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NEW DEVELOPMENT OF STOCHASTIC HOPF BIFURCATION ANALYSIS IN A NOVEL TWO-DIMENSIONAL CHAOTIC SYSTEM

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Abstract. *In this paper, we proposed a two-dimensional chaotic system with random parameter. Stochastic Hopf bifurcation in the two-dimensional chaotic system is studied by the orthogonal polynomial approximation method, which reduces the stochastic nonlinear dynamical system into its equal deterministic nonlinear dynamical system. The parameter condition to ensure the appearance of Hopf bifurcation in this two-dimensional chaotic system is obtained by the Hopf bifurcation theorem with the aid of the Maple program. The direction and stability of the Hopf bifurcation are studied by the calculation of the first Lyapunov coefficient. The critical value of stochastic Hopf bifurcation is determined by deterministic parameters and the intensity of random parameter in stochastic system. As the intensity of random parameter is increased, the critical value of stochastic Hopf bifurcation is also increased. At last, numerical simulations results show the effectiveness of the method and the correctness of the theoretical results in the paper.*
Keywords: Stability, Stochastic chaos, Stochastic Hopf bifurcation, Chebyshev polynomial approximation

1. Introduction.

Stochastic bifurcation and chaos are a hot topic in the area of nonlinear dynamics in the past few decades. In recent years, many scholars have carried on the thorough research in many different systems. It has been applied to chemical plants [1], mechanical systems [2], ecosystems [3], economics [4], and biology [5,6], and computer network [7]. Compared with large numbers of investigations on deterministic bifurcation phenomena, stochastic bifurcations are still in its infancy in the sense of theory, methods and applications [8]. Unlike deterministic bifurcations concerning the sudden change of topological properties of the portrait of phase trajectories, stochastic bifurcations pay attention to the qualitative changes of the stationary probability density. However, because of the complexity of the system, these studies are just limited to qualitative stage, and the quantitative research of system is not too much. The stochastic systems are widespread in nature, and the demand to the veracity and accuracy of the actual model become higher and higher. Therefore, more and more random systems are used to depict the dynamic relationship among things, especially stochastic system with random parameter. To solve the problems of the system with random parameters, there are several basic mathematical methods available: one is Monte-Carlo method [9]. The second is stochastic perturbation method. And the third is orthogonal polynomial approximation method, which was introduced in [10] and improved by Li [11], and is an effective analytical method [12]. However, Monte-Carlo method can give the results of higher accuracy only in the case of completing a great amount of calculation. Stochastic perturbation method requires that it must be a small amount of the random parameter's variation of stochastic system; otherwise, calculation accuracy will be collapsed. Orthogonal polynomial approximation method does not need to assume small random disturbance, and amount of calculation relative random simulation method smaller, so it is a more practical method.

Recently, stochastic bifurcation and chaos in some typical dynamical models were successfully analyzed by the Chebyshev polynomial approximation [12-15]. Fang et al. [16] studied the stochastic parameter system with bounded random variables by the Chebyshev polynomial approximation method, and then further applied this method to study bifurcation and chaos of stochastic Duffing system. Li and Li [17] also applied this method to discuss the bifurcation and chaos phenomena, control and synchronization problems [18-21] of random dynamical system. Ma [12] explored the stochastic Hopf bifurcation in Brusselator system with random parameter. The results show that orthogonal polynomial approximation method is not only effective for random dynamic problems with stochastic parameters, but also found some characteristics of the random dynamic system. We will use the same strategy to explore the stochastic chaos and Hopf bifurcation in a two-dimensional chaotic system. The contribution of this paper is giving a common method for the biological systems, the ecosystems, and the financial system and so on to study the bifurcation and chaos phenomena. This method can be better to predict what may occur in different periods, so that we can take preventive measures to avoid the bad development direction for us. The paper discussed Hopf bifurcation of a two-dimensional chaotic system in detail, and the condition for the existence of Hopf bifurcation is obtained.

The rest of this paper is organized as follows. To be specific, we first transform the original stochastic two-dimensional chaotic system into its equivalent deterministic one by orthogonal polynomial approximation in Section 2. Section 3 is devoted to studying existence, direction and stability of Hopf bifurcation of stochastic two-dimensional chaotic system. The numerical simulations about the stochastic two-dimensional chaotic system are given in Section 4. Section 5 concludes the paper.

2. **Chebyshev Polynomial Approximation of a Novel Two-Dimensional Chaotic System.** The deterministic two-dimensional chaotic system is as follows:

$$\begin{cases} \dot{x} = Ay \\ \dot{y} = x - \bar{u}y - x^2 y \end{cases} \tag{1}$$

where A and $\bar{u}$ are deterministic parameters. It is easy to know that Equation (1) has a unique equilibrium $(0, 0)$.

If A is a deterministic parameter, and $\bar{u}$ is a random parameter, then Equation (1) is a stochastic two-dimensional Hopf bifurcation model. Suppose that $\bar{u}$ can be expressed as

$$\bar{u} = a + \delta U \tag{2}$$

where a and δ are the deterministic parameters of $\bar{u}$, and δ is regarded as intensity of the random parameter $\bar{u}$; U is the random variable defined on $(-\infty, +\infty)$ with some probability density function $\rho_U(u)$. It follows from the orthogonal polynomial approximation that the responses of system (1) can be expressed approximately by the following series under condition of the convergence in mean square

$$\begin{cases} x(t, u) = \sum\limits_{i=0}^{M} x_i(t) P_i(u) \\ y(t, u) = \sum\limits_{i=0}^{M} y_i(t) P_i(u) \end{cases} \tag{3}$$

where

$$\begin{cases} x_i(t) = \int_{-\infty}^{+\infty} \rho_U(u)x(t,u)P_i(u)du \\ y_i(t) = \int_{-\infty}^{+\infty} \rho_U(u)y(t,u)P_i(u)du \end{cases}$$

We choose the random variable defined on $[-1,1]$ with arch-like probability density function $\rho_U(u)$ which is usually characterizing the uncertainty in real word,

$$\rho_U(u) = \begin{cases} \frac{\pi}{2}\sqrt{1-\mu^2}, & \text{as } |\mu| \le 1, \\ 0, & \text{as } |\mu| > 1. \end{cases}$$

Corresponding to this random variable, the orthogonal polynomial of Equation (3) is chosen as the second kind of Chebyshev polynomial [15], where $\rho_U(u)$ represents the ith orthogonal polynomial, and M represents the largest order of the polynomials we have taken.

