International Journal of Innovative Computing, Information and Control
Volume **11**, Number **6**, December **2015**
ICIC International ©2015 ISSN 1349-4198
pp. 1869–1880

SOME THEORETICAL CONSIDERATIONS RELEVANT TO LONG-TERM RATE OF RETURNS ON RISKY ASSETS

Xuefeng Wang[1] and Heng Sun[2]

[1]School of Management
Harbin Institute of Technology
No. 92, West Dazhi Street, Nangang District, Harbin 150001, P. R. China
wangxuefeng1125@126.com

[2]Department of Statistics
Columbia University
116th Street and Broadway, New York, NY 10027, The United States
hs2793@columbia.edu

Received April 2015; revised September 2015

Abstract. *This paper identifies and describes some theoretical considerations relevant to the long-term rate of return on risky assets. Relying on the assumption of lognormality for the short-term rate of return on risky assets, we deduced the theoretical formulae of the probability density function (p.d.f.) as well as the corresponding mean and variance of the long-term rate of return on risky assets. Then we further provided the proofs of some investment characteristics of the formulae obtained, discussed the relationship between the means and variances respectively of the short-term rate of return and long-term rate of return, and deduced the mathematical formulae for the critical curve of the mean and variance of the short-term rate of return. We also derived the approximation formulae for the p.d.f. and the corresponding mean and variance of the long-term rate of return on risky assets when the short-term rate of return on risky assets follows a general probability distribution and the holding period n is comparatively large.*

Keywords: Risky assets, The lognormal distribution, Short-term rate of return, Long-term rate of return

1. **Introduction.** The relationship between the short-term and long-term rates of return is well known. The long-term rate of return is the geometric mean of the short-term rate of return over a sequence of time periods; the arithmetic mean of these short-term rates of return can be considered as an estimate of the expectation of the short-term rate of return. In investment analysis, people are accustomed to measure the earnings accomplished using the expectation of the short-term rate of return and the relevant risk (uncertainty) using the variance. However, if one plans to make long-term investments, judging the level of earnings and the risk magnitude merely based on the mean and variance of the short-term rate of return as stated above is blind and error-prone. The long-term rate of return on risky assets is far below the expectation of the short-term rate of return, but the long-term rate of return is far higher than the risk-free rate of return in stock market. In [4], Mehra and Prescott indicate that the abnormal premium rate of return of stocks exists compared with the risk-free rate of return. In [5], Fama and French explained the abnormal premium rate of return of stocks using the expanded samples and the DDM model. The short-term rate of return on risky assets for a single period is random; therefore, the series of the rate of return over a sequence of consecutive time periods is necessarily random and there is no appropriate straight-forward theoretical formula to

demonstrate the relationship between the means and variances respectively between the short-term and long-term rates of return.

In [9], the authors obtained some numerical results reflecting the relationship between the short-term rates of return and the average annual rate of return for long-term on risky assets by implementing Monte Carlo simulation method. In [10], we made further efforts to describe such relationship again using Monte Carlo simulation method and obtained the critical curve of the expectation and variance of the short term rate of return, along with some empirical results identifying conditions under which long-term risky assets would earn positive or negative earnings. In [11], the authors made more thorough theoretical analysis when the holding period is indefinite and gained some theoretical formulae. There were some shortcomings in the authors' previous works, among which the main drawback is that instead of concluding theoretical formulae under general circumstances we only reached some numerical results when the holding period is n. The significance of in-depth theoretical research is beyond suspicion; however, such kind of studies involves relatively complex mathematical deduction. Focusing on the particular assumption that the short-term rate of return on risky assets is log-normally distributed, we made more detailed discussions based on the authors' previous works and obtained some theoretical results that roughly meet our expectation.

If an investor plans to hold a long-term risky asset, he or she needs to have an idea of the average annual rate of return and the corresponding variance on long-term risky assets rather than just knowing the probability distribution of the short-term rate of return. The main objective of this paper is to theoretically study the relationship between the means and variances respectively of the short-term and long-term rate of return. We try to derive the theoretical formulae and the corresponding properties under the assumption that the short-term rate of return is log-normally distributed. These problems have not been thoroughly studied in the existing investment theories. The average annual rate of return on the long-term risky assets as an indicator definitely has critical instructive significance for investors.

