extend the HIFLEs by adding the minimum IFLEs until they have the same number of IFLEs for the same attribute and the results can be got as in Table 2.

Step 2. The HIFLPIS and the HIFLNIS can be determined as follows:

$$\text{HIFLPIS: } H^+ = \Big(\{(s_7,(0.8,0.2)),(s_8,(0.6,0.3))\},\{(s_7,(0.8,0.1)),(s_8,(0.7,0.3)),(s_6,(0.6,0.2))\},\{(s_7,(0.7,0.1)),(s_8,(0.6,0.2)),(s_5,(0.5,0.2))\},\{(s_6,(0.8,0.1)),(s_6,(0.5,0.4))\}\Big).$$

$$\text{HIFLNIS: } H^- = \Big(\{(s_3,(0.5,0.2)),(s_3,(0.5,0.2))\},\{(s_2,(0.6,0.2)),(s_2,(0.6,0.2)),(s_2,(0.6,0.2))\},\{(s_2,(0.5,0.4)),(s_2,(0.5,0.4)),(s_2,(0.5,0.4))\},\{(s_3,(0.6,0.2)),(s_3,(0.6,0.2))\}\Big).$$

Step 3. The generalized hesitant intuitionistic fuzzy linguistic weighted distances can be calculated by using Equations (33) and (34). Here we consider $\lambda = 1, 2, 3, 5, 10$, respectively. The results are shown in Table 3.

Step 4. Determine the relative closeness of the alternatives by using Equation (35). The results are also shown in Table 3.

TABLE 3. The results obtained by generalized HIFL weighted distance

		A_1	A_2	A_3	A_4	A_5	Rankings
λ=1	d_i^+	0.1511	0.1694	0.2058	0.1627	0.1460	
	d_i^-	0.0960	0.1006	0.0779	0.1051	0.1018	$A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$
	CC_i	0.3886	0.3724	0.2745	0.3926	0.4107	
λ=2	d_i^+	0.2264	0.2143	0.2772	0.2131	0.2419	
	d_i^-	0.2079	0.2077	0.1593	0.2074	0.1996	$A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$
	CC_i	0.4787	0.4921	0.3650	0.4932	0.4520	
λ=3	d_i^+	0.2862	0.2728	0.3266	0.2647	0.3042	
	d_i^-	0.2759	0.2707	0.2265	0.2639	0.2675	$A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$
	CC_i	0.4908	0.4980	0.4094	0.4993	0.4679	
λ=5	d_i^+	0.3706	0.3651	0.4006	0.3497	0.3846	
	d_i^-	0.3333	0.3229	0.3867	0.3030	0.3292	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$
	CC_i	0.4735	0.4693	0.4172	0.4642	0.4612	
λ=10	d_i^+	0.4753	0.4816	0.4936	0.4746	0.4807	
	d_i^-	0.4581	0.4460	0.3975	0.3973	0.4577	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$
	CC_i	0.4908	0.4808	0.4460	0.4557	0.4877	

Step 5. Rank the alternatives according to the ranking of the relative closeness of the alternatives. The results are also shown in Table 3.

From the results we can see that different rankings of the alternatives can be got by using different λ. If $\lambda = 1$, the best alternative is A_5. The optimal alternative is A_4 if $\lambda = 2$ or $\lambda = 3$. For $\lambda = 5$ and $\lambda = 10$, the best alternative becomes A_1. A_5 has become suboptimal alternative in the case that $\lambda = 10$ since it has the largest evaluation value. The λ can be seen as the risk attitude of the decision maker. The larger the λ is, the more risk-seeking the decision maker is. This is due to that the large evaluation values play more and more important role in the results and the effects of relative small values have been reduced during aggregation process. The decision maker can select the corresponding λ according to his/her risk attitude and real needs. We also use the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance and the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance. The relative closeness values of the alternatives and the rankings of the alternatives are shown in Tables 4 and 5, respectively.