The orthogonality of the second kind of Chebyshev polynomial can be expressed as

$$\int_{-\infty}^{+\infty} \rho_U(u)x(t,u)P_i(u)du = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4}$$

And the recurrent formula for the second kind of Chebyshev polynomial is

$$uP_i(u) = \frac{1}{2}[P_{i+1}(u) + P_{i-1}(u)]$$

By the recurrent formulas, the triple product polynomial of Equation (1) can be further reduced into a linear combination of related single polynomials. By denoting the coefficient of $\rho_U(u)$ in the linear combination as $X_i(t)$, the nonlinear term

$$\left(\sum_{i=0}^{2} x_i(t)P_i(u)\right)^2 \sum_{i=0}^{2} y_i(t)P_i(u),$$

on the right side of Equation (1) can be expanded into

$$\left(\sum_{i=0}^{2} x_i(t)P_i(u)\right)^2 \sum_{i=0}^{2} y_i(t)P_i(u) = X_0(t)P_i(u) + \cdots + X_{3M}(t)P_{3M}(u) = \sum_{i=0}^{3M} X_i(t)P_i(u)$$

Substituting Equation (2) and Equation (3) into Equation (1), we have Equation (5) as follows,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\sum\limits_{i=0}^{M} x_i(t)P_i(u) = A\sum\limits_{i=0}^{M} y_i(t)P_i(u) \\ \frac{\mathrm{d}}{\mathrm{d}t}\sum\limits_{i=0}^{M} y_i(t)P_i(u) = \sum\limits_{i=0}^{M} x_i(t)P_i(u) - a\sum\limits_{i=0}^{M} y_i(t)P_i(u) \\ \qquad -\frac{\delta}{2}\sum\limits_{i=0}^{M} P_i(u)[y_{i+1}(t) + y_{i-1}(t)] - \sum\limits_{i=0}^{3M} X_i(t)P_i(u) \end{cases} \tag{5}$$

Multiplying both sides of Equation (5) by $P_i(u)$, $i = 0,1,2,\ldots,3M$; M in sequence and taking expectation with respect to U, owing to the orthogonality of the second kind of Chebyshev polynomials, we can finally obtain the equivalent deterministic equation. Remember that if $M \to \infty$, Equation (3) is strictly established. Otherwise, if M is finite, Equation (3) is just approximate value. According to request of computational precision in the following numerical analysis, we take $M = 2$ and obtain the equivalent deterministic equation approximately as follows:

$$\begin{cases} \frac{\mathrm{d}x_0}{\mathrm{d}t} = Ay_0 \\ \frac{\mathrm{d}y_0}{\mathrm{d}t} = x_0 - ay_0 - \frac{\delta}{2}y_1 - X_0 \\ \frac{\mathrm{d}x_1}{\mathrm{d}t} = Ay_1 \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} = x_1 - ay_1 - \frac{\delta}{2}(y_0 + y_2) - X_1 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = Ay_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = x_2 - ay_2 - \frac{\delta}{2}y_1 - X_2 \end{cases} \tag{6}$$

3. The Stochastic Hopf Bifurcation Analysis.

3.1. Existence of Hopf bifurcation. The supercritical Hopf bifurcation will appear at $(x, y) = (0, 0)$ in Equation (1) when the parameter $a_0 = \frac{\sqrt{2}}{2}\delta$. The parameters A and δ are assumed to satisfy $-\frac{1}{8}\delta^2 < A < 0$, and there is only one equilibrium $(0, 0, 0, 0, 0, 0)$ of system (1). We let a as the bifurcation parameter.

The Jacobian matrix J of the system (6) at the equilibrium $(0, 0, 0, 0, 0, 0)$ is

$$\mathrm{J}|_{(0,0,0,0,0,0)} = \begin{bmatrix} 0 & A & 0 & 0 & 0 & 0 \\ 1 & -a & 0 & -\frac{\delta}{2} & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 \\ 0 & -\frac{\delta}{2} & 1 & -a & 0 & -\frac{\delta}{2} \\ 0 & 0 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & -\frac{\delta}{2} & 1 & -a \end{bmatrix}$$

With the aid of Maple, we obtain the characteristic equation:

$$\begin{aligned} f(\lambda) = {} & \lambda^6 + 3a\lambda^5 + \left(-\frac{1}{2}\delta^2 - 3A + 3a^2\right)\lambda^4 + \left(-\frac{1}{2}\delta^2 a - 6aA + a^3\right)\lambda^3 \\ & + \left(\frac{1}{2}\delta^2 A + 3A^2 - 3Aa^2\right)\lambda^2 + 3aA^2\lambda - A^3 = 0 \end{aligned} \tag{7}$$

According to the Hopf bifurcation theorem, the system (6) has a pair of pure imaginary root $\lambda_{1,2} = \pm i\omega_0$, and $\frac{\mathrm{d}Re\lambda}{\mathrm{d}a}\big|_{a=a_0} \neq 0$, the Hopf bifurcation will appear at equilibrium when the parameter $a = a_0 = \frac{\sqrt{2}}{2}\delta$. When $a_0 = \frac{\sqrt{2}}{2}\delta$, using Maple we can obtain the all eigenvalues:

$$\lambda_1 = \sqrt{-A}i,\ \lambda_2 = -\sqrt{-A}i,\ \lambda_3 = -\frac{\sqrt{2}}{2}\delta + \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 2A},\ \lambda_4 = -\frac{\sqrt{2}}{2}\delta - \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 2A}$$
$$\lambda_5 = -\frac{\sqrt{2}}{4}\delta + \frac{\sqrt{2}}{4}\sqrt{\delta^2 + 8A},\ \lambda_6 = -\frac{\sqrt{2}}{4}\delta - \frac{\sqrt{2}}{4}\sqrt{\delta^2 + 8A}.$$