In Section 2, we derived the general mathematical formulae of the p.d.f. of the rate of return on long-term risky assets. In Section 3, we gave the mathematical formulae of the p.d.f. of the rate of return on long-term risky assets along with the corresponding mean and variance under the condition that the short-term rate of return is log-normally distributed. In Section 4, we deduced and proved some characteristics relevant to the short-term and long-term rates of return based on the theoretical analysis and formulae obtained in Section 3. In Section 5, we discussed the situation under which the short-term rate of return follows a general form of distribution, and provided the approximation formulae for the p.d.f. of the long-term rate of return and the corresponding mean and variance when the holding period n is relatively large. The last section is conclusions.

2. **The Theoretical Formulae of the Rate of Return on Long-Term Risky Assets.** Consider a risky asset K, a stock for example, and we call the annual rate of return on asset K the short-term rate of return for convenience. In fact, the concepts of long-term and short-term are just relative, e.g., short-term rate of return can be daily rate of return, weekly rate of return or monthly rate of return, etc. $\xi_1, \xi_2, \ldots, \xi_n$ are the rate of return for each year over n consecutive years, and the sequence of random variables ξ_i are identically distributed random variables with the identical p.d.f. $p(x)$. According to the characteristics of risky assets in financial markets and the well-known efficient-market hypothesis (EMH), as long as the time intervals are not too short, e.g., if longer than 1 day, the dependence between two adjacent returns is very weak. Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent and identically distributed (i.i.d.) random variables and such hypothesis

is consistent with the characteristics of random walk. Because ξ_i denotes rate of return, it is required that $\xi_i > -1$, $i = 1, 2, \ldots, n$. In other words, when $x \le -1$, $p(x) = 0$. The average annual rate of return η_n on asset K over n consecutive years is given by the following expression:

$$\eta_n = ((1+\xi_1)(1+\xi_2)\cdots(1+\xi_n))^{1/n} - 1$$

We can use the following expression:

$$\eta_n + 1 = ((1+\xi_1)(1+\xi_2)\cdots(1+\xi_n))^{1/n} \tag{1}$$

In this paper we also call η_n the long-term rate of return for convenience. The main objective of this paper is to theoretically study the expectation and variance of the random variable η_n and also its probability distribution, and we have also included study of the circumstances under which $n \to +\infty$.

In order to illustrate the properties of the rate of return on long-term risky assets, we consider a special case that the price of the risky assets will return to original level after n periods, and the prices of assets at each time point constitute a time series $P_0, P_1, P_2, \ldots, P_n$ with $P_n = P_0$. Accordingly, we obtain the time series $\xi_1, \xi_2, \ldots, \xi_n$ constituted by the rate of return at the end of these n periods.

$$\xi_i = \frac{P_i - P_{i-1}}{P_{i-1}}, \quad i = 1, 2, \ldots, n$$

Since $P_n = P_0$, the overall rate of return over these n successive periods is 0. Let A_0 denote the amount of initial investment, let A_n denote the amount of the investment achieved at the end of period n, according to the assumption we have $A_n = A_0$, so

$$A_0 = A_0(1+\xi_1)(1+\xi_2)\cdots(1+\xi_n)$$

Therefore, there is $\prod_{i=1}^{n}(1+\xi_i) = 1$. We have the following theorem.

Theorem 2.1. *Let $P_0, P_1, P_2, \ldots, P_n$ be the time series specified by the prices at the end of a sequence of consecutive time periods, with $P_i > 0$ and P_i is not identically equal to any constant; furthermore, $P_n = P_0$. The time series constituted by the short-term rate of return is $\xi_1, \xi_2, \ldots, \xi_n$, where*

$$\xi_i = \frac{P_i - P_{i-1}}{P_{i-1}}, \quad i = 1, 2, \ldots, n$$

Then it must hold that $\frac{1}{n}\sum_{i=1}^{n}\xi_i > 0$.