From the ranking results we can see that most rankings are the same or change slightly in different distance measures. For example, the alternatives have the same ranking in three distance measures if $\lambda = 2$. If $\lambda = 3$, the alternatives have the same ranking in generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance and the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance and the suboptimal alternative in the above distance measures becomes the optimal alternative in generalized hesitant intuitionistic fuzzy linguistic weighted distance. For $\lambda = 1$, A_5 is the best alternative if the generalized hesitant intuitionistic fuzzy linguistic weighted distance

TABLE 4. The results obtained by generalized HIFL weighted Hausdorff distance

	CC_1	CC_2	CC_3	CC_4	CC_5	Rankings
λ=1	0.4470	0.4362	0.3872	0.4589	0.4647	$A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$
λ=2	0.4699	0.4823	0.4173	0.4831	0.4607	$A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$
λ=3	0.4765	0.4941	0.4345	0.4859	0.4693	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
λ=5	0.4874	0.4944	0.4510	0.4788	0.4857	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$
λ=10	0.5001	0.4936	0.4608	0.4659	0.5002	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$

TABLE 5. The results obtained by generalized HIFL hybrid weighted distance

	CC_1	CC_2	CC_3	CC_4	CC_5	Rankings
λ=1	0.4235	0.4232	0.3447	0.4448	0.4430	$A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$
λ=2	0.4595	0.4800	0.4028	0.4858	0.4548	$A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$
λ=3	0.4722	0.4931	0.4296	0.4929	0.4668	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
λ=5	0.4866	0.4941	0.4514	0.4859	0.4843	$A_2 \succ A_1 \succ A_4 \succ A_5 \succ A_3$
λ=10	0.5552	0.5513	0.5369	0.5470	0.5536	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

TABLE 6. Hesitant linguistic decision matrix D

	C_1	C_2	C_3	C_4
A_1	$\{s_3\}$	$\{s_7,s_8\}$	$\{s_6\}$	$\{s_4,s_6\}$
A_2	$\{s_6,s_7\}$	$\{s_2\}$	$\{s_7,s_8\}$	$\{s_5\}$
A_3	$\{s_4\}$	$\{s_5\}$	$\{s_2,s_3,s_4\}$	$\{s_8\}$
A_4	$\{s_7,s_8\}$	$\{s_4\}$	$\{s_6,s_5\}$	$\{s_3\}$
A_5	$\{s_3\}$	$\{s_6,s_7,s_8\}$	$\{s_2\}$	$\{s_6\}$

or the generalized hesitant intuitionistic fuzzy linguistic weighted Hausdorff distance is used and A_4 becomes the first choice if the generalized hesitant intuitionistic fuzzy linguistic hybrid weighted distance is used.

In order to illustrate the practical advantages of the proposed method, we compare it with the existing method in reference [15]. If only linguistic arguments are considered, the decision matrix D is got as in Table 6. Extend decision matrix by adding the minimum linguistic evaluation value in each attribute evaluation value until each attribute having the same number of linguistic evaluation values and the results are shown in Table 7. Let $S^+ = (s_1^+, s_2^+, s_3^+, s_4^+) = (\max_i s_{i1}, \max_i s_{i2}, \max_i s_{i3}, \max_i s_{i4})$ be hesitant linguistic positive ideal solution (HLPIS) and $S^- = (s_1^-, s_2^-, s_3^-, s_4^-) = (\min_i s_{i1}, \min_i s_{i2}, \min_i s_{i3}, \min_i s_{i4})$ be hesitant linguistic negative ideal solution (HLNIS), respectively. Assume the attribute weight vector is $(0.30, 0.20, 0.15, 0.35)$, which is the same as that in above example in order to facilitate comparison. Calculate the weighted distances of each alternatives' evaluation values to the HLPIS and the HLNIS by using the generalized hesitant linguistic weighted distance (GHLWD) as follows