By the calculation, we get

$$\omega_0 = \sqrt{-A},$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}a} = \frac{3\lambda^5 + 6a\lambda^4 - \frac{1}{2}\delta^2\lambda^3 - 6A\lambda^3 + 3a^2\lambda^3 - 6Aa\lambda^2 + 3A^2\lambda}{6\lambda^5 + 15a\lambda^4 + 4\left(-\frac{1}{2}\delta^2 - 3A + 3a^2\right)\lambda^3 + 3\left(-\frac{1}{2}\delta^2 a - 6aA + a^3\right)\lambda^2 + 2\left(\frac{1}{2}\delta^2 A + 3A^2 - 3Aa^2\right)\lambda + 3aA^2}$$

$$\left.\frac{\mathrm{d}Re\lambda}{\mathrm{d}a}\right|_{a=a_0} = -\frac{2\delta^4 A^3}{\frac{9}{4}\delta^4 A^2 + \frac{9}{8}\delta^6 A^2 - \frac{9\sqrt{2}}{16}\delta^5 A^2 - 4\delta^4 A^3} \neq 0.$$

Therefore, $a_0 = \frac{\sqrt{2}}{2}\delta$ is the Hopf bifurcation critical value of the stochastic system (6). When a passes through the critical value $a_0 = \frac{\sqrt{2}}{2}\delta$, the Hopf bifurcation will appear at equilibrium $0(0, 0, 0, 0, 0, 0)$ of system (6).

3.2. **Direction and stability of the Hopf bifurcation.** In this section, we further investigate the Hopf bifurcation of the system (6) by the calculation of the first Lyapunov coefficient.

When $L_1 < 0$, the equilibrium point is asymptotically stable, there is a supercritical stochastic Hopf bifurcation, and there exists stable limit cycle near the equilibrium point. When $L_1 > 0$, the equilibrium point is unstable, there is a subcritical stochastic Hopf bifurcation and there exists unstable limit cycle near the equilibrium point.

Let C^n be a linear space defined on the complex number field C. Let $q \in \mathrm{C}^n$ be a complex eigenvector corresponding to λ_1 and $p \in \mathrm{C}^n$ be an adjoint eigenvector which satisfy the following properties

$$Jq = I\omega q, \quad J\bar{q} = -I\omega\bar{q}, \quad J^{\mathrm{T}}p = -I\omega p, \quad J^{\mathrm{T}}\bar{p} = I\omega\bar{p}, \quad \langle p, q\rangle = \sum_{i=0}^{n} \bar{p}_i q_i = 1.$$

When $a = a_0$, $0(0,0,0,0,0,0)$ is the equilibrium of Equation (6), and Equation (6) can be written as:

$$F(x) = Jx + N(x),$$
$$N(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + o(||x||)^2,$$

where $x = (x_0, y_0, x_1, y_1, x_2, y_2)$, $B(x,y)$ and $C(x,y,z)$ are bilinear and trilinear functions respectively which can be written as

$$B_i(x,y) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial\xi_j \partial\xi_k}\right|_{\xi=0} x_j y_k, \quad i = 1,2,\dots,n.$$

$$C_i(x,y,z) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l}\right|_{\xi=0} x_j y_k z_l, \quad i = 1,2,\dots,n.$$

The first Lyapunov coefficient [22] at the origin is defined by

$$L_1(0) = \frac{1}{2\omega^2} Re\left(iG_{20}G_{11} + \omega G_{21}\right).$$

We also define the following coefficients

$$G_{20} = \langle p, B(q,q)\rangle, \quad G_{11} = \langle p, B(q,\bar{q})\rangle, \quad G_{02} = \langle p, B(\bar{q},\bar{q})\rangle,$$
$$G_{21} = \langle p, C(q,q,\bar{q})\rangle - 2\left\langle p, B(q, J^{-1}B(q,\bar{q}))\right\rangle + \left\langle p, B(\bar{q}, (2i\omega E - J)^{-1} B(q,q))\right\rangle$$
$$+ \frac{1}{i\omega}\langle p, B(q,q)\rangle \langle p, B(q,\bar{q})\rangle - \left.\frac{2}{i\omega}\right| |\langle p, B(q,\bar{q})\rangle|^2 - \frac{1}{3i\omega}|\langle p, B(\bar{q},\bar{q})\rangle|^2.$$

When $\lambda_1 = \sqrt{-A}i$, using Maple, we can obtain q, $\bar{p} \in \mathrm{C}^n$:

$$q = \Bigg(\sqrt{-A}i, -\sqrt{-A}i, \frac{\sqrt{2}}{2}\delta + \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 2A}, \frac{\sqrt{2}}{2}\delta - \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 2A},$$
$$-\frac{\sqrt{2}}{2}\delta - \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 8A}, -\frac{\sqrt{2}}{2}\delta + \frac{\sqrt{2}}{2}\sqrt{\delta^2 + 8A}\Bigg)^{\mathrm{T}}$$

$$\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5, \bar{p}_6)^{\mathrm{T}}$$
$$= \Bigg(\frac{1}{K}, -\frac{\sqrt{-A}}{K}, \frac{4\sqrt{-A} - 2A}{\delta K}, \frac{2A\sqrt{-A} + 4A}{\delta K}, \frac{4Aa - 16A + \delta^2 - 8A\sqrt{-A} - 8a\sqrt{-A}}{\delta^2 K},$$
$$\frac{-8Aa - 8A^2 - 4aA\sqrt{-A} + 16A\sqrt{-A} - \delta^2\sqrt{-A}}{\delta^2 K}\Bigg)^{\mathrm{T}}$$

where K is shown in appendix.