Proof: For $a_i > 0$, $i = 1, 2, \ldots, n$ with a_i not identically equal to any constant, the following inequality holds:

$$(a_1 a_2 \cdots a_n)^{1/n} < \frac{1}{n}\sum_{i=1}^{n} a_i$$

Substitute $a_i = 1 + \xi_i$ in the expression above and it becomes

$$\left(\prod_{i=1}^{n}(1+\xi_i)\right)^{1/n} < \frac{1}{n}\sum_{i=1}^{n}(1+\xi_i) = 1 + \frac{1}{n}\sum_{i=1}^{n}\xi_i$$

According to the assumptions of the theorem and the previous discussions, we have $\prod_{i=1}^{n}(1+\xi_i) = 1$, substituted in inequality above and we will obtain $\frac{1}{n}\sum_{i=1}^{n}\xi_i > 0$. The theorem gets proved.

The conclusion of Theorem 2.1 is inconsistent with our intuition. Theorem 2.1 asserts that if the initial and the terminal price of a risky assets are equal (which indicates the gross rate of return over the consecutive periods is 0), no matter what path the price movements takes, if price fluctuations exist, the arithmetic mean of the short-term rate of return determined by the price series must be positive.

Taking the logarithm of both sides of expression (1), we shall get

$$\ln(\eta_n + 1) = \frac{1}{n}\sum_{i=1}^{n}\ln(1+\xi_i)$$

Designate the p.d.f. of the random variable $\omega_i = 1 + \xi_i$ as $g(x)$ and according to the probability theory we have $g(x) = p(x-1)$. Then solve for the p.d.f. $d(y)$ of the function of random variable $\ln\omega_i = \ln(1+\xi_i)$, since $y = \ln(x)$ is a monotone increasing function, and the inverse function is $x = e^y$, so $\frac{dx}{dy} = e^y$. We have

$$d(y) = g(e^y)\frac{dx}{dy} = g(e^y)e^y \tag{2}$$

Now, we shall start solving for the p.d.f. $M_n(y)$ of the sum of random variables $\sum_{i=1}^{n}\ln(1+\xi_i)$, since $\xi_1, \xi_2, \ldots, \xi_n$ are independent and identically distributed (i.i.d.) random variables, $\ln(1+\xi_1), \ln(1+\xi_2), \ldots, \ln(1+\xi_n)$ are also i.i.d. random variables. According to probability theory, the p.d.f. $M_n(y)$ of the sum $\sum_{i=1}^{n}\ln(1+\xi_i)$ is the $(n-1)$ fold iteration of the convolution of the p.d.f. $d(y) = g(e^y)e^y$ with itself. The p.d.f. of $\ln(1+\xi_1) + \ln(1+\xi_2)$ is $d(y) * d(y)$, and we have

$$d(y) * d(y) = \int_{-\infty}^{+\infty} d(t)d(y-t)dt$$

The p.d.f. of the sum of n items $\sum_{i=1}^{n}\ln(1+\xi_i)$ is

$$M_n(y) = ((\cdots((d(y) * d(y)) * d(y)) * \cdots) * d(y)$$

The above expression demonstrates the result of the $(n-1)$ fold iteration of the convolution of $d(y)$ with itself. Theoretically, there are two methods of solving $M_n(y)$: the first one is to directly solve the analytical expression of the $(n-1)$ fold iteration of the convolution with itself for $M_n(y)$; the second is to utilize integral transformation such as Fourier transform to obtain $M_n(y)$. Since $d(y)$ is nonnegative continuously differentiable function with $\int_{-\infty}^{+\infty} d(y)dy = 1$, the Fourier transform of $d(y)$ must exist. Designate the Fourier transform of $d(y)$ as $D(z)$, i.e.,

$$F(d(y)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d(y)e^{izy}dy = D(z)$$

According to the properties of the convolution calculation, we have

$$F(d(y) * d(y)) = D^2(z)$$

$$F\left\{((\cdots((d(y) * d(y) * d(y)) * \cdots) * d(y)\right\} = D^n(z)$$

If we manage to get the analytical expression of the Fourier inverse transform of the function $D^n(z)$, we shall obtain the p.d.f. $M_n(y)$ of the sum of variables $\sum_{i=1}^{n}\ln(1+\xi_i)$, which is

$$M_n(y) = F^{-1}(D^n(z)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} D^n(z) e^{-izy} dz$$