$$d_{GHLWD}(\mathbb{H}_1, \mathbb{H}_2) = \left(\sum_{j=1}^{n} \frac{w_j}{l} \sum_{i=1}^{l} \left(\left| \theta^{(1)}_{(\sigma(i)j)} - \theta^{(2)}_{(\sigma(i)j)} \right|^{\lambda} / g^{\lambda} \right) \right)^{1/\lambda},$$

where $s_{\theta^{(k)}_{(\sigma(i)j)}} \in \tilde{h}_j^{(k)}$, $s_{\theta^{(k)}_{(\sigma(i)j)}} \geq s_{\theta^{(k)}_{(\sigma(i+1)j)}}$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, n$, $k = 1, 2$. Then calculate the closeness coefficients and rank the alternatives accordingly. The results are shown in Table 8. Comparing the results with that in Table 3, we can see that different results can be got if only linguistic evaluation values are considered and the intuitionistic fuzzy memberships are not considered. Alternative A_2 becomes the optimal alternative in most cases. However, if we consider the intuitionistic fuzzy memberships of linguistic arguments, A_2 is not the optimal alternative for different λ. Intuitionistic fuzzy memberships can be used to reflect hesitation in evaluation process. By using hesitant fuzzy linguistic information, more accurate and scientific results can be got.

5. **Conclusions.** In this paper, we first develop hesitant intuitionistic fuzzy linguistic elements by generalizing the hesitant fuzzy set to accommodate linguistic arguments and intuitionistic fuzzy values. Then we investigate several types of the distance measures and

TABLE 7. Extended hesitant linguistic decision matrix D'

	C_1	C_2	C_3	C_4
A_1	$\{s_3,s_3\}$	$\{s_7,s_8,s_2\}$	$\{s_6,s_2,s_2\}$	$\{s_4,s_6\}$
A_2	$\{s_6,s_7\}$	$\{s_2,s_2,s_2\}$	$\{s_7,s_8,s_2\}$	$\{s_5,s_3\}$
A_3	$\{s_4,s_3\}$	$\{s_5,s_2,s_2\}$	$\{s_2,s_3,s_4\}$	$\{s_8,s_3\}$
A_4	$\{s_7,s_8\}$	$\{s_4,s_2,s_2\}$	$\{s_6,s_5,s_2\}$	$\{s_3,s_3\}$
A_5	$\{s_3,s_3\}$	$\{s_6,s_7,s_8\}$	$\{s_2,s_2,s_2\}$	$\{s_6,s_3\}$

TABLE 8. The results obtained by GHLWD

	CC_1	CC_2	CC_3	CC_4	CC_5	Rankings
λ=1	0.3904	0.4815	0.2794	0.4527	0.3765	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
λ=2	0.4897	0.5143	0.3976	0.5004	0.4695	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
λ=3	0.4955	0.5009	0.4143	0.5003	0.4657	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
λ=5	0.5434	0.5475	0.4955	0.5068	0.5320	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$
λ=10	0.5036	0.4936	0.4587	0.4683	0.5005	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

similarity measures between HIFLEs based on the generalized mean, the Hamming distance, the Euclidean distance and the Hausdorff distance. It should be pointed out that the number of intuitionistic fuzzy linguistic elements are different for different HIFLEs in most cases. They should be extended until they have the same number of IFLEs in each HIFLE according to the decision maker's risk attitude. We focus on distance measures only since similarity measure can be easily obtained by using the relationship of distance measure and similarity measure. We present several generalized weighted distance measures between two collections of HIFLEs. Based on the proposed distance measures and TOPSIS, we develop a new hesitant intuitionistic fuzzy linguistic multiple attribute decision making method. A practical example of the metro project risk assessment is presented to illustrate the feasibility and efficiency of the proposed method. From the numerical results we can see that different distance measures focus on different aspects of the decision problem and λ can be seen as the risk attitude of the decision makers. As a result, the decision maker can select the proper distance measure and λ according to his/her risk preference and the real needs. We also compare the new method with some existing methods. More accurate and scientific results can be got by using the new method.

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