We also have

$$B(q,q)=0, \quad B(q,\bar{q})=0, \quad B(\bar{q},\bar{q})=0, \quad C(q,q,\bar{q})=[H_1,H_2,H_3,H_4,H_5,H_6],$$

where

$$\begin{aligned}
H_1 =& -6A\sqrt{-A}i-\frac{5}{2}\delta^2\sqrt{-A}i-2\delta\sqrt{-\delta^2A-2A^2}-\frac{1}{2}\delta\sqrt{-\delta^2A-8A^2}+3\sqrt{2}\delta A\\
&+6\sqrt{2}A\sqrt{\delta^2+8A}-\frac{3\sqrt{2}}{2}\delta^3-\frac{3\sqrt{2}}{2}\delta^2\sqrt{\delta^2+2A}+\frac{3\sqrt{2}}{2}\delta^2\sqrt{\delta^2+8A}\\
&+\frac{3\sqrt{2}}{2}\delta\sqrt{\delta^4+10\delta^2A+16A^2},\\
H_2 =& -\delta^2\sqrt{-A}i-\frac{15\sqrt{2}}{2}\delta A+\delta\sqrt{-\delta^2A-2A^2}i+\delta\sqrt{-\delta^2A-8A^2}i-\frac{31\sqrt{2}}{2}A\sqrt{\delta^2+2A}\\
&+3\sqrt{-\delta^4A-10\delta^2A^2-16A^3}A-\frac{3\sqrt{2}}{4}\delta^3+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+2A}+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+8A}\\
&-\frac{3\sqrt{2}}{4}\delta\sqrt{\delta^4+10\delta^2A+16A^2},\\
H_3 =& -10A\sqrt{-A}i-\frac{5}{4}\delta^2\sqrt{-A}i+\frac{9\sqrt{2}}{2}\delta A-\delta\sqrt{-\delta^2A-2A^2}i-\frac{1}{4}\delta\sqrt{-\delta^2A-8A^2}i\\
&+\frac{17\sqrt{2}}{2}A\sqrt{\delta^2+8A}-\frac{3\sqrt{2}}{2}\delta^3-\frac{3\sqrt{2}}{2}\delta^2\sqrt{\delta^2+2A}+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+8A}\\
&+\frac{3\sqrt{2}}{2}\delta\sqrt{\delta^4+10\delta^2A+16A^2},\\
H_4 =& -\delta^2\sqrt{-A}i-\frac{21\sqrt{2}}{2}\delta A+\delta\sqrt{-\delta^2A-2A^2}i+\delta\sqrt{-\delta^2A-8A^2}i\\
&+3\sqrt{-\delta^4A-10\delta^2A^2-16A^3}i-\frac{33\sqrt{2}}{2}A\sqrt{\delta^2+2A}-\frac{3\sqrt{2}}{4}\delta^3\\
&+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+2A}+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+8A}-\frac{3\sqrt{2}}{4}\delta\sqrt{\delta^4+10\delta^2A+16A^2},\\
H_5 =& -5A\sqrt{-A}i-\frac{1}{4}\delta^2\sqrt{-A}i+3\sqrt{2}\delta A-\frac{1}{4}\delta\sqrt{-\delta^2A-8A^2}i+\frac{18\sqrt{2}}{4}A\sqrt{\delta^2+8A}\\
&-\frac{3\sqrt{2}}{4}\delta^3-\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+2A}+\frac{3\sqrt{2}}{4}\delta^2\sqrt{\delta^2+8A}+\frac{3\sqrt{2}}{4}\delta\sqrt{\delta^4+10\delta^2A+16A^2},\\
H_6 =& -\frac{9\sqrt{2}}{2}\delta A-\frac{15\sqrt{2}}{2}A\sqrt{\delta^2+2A}-\frac{3\sqrt{2}}{8}\delta^3+\frac{3\sqrt{2}}{8}\delta^2\sqrt{\delta^2+2A}\\
&+\frac{3\sqrt{2}}{8}\delta^2\sqrt{\delta^2+8A}-\frac{3\sqrt{2}}{8}\delta\sqrt{\delta^4+10\delta^2A+16A^2}.
\end{aligned}$$

$$G_{20}=0,\ G_{11}=0,\ G_{02}=0,$$

$$G_{21}=\bar{p}_1H_1+\bar{p}_2H_2+\bar{p}_3H_3+\bar{p}_4H_4+\bar{p}_5H_5+\bar{p}_6H_6=\frac{V}{K},$$

$$L_1(0)=\frac{1}{2A}Re\left(\sqrt{-A}G_{21}\right)=\frac{L}{2AR}$$

where V, L, R are shown in appendix.

As we choose the parameters $0<\delta\le 0.376$, $A\to-\infty$, the first Lyapunov coefficient $L_1(0)<0$ corresponding to different random intensities. By the calculation, we have

$L_1(0) < 0$ if the random intensity δ is changed from 0 to 0.376. It is to say that as $a_0 = \frac{\sqrt{2}}{2}\delta$ and $0 < \delta \leq 0.376$ there is a supercritical stochastic Hopf bifurcation at the equilibrium $(x, y) = (0, 0)$ for stochastic two-dimensional chaotic system with random parameter, Equation (1). In next section we will verify the theoretical analysis by numerical simulation.