Suppose we have already deduced the analytical expression of p.d.f. $M_n(y)$, according to the probability theory we know the form of p.d.f. $m_n(y)$ of the random variable $\frac{1}{n}\sum_{i=1}^{n} \ln(1+\xi_i)$ is as follows:

$$m_n(y) = nM_n(ny) \tag{3}$$

We solve the p.d.f. $N_n(x)$ of the random variable $\eta_n + 1$ by taking the following steps:

$$\ln(\eta_n + 1) = \frac{1}{n}\sum_{i=1}^{n} \ln(1+\xi_i)$$

$$\eta_n + 1 = e^{\frac{1}{n}\sum_{i=1}^{n} \ln(1+\xi_i)}$$

Since $x = e^y$ is a monotone increasing function, its inverse function is $y = \ln(x)$, and $\frac{dy}{dx} = \frac{1}{x}$. Therefore, we have

$$N_n(x) = \frac{dy}{dx} m_n(\ln(x)) = \frac{1}{x} m_n(\ln(x)) \tag{4}$$

According to the probability theory, the p.d.f. $H_n(x)$ of the random variable η_n is

$$H_n(x) = \frac{1}{x+1} m_n(\ln(x+1)) \tag{5}$$

In order to obtain the p.d.f. $H_n(x)$ of the random variable η_n, we only need to solve for the p.d.f. $M_n(y)$. It is thus evident that the derivation of the analytical expression of the p.d.f. $M_n(y)$ is a crucial step.

The procedure of solving the p.d.f. $H_n(x)$ of random variable η_n is to first identify $g(x) = p(x-1)$ based on $p(x)$, then determine $d(y) = g(e^y)e^y$ according to $g(x)$, then find $M_n(y)$ based on $d(y)$, $M_n(y)$ is the $(n-1)$ fold iteration of the convolution of $d(y)$ with itself, and subsequently apply $m_n(y) = nM_n(ny)$; finally, the p.d.f. of the random variable η_n is denoted as $H_n(x) = \frac{1}{x+1} m_n(\ln(x+1))$.

3. **The Theoretical Formulae under the Circumstances that the Short-Term Rate of Return is Log-Normally Distributed.** In this section we assume that the short-term rate of return ξ of asset K follows a specific form of lognormal distribution, and we derive the corresponding theoretical formulae under such assumption. A log-normally distributed random variable takes only positive values, the range of the short-term rate of return ξ is $(-1, +\infty)$; therefore, the range of $1+\xi$ is $(0, +\infty)$ which meets the requirements of lognormal distribution. Assume the random variable $1+\xi$ determined by the short-term rate of return ξ is log-normally distributed with parameters (μ, σ), i.e., the p.d.f. of the random variable $1+\xi$ is

$$q(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \tag{6}$$

Based on the properties of lognormal distribution, the expectation of the random variable $1+\xi$ is

$$E(1+\xi) = e^{\mu + \frac{\sigma^2}{2}} \tag{7}$$

The variance of the random variable $1+\xi$ is

$$D(1+\xi) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{8}$$

According to the probability theory, the p.d.f. of the random variable ξ is

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma(x+1)} e^{-\frac{(\ln(x+1)-\mu)^2}{2\sigma^2}}$$

The expectation of the random variable ξ is

$$E(\xi) = e^{\mu+\frac{\sigma^2}{2}} - 1 \tag{9}$$

The variance of the random variable ξ is the same as expression (8), i.e.,

$$D(\xi) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{10}$$

Through a vast amount of statistical analysis people know the p.d.f. of the short-term rate of return on regular risky assets roughly takes the symmetric unimodal curve. For symmetric p.d.f., the mode coincides with mean. For the p.d.f. of lognormal distribution, according to the expression $dp(x)/dx = 0$ it is not difficult to obtain $x = e^{\mu-\sigma^2} - 1$, which is the mode. Let us now measure the deviation of the mode from the mean of ξ, let ER denote this value, the smaller ER is the less the lognormal distribution deviates from the symmetric distribution form, and combined with (9) we get

$$ER = \left(e^{\mu+\frac{\sigma^2}{2}} - 1\right) - \left(e^{\mu-\sigma^2} - 1\right) = e^{\mu}\left(e^{\frac{\sigma^2}{2}} - e^{-\sigma^2}\right)$$

Under the circumstance that σ^2 is small, we have $ER \approx \frac{3}{2}\sigma^2 e^{\mu}$.