4. **The Numerical Simulation Example.** We fixed $A = -1$, and chose the initial value $x_0 = [0.32, 0.08, 0.23, 0.32, 1.55, 0.43]^{\mathrm{T}}$. As the random intensity $\delta = 0$, $\bar{\mu} = a$, Equation (1) is a deterministic chaotic system. When the parameter $A = -1$, we can get the critical value $a = 0$, and the deterministic chaotic system undergoes the supercritical Hopf bifurcation at the equilibrium. When $a = 0.27$, δ is chosen as 0.0, 0.126, 0.252, 0.376 respectively, the phase trajectories of two-dimensional chaotic system with random parameter converge at zero, as shown in Figure 1(a). Figure 1(b) is time history diagram corresponding to Figure 1(a). As the parameter $a = 1.7$, the phase trajectories of two-dimensional chaotic system with random parameter converged at their limit cycle which is shown in Figure 1(c). Figure 1(d) is local portrait of Figure 1(c). Figure 1(e) is time history diagram corresponding to Figure 1(c).

From Figure 1(a) we can know that as the bifurcation parameter a is far from the critical value, the phase trajectories of the deterministic system accord with the phase trajectories of stochastic chaotic system. The supercritical Hopf bifurcation occurs in both two systems.

From Figure 1(a) to Figure 1(e), the results of numerical simulation and theoretical calculation results can be found to be consistent. Time history diagram shows periodic oscillation, and the phase diagrams of convergence for the limit cycle when the supercritical Hopf bifurcation occurs. Those results suggest when the system satisfies some conditions, from single state into a stable state of cycle. We can change the parameters of the system operation appropriately according to the need to avoid the drastic fluctuations. The length of the duration, throughout all phases of the turning point of the specific time and the strength of the expansion and contraction can be grasped through prediction and monitoring to periodic fluctuation effectively so that we can accord to different cycle characteristics, formation mechanism to make corresponding countermeasures, as far as possible to slow the progress of the cycle, and reduce the damage of the cycle fluctuation extent. This paper not only can make us know and solve the problem of stochastic system, but also can explain and predict some practical problems. Then we can make them develop to our hope's direction.

5. **Conclusions.** Orthogonal polynomial approximation is applied to study the stochastic Hopf bifurcation phenomena in stochastic two-dimensional chaotic system with random parameter. Analysis shows that orthogonal polynomial approximation is effective to reduce the stochastic two-dimensional chaotic system into its equivalent deterministic system. Then the first Lyapunov coefficient method is applied to study the Hopf bifurcation in equivalent deterministic system. We found that the stochastic Hopf bifurcation in stochastic two-dimensional chaotic system not only is similar to the conventional Hopf bifurcation, but has its unique feature. For instance, the stochastic Hopf bifurcation can result from the variation of intensity of the random parameter alone. Theoretical results are verified by numerical simulations. Apparently, there are more interesting problems about this two-dimensional chaotic system with random parameter in terms of complexity, control, and synchronization, which deserve further investigation.

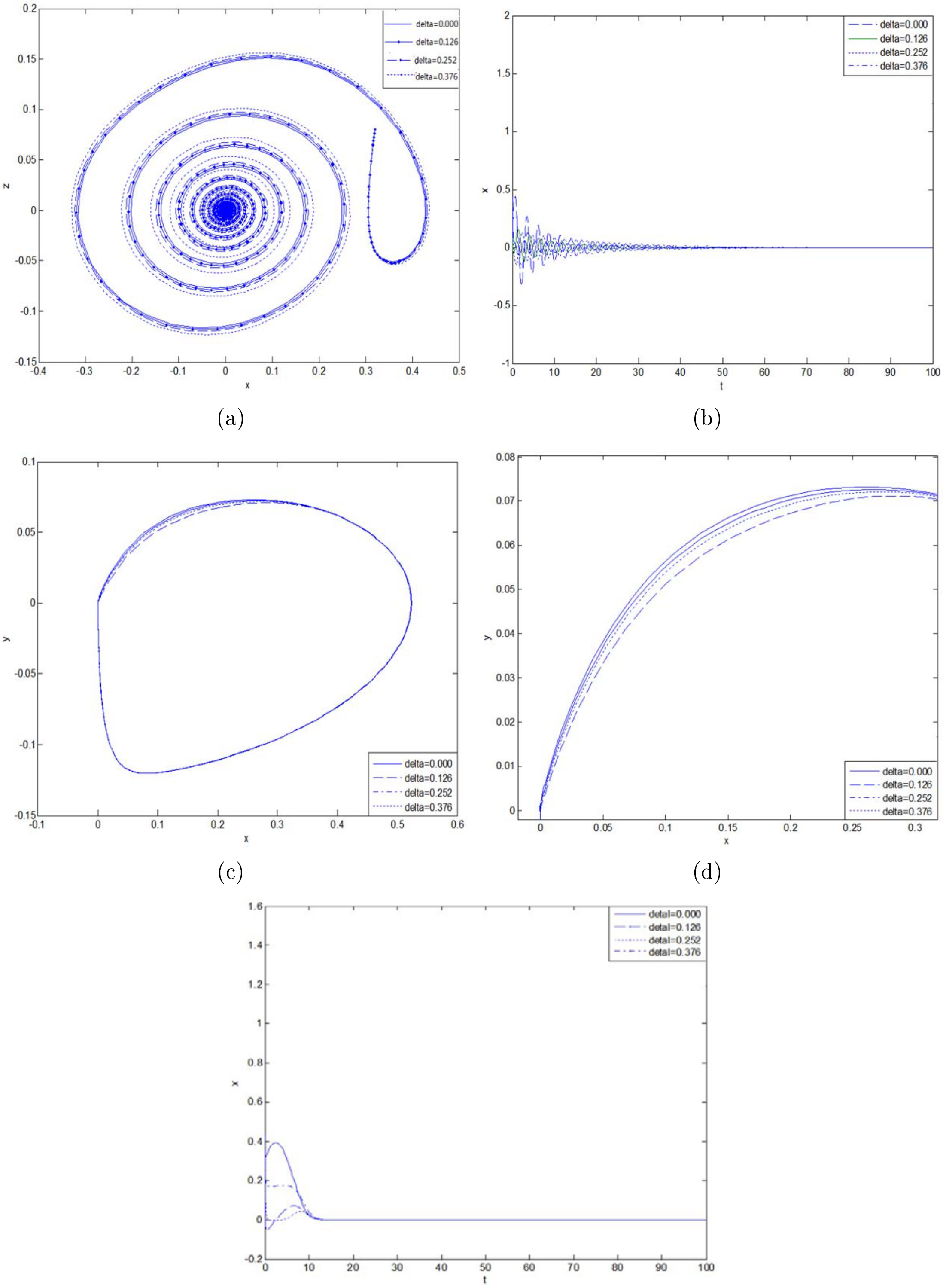

FIGURE 1. Phase trajectories and time history diagrams of Hopf bifurcation

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REFERENCES

[1] G. Continillo, A. Grabski, E. Mancusi and L. Russo, Parallel tools for the bifurcation analysis of large-scale chemically reactive dynamical systems, *Computers and Chemical Engineering*, vol.38, pp.94-105, 2012.

[2] W. V. Wedig, Stability and bifurcation in multi-scaled stochastic mechanics, *Procedia IUTAM*, vol.6, pp.169-179, 2013.

[3] W. Ebeling, R. Feistel and S. Feistel, On bifurcations in complex ecological systems with diffusion and noise, *Ecological Complexity*, vol.14, pp.2-7, 2013.

[4] Q. Zhang, Z. Xu, T. Feng and J. Jiao, A dynamic stochastic frontier model to evaluate regional financial efficiency: Evidence from Chinese county-level panel data, *European Journal of Operational Research*, vol.241, pp.907-916, 2015.

[5] N. Li, H. Sun and Q. Zhang, The dynamics and bifurcation control of a singular biological economic model, *International Journal of Automation and Computing*, vol.9, no.1, pp.1-7, 2012.

[6] J. Wang and S. Chen, The stability and Hopf bifurcation for a kind of predator system with time delay, *Journal of Northwest Normal University (Natural Science)*, vol.48, no.1, pp.1-7, 2012.

[7] M. Liu, X. Xu and C. Zhang, Stability and global Hopf bifurcation for neutral BAM neural network, *Neurocomputing*, vol.145, pp.122-130, 2014.

[8] M. Zumsande and T. Gross, Bifurcations and chaos in the MAPK signaling cascade, *J. Theoret. Biol.*, vol.265, no.3, pp.481-491, 2010.

[9] H. Zhao, X. Huang and X. Zhang, Hopf bifurcation and harvesting control of a bioeconomic plankton model with delay and diffusion terms, *Physica A*, vol.421, pp.300-315, 2015.

[10] Y. Zhang, W. Xu and T. Fang, Stochastic Hopf bifurcation and chaos of stochastic Bonhoeffer-Vander Pol system via Chebyshev polynomial approximation, *Applied Mathematics and Computation*, vol.190, pp.1225-1236, 2007.

[11] J. Li, The expanded order system method of combined random vibration analysis, *Acta Mech. Sin.*, vol.28, pp.63-68, 1996.

[12] S. Ma, The stochastic Hopf bifurcation analysis in Brusselator system with random parameter, *Applied Mathematics and Computation*, vol.219, pp.306-319, 2012.

[13] R. K. Pandey, S. Suman, K. K. Singh and O. P. Singh, An approximate method for Abel inversion using Chebyshev polynomials, *Applied Mathematics and Computation*, vol.237, pp.120-132, 2014.

[14] M. R. Eslahchi, M. Dehghan and S. Amani, The third and fourth kinds of Chebyshev polynomials and best uniform approximation, *Mathematical and Computer Modelling*, vol.55, pp.1746-1762, 2012.

[15] F. Martel, D. Rancourt, C. Chochol, Y. St-Amant, S. Chesne and D. Remond, Time-varying torsional stiffness identification on a vertical beam using Chebyshev polynomials, *Mechanical Systems and Signal Processing*, vols.54-55, pp.481-490, 2015.

[16] T. Fang, X. L. Leng and C. Q. Song, Chebyshev polynomials approximation for dynamical response problem of random system, *Journal of Sound and Vibration*, vol.226, pp.198-206, 2003.

[17] Y. Li and C. Li, Stability and Hopf bifurcation analysis on a delayed Leslie-Gower predator – Prey system incorporating a prey refuge, *Applied Mathematics and Computation*, vol.219, pp.4576-4589, 2013.

[18] Z. Hu, Z. Teng and L. Zhang, Stability and bifurcation analysis in a discrete SIR epidemic model, *Mathematics and Computers in Simulation*, vol.97, pp.80-93, 2014.

[19] C. Anton, J. Deng and Y. S. Wong, Hopf bifurcation analysis of an aeroelastic model using stochastic normal form, *Journal of Sound and Vibration*, vol.331, pp.3866-3886, 2012.

[20] G. Zhang, Y. Shen and B. Chen, Bifurcation analysis in a discrete differential algebraic predator prey system, *Applied Mathematical Modelling*, vol.38, pp.4835-4848, 2014.

[21] B. Wang, Existence, stability and bifurcation of random complete and periodic solutions of stochastic parabolic equations, *Nonlinear Analysis*, vol.103, pp.9-25, 2014.

[22] B. Hassard, N. Kazarinoff and Y. Wan, *Theory and Application of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.