For the risky assets in reality, such as the foreign exchange, blue-chip stocks, floating rate deposit, treasury bonds, corporate bonds, the standard deviation σ of the annual rate of return are greatly diversified, for stock and foreign exchange $\sigma \approx 0.35$, for corporate bonds $\sigma \approx 0.08$; for bank deposit and treasury bonds about $\sigma \approx 0.04$. According to expression (10), when $\mu = 0$, $D(\xi) = e^{\sigma^2}\left(e^{\sigma^2} - 1\right) \approx \sigma^2 + \sigma^4$, so if σ^2 is small, σ^4 is smaller, and we have $D(\xi) \approx \sigma^2$. We shall approximate $D(\xi)$ using σ^2 in the expression of ER, and approximately take $\mu = 0$, we have $ER(\text{Stock}) \approx 0.18$, $ER(\text{Bonds}) \approx 0.01$, and $ER(\text{Treasury Bonds}) \approx 0.002$. When parameter σ^2 is very small, the lognormal distribution becomes closer to symmetrical form, but when parameter σ^2 is relatively large, the lognormal distribution deviates further from the symmetrical form. Theoretically, the lognormal distribution is more appropriate for describing the p.d.f. of the rate of return on less risky asset, for riskier investments such as stock and foreign exchange, the p.d.f. of their rate of return is difficult to describe precisely using the lognormal distribution.

Let us derive the p.d.f. $d(y)$ of the random variable $\ln(1+\xi)$, according to expression (2) we have $d(y) = q(e^y)e^y$, among which $q(x)$ is identified by expression (6), substituting and rearranging the expression we get

$$d(y) = \frac{e^y}{\sqrt{2\pi}\sigma e^y} e^{-\frac{(\ln e^y - \mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Therefore, $\ln(1+\xi)$ is normally distributed with expectation μ and variance σ^2, i.e., $\ln(1+\xi) \sim N(\mu, \sigma^2)$.

Next we shall derive the p.d.f. $M_n(y)$ of the sum of random variables $\sum_{i=1}^{n} \ln(1+\xi_i)$. According to the features of the p.d.f. of normal distribution, the sum of normally distributed random variables is still normally distributed, and calculating the sums of expectations and the variances respectively, we have

$$M_n(y) = \frac{1}{\sqrt{2\pi(\sigma^2 n)}} e^{-\frac{(y-n\mu)^2}{2(n\sigma^2)}}$$

According to probability theory, the p.d.f. $m_n(y)$ of random variable $\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)$ is

$$m_n(y) = nM_n(ny) = \frac{n}{\sqrt{2\pi(\sigma^2 n)}}e^{-\frac{(ny-n\mu)^2}{2(n\sigma^2)}}$$

Simplifying the expression above, we get

$$m_n(y) = \frac{1}{\sqrt{2\pi}\,(\sigma/\sqrt{n})}e^{-\frac{(y-\mu)^2}{2(\sigma/\sqrt{n})^2}} \tag{11}$$

The p.d.f. $m_n(y)$ identifies the normal distribution with expectation μ and variance σ^2/n.

We shall now derive the p.d.f. $N_n(x)$ of the random variable $\eta_n + 1$. $\eta_n + 1 = \exp\left(\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)\right)$, according to expression (4) we have

$$N_n(x) = \frac{1}{x}m_n(\ln(x)) = \frac{1}{x\sqrt{2\pi}(\sigma/\sqrt{n})}e^{-\frac{(\ln x-\mu)^2}{2(\sigma/\sqrt{n})^2}} \tag{12}$$

This is a p.d.f. with the form of lognormal distribution with parameters $(\mu, \sigma^2/n)$. Next we shall deduce the p.d.f. $H_n(x)$ of the random variable η_n, and according to expression (5) we have

$$H_n(x) = \frac{1}{x+1}m_n\left(\ln(x+1)\right) = \frac{1}{(x+1)\sqrt{2\pi}(\sigma/\sqrt{n})}e^{-\frac{(\ln(x+1)-\mu)^2}{2(\sigma/\sqrt{n})^2}} \tag{13}$$