Appendix A. In this paper, $X_i(t)$, $(i = 0, 1, 2, 3, 4, 5, 6)$ are as follows:

$$X_0(t) = x_0^2y_0 + x_1^2y_0 + x_2^2y_0 + 2x_0x_1y_1 + 2x_1x_2y_1 + 2x_0x_2y_2 + x_1^2y_2 + x_2^2y_2,$$
$$X_1(t) = x_0^2y_1 + 2x_1^2y_1 + 2x_2^2y_1 + 2x_0x_1y_0 + 2x_1x_2y_0 + 2x_0x_2y_1 + 2x_0x_1y_2 + 4x_1x_2y_2,$$
$$X_2(t) = x_0^2y_2 + x_1^2y_0 + x_2^2y_0 + 2x_0x_2y_0 + 4x_1x_2y_1 + 2x_0x_1y_1 + 2x_0x_2y_2 + 2x_1^2y_2 + 3x_2^2y_2,$$
$$X_3(t) = x_1^2y_1 + 2x_2^2y_1 + 2x_0x_2y_1 + 2x_1x_2y_0 + 2x_0x_1y_2 + 4x_1x_2y_2 + 2x_1^2y_2 + 3x_2^2y_2,$$
$$X_4(t) = x_1^2y_2 + x_2^2y_0 + 2x_0x_2y_2 + 2x_1x_2y_1 + 4x_1x_2y_2 + x_2^2y_2,$$
$$X_5(t) = 2x_1x_2y_2 + x_2^2y_1,$$
$$X_6(t) = x_2^2y_2.$$

$$\begin{aligned}
K = {} & \sqrt{-A}i - Ai + 2\sqrt{2}\sqrt{-A} + \sqrt{2}A - \frac{3\sqrt{2}}{\delta}A\sqrt{\delta^2+2A} + \frac{2\sqrt{2}}{\delta}\sqrt{-\delta^2A-2A^2} \\
& + \sqrt{2}A\sqrt{-A} - \frac{\sqrt{2}}{\delta}A\sqrt{-\delta^2A-2A^2} + \frac{4\sqrt{2}}{\delta}A - \frac{\sqrt{2}}{4}\delta - \frac{2\sqrt{2}}{\delta}A\sqrt{-A} + \frac{2\sqrt{2}}{\delta}a\sqrt{-A} \\
& + \frac{2\sqrt{2}}{\delta}aA + \frac{4\sqrt{2}}{\delta^2}A\sqrt{\delta^2+8A} - \frac{\sqrt{2}}{4}\sqrt{\delta^2+8A} + \frac{6\sqrt{2}}{\delta^2}A\sqrt{-\delta^2A-8A^2} \\
& + \frac{2\sqrt{2}}{\delta^2}a\sqrt{-\delta^2A-8A^2} - \frac{3\sqrt{2}}{\delta^2}aA\sqrt{\delta^2+8A} - \frac{\sqrt{2}}{\delta^2}aA\sqrt{-\delta^2A-8A^2} \\
& + \frac{\sqrt{2}}{4}\delta\sqrt{-A} + \frac{2\sqrt{2}}{\delta}A^2 + \frac{\sqrt{2}}{\delta}aA\sqrt{-A} + \frac{\sqrt{2}}{4}\sqrt{-\delta^2A-8A^2} - \frac{2\sqrt{2}}{\delta^2}A^2\sqrt{\delta^2+8A},
\end{aligned}$$

$$\begin{aligned}
V = {} & -7A\sqrt{-A}i - \frac{11}{4}\delta^2\sqrt{-A}i - 2(1+3\sqrt{2})\delta\sqrt{-\delta^2A-2A^2} \\
& + \left(6\sqrt{2}-\frac{1}{2}\right)\delta\sqrt{-\delta^2A-8A^2} + 18\sqrt{2}\delta A - \frac{3\sqrt{2}}{2}A\sqrt{\delta^2+8A} - \frac{9\sqrt{2}}{4}\delta^3 \\
& - \frac{9\sqrt{2}}{4}\delta^2\sqrt{\delta^2+2A} + \frac{9\sqrt{2}}{4}\delta^2\sqrt{\delta^2+8A} + \frac{9\sqrt{2}}{4}\delta\sqrt{\delta^4+10\delta^2A+16A^2} - \delta^2Ai \\
& + \frac{27\sqrt{2}}{4}\delta A\sqrt{-A} + \delta A\sqrt{\delta^2+2A}i + \delta A\sqrt{\delta^2+8A}i + \frac{119\sqrt{2}}{4}A\sqrt{-\delta^2A-2A^2} \\
& - \frac{9\sqrt{2}}{8}\delta^2\sqrt{-\delta^2A-8A^2} - \frac{9\sqrt{2}}{8}\delta^2\sqrt{-\delta^2A-2A^2} + \left(3-6\sqrt{2}\right)A\sqrt{\delta^4+10\delta^2A+16A^2} \\
& + \frac{3\sqrt{2}}{4}\delta^3\sqrt{-A} + \frac{9\sqrt{2}}{8}\delta\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{40}{\delta}A^2i + 5\delta Ai \\
& + 18\sqrt{2}A\sqrt{-A} + 4A\sqrt{\delta^2+2A}i + A\sqrt{\delta^2+8A}i + \frac{34\sqrt{2}}{\delta}A\sqrt{-\delta^2A-8A^2} \\
& - 6\sqrt{2}\delta^2\sqrt{-A} + 6\sqrt{2}\sqrt{-\delta^4A-10\delta^2A^2-16A^3} + \frac{20}{\delta}A^2\sqrt{-A}i + \frac{9}{2}A\sqrt{-\delta^2A-8A^2}i \\
& - 51\sqrt{2}A^2 - \frac{3}{2}\delta A\sqrt{-A}i + 6A\sqrt{-\delta^2A-2A^2}i - \frac{17\sqrt{2}}{\delta}A^2\sqrt{\delta^2+8A} - \sqrt{2}\delta^2A \\
& + 6\sqrt{2}\delta A\sqrt{\delta^2+2A} + \frac{12A}{\delta}\sqrt{-\delta^4A-10\delta^2A^2-16A^3}i \\
& - \frac{3\sqrt{2}}{4\delta}A\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{66\sqrt{2}}{\delta}A^2\sqrt{\delta^2+2A} + 2A^2\delta i + 3\sqrt{2}A^2\delta \\
& - 21\sqrt{2}A^2\sqrt{-A} - 2A^2\sqrt{\delta^2+2A}i - 3\sqrt{2}A^2\sqrt{\delta^2+2A} - 2A^2\sqrt{\delta^2+8A}i
\end{aligned}$$

$$
\begin{aligned}
&- 3\sqrt{2}A^2\sqrt{\delta^2+8A} - \frac{6}{\delta}A^2\sqrt{\delta^4+10\delta^2A+16A^2} + \frac{3\sqrt{2}}{\delta}A^2\sqrt{\delta^4+10\delta^2A+16A^2}\\
&- \frac{33\sqrt{2}}{\delta}A^2\sqrt{-\delta^2A-2A^2} - \frac{3\sqrt{2}}{2}\delta^2A\sqrt{-A} + \frac{3\sqrt{2}}{2}\delta A\sqrt{-\delta^2A-8A^2}\\
&+ \frac{3\sqrt{2}}{2}\delta A\sqrt{-\delta^2A-2A^2} - \frac{3\sqrt{2}}{2}A\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{20}{\delta^2}aA^2\sqrt{-A}i\\
&- aA\sqrt{-A}i + \frac{12\sqrt{2}}{\delta}aA^2 - \frac{1}{\delta}aA\sqrt{-\delta^2A-8A^2}i + \frac{18\sqrt{2}}{\delta^2}aA^2\sqrt{\delta^2+8A}\\
&+ \frac{60\sqrt{2}}{\delta^2}aA^2\sqrt{\delta^2+2A} + \frac{30\sqrt{2}}{\delta^2}aA^2\sqrt{-\delta^2A-2A^2} - 6\sqrt{2}aA\sqrt{\delta^2+2A}\\
&+ \frac{3\sqrt{2}}{\delta}aA\sqrt{\delta^4+10\delta^2A+16A^2} - \frac{36\sqrt{2}}{\delta^2}aA\sqrt{\delta^2+8A} + \frac{80}{\delta^2}A^2\sqrt{-A}i - \frac{48\sqrt{2}}{\delta}A^2\\
&+ \frac{4}{\delta}A\sqrt{-\delta^2A-8A^2}i - \frac{72\sqrt{2}}{\delta^2}A^2\sqrt{\delta^2+8A} + 12\sqrt{2}A\sqrt{\delta^2+2A} - 12\sqrt{2}A\sqrt{\delta^2+8A}\\
&- \frac{12\sqrt{2}}{\delta^2}A\sqrt{\delta^4+10\delta^2A+16A^2} - \frac{1}{4}\delta\sqrt{-\delta^2A-8A^2}i - \frac{5}{\delta^2}A^3i - \frac{7}{4}A^2i\\
&- \frac{99\sqrt{2}}{\delta}A^2\sqrt{-A} - \frac{1}{4\delta}aA\sqrt{\delta^2+8A}i - \frac{9\sqrt{2}}{2\delta^2}A^2\sqrt{-\delta^2A-8A^2}\\
&+ \frac{21\sqrt{2}}{4}A\sqrt{-\delta^2A-8A^2} - \frac{40}{\delta^2}aA^2i - \frac{2}{\delta}aA\sqrt{\delta^2+8A}i + 6\sqrt{2}a\delta\sqrt{-A}\\
&- \frac{6\sqrt{2}}{\delta}a\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{120\sqrt{2}}{\delta^2}A^2\sqrt{-\delta^2A-2A^2}\\
&- \frac{6\sqrt{2}}{\delta}A\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{3\sqrt{2}}{8}\delta^3\sqrt{-A}\\
&+ \frac{3\sqrt{2}}{8}\delta\sqrt{-\delta^4A-10\delta^2A^2-16A^3} - \frac{36\sqrt{2}}{\delta^2}A^2 + \frac{60\sqrt{2}}{\delta^2}A^3\sqrt{\delta^2+2A} + \frac{36\sqrt{2}}{\delta}aA^2\\
&+ \frac{18\sqrt{2}}{\delta}aA^2\sqrt{-A} + \frac{3\sqrt{2}}{2}\delta aA\sqrt{-A} - 3\sqrt{2}aA\sqrt{-\delta^2A-8A^2}\\
&+ \frac{3\sqrt{2}}{2\delta}aA\sqrt{-\delta^4A-10\delta^2A^2-16A^3},
\end{aligned}
$$

$$R = K - \sqrt{-A}i + Ai,$$

$$
\begin{aligned}
L = {} & V + 7A\sqrt{-A}i + \frac{11}{4}\delta^2\sqrt{-A}i + \delta^2 i - \delta A\sqrt{\delta^2+2A}i - \delta A\sqrt{\delta^2+8A}i + \frac{40}{\delta}A^2i\\
&- 5\delta Ai - 4A\sqrt{\delta^2+2A}i - A\sqrt{\delta^2+8A}i - \frac{20}{\delta}A^2\sqrt{-A}i + \frac{9}{2}A\sqrt{-\delta^2A-8A^2}i\\
&+ \frac{3}{2}\delta A\sqrt{-A}i - 6A\sqrt{-\delta^2A-2A^2}i - \frac{12A}{\delta}\sqrt{-\delta^4A-10\delta^2A^2-16A^3}i\\
&- 2A^2\delta i + 2A^2\sqrt{\delta^2+2A}i + \frac{20}{\delta^2}aA^2\sqrt{-A}i + aA\sqrt{-A}i + \frac{1}{\delta}aA\sqrt{-\delta^2A-8A^2}i\\
&- \frac{80}{\delta^2}A^2\sqrt{-A}i - \frac{4}{\delta}A\sqrt{-\delta^2A-8A^2}i + \frac{1}{4}\delta\sqrt{-\delta^2A-8A^2}i\\
&+ \frac{5}{\delta^2}A^3i + \frac{7}{4}A^2i + \frac{1}{4\delta}aA\sqrt{\delta^2+8A}i + \frac{40}{\delta^2}aA^2i + \frac{2}{\delta}aA\sqrt{\delta^2+8A}i.
\end{aligned}
$$