This is a p.d.f. of lognormal distribution with parameters μ and $\frac{\sigma^2}{n}$. Designating the expectation of the random variable η_n as r_n and variance as D_n^2, according to expression (9) and expression (10), we have

$$r_n = e^{\mu+\frac{\sigma^2}{2n}} - 1 \tag{14}$$

$$D_n^2 = e^{2\mu+\frac{\sigma^2}{n}}\left(e^{\frac{\sigma^2}{n}} - 1\right) \tag{15}$$

4. **Some Theoretical Results of the Long-Term Rate of Return with Log-Normally Distribution on Risky Assets.** In this section we assume that the short-term rate of return ξ on risky asset K is log-normally distributed, and the p.d.f. is $p(x)$, i.e.,

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma(x+1)}e^{-\frac{(\ln(x+1)-\mu)^2}{2\sigma^2}}$$

Based on expressions (14) and (15) obtained in Section 3, we have the following theorem.

Theorem 4.1. *Let r_∞ denote the limit of the expectation of the rate of return η_n on risky asset K when n approaches infinity, and we have*

$$r_\infty = e^\mu - 1 \tag{16}$$

Proof: According to expression (14), we have

$$r_n = e^{\mu+\frac{\sigma^2}{2n}} - 1$$

It is easy to obtain $r_\infty = \lim\limits_{n\to+\infty} r_n = \lim\limits_{n\to+\infty}\left(e^{\mu+\frac{\sigma^2}{2n}} - 1\right) = e^\mu - 1$, and the theorem gets proved.

Theorem 4.2. *Let D_∞^2 denote the limit of the variance of the rate of return η_n on risky asset K when n approaches infinity, and we have*

$$D_\infty^2 = 0 \tag{17}$$

Proof: According to expression (15), we have

$$D_n^2 = e^{2\mu+\frac{\sigma^2}{n}}\left(e^{\frac{\sigma^2}{n}}-1\right)$$

It is easy to come up with $D_\infty^2 = \lim_{n\to+\infty} D_n^2 = \lim_{n\to+\infty}\left(e^{2\mu+\frac{\sigma^2}{n}}\left(e^{\frac{\sigma^2}{n}}-1\right)\right) = 0$. The theorem gets approved.

The conclusions of Theorem 4.1 and Theorem 4.2 indicate that when the holding period n of risky asset K is long enough, the average rate of return η_n is approximately $e^\mu - 1$, and the uncertainty of η_n gradually vanishes because D_n^2 approaches 0.

Theorem 4.3. *As the holding period n of the risky asset approaches infinity, the expectation $E(\xi)$ of the short-term rate of return must be greater than the average rate of return r_∞ when the asset is held indefinitely, the difference between these two is $e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right)>0$, and therefore we have*

$$E(\xi) - r_\infty = e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right) \tag{18}$$

Or equivalently

$$r_\infty = E(\xi) - e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right)$$

Proof: According to expression (16), we have $r_\infty = e^\mu - 1$; based on expression (9), there is $E(\xi) = e^{\mu+\frac{\sigma^2}{2}} - 1$ and we come up with

$$E(\xi) - r_\infty = e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right)$$

Since $e^\mu > 0$ and $\frac{\sigma^2}{2} > 0$, there is $e^{\frac{\sigma^2}{2}} > 1$, which means $e^{\frac{\sigma^2}{2}} - 1 > 0$, then

$$E(\xi) - r_\infty = e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right) > 0$$

The theorem gets proved.

The conclusion of Theorem 4.3 shows that the average rate of return r_∞ of the indefinitely held risky asset must be less than the expectation $E(\xi)$ of the short-term rate of return on risky asset and the difference of these two is $e^\mu\left(e^{\frac{\sigma^2}{2}}-1\right)$. From expression (18) we conclude that for fixed $\mu > 0$, as σ^2 becomes larger the difference $E(\xi) - r_\infty$ becomes greater, which indicates that the magnitude of decline of the long-term rate of return $E(\xi) - r_\infty$ is positively correlated to the factor σ^2. Moreover, expression (18) shows that if $\sigma^2 = 0$, then $E(\xi) - r_\infty = 0$, which indicates that for risk-free assets, the short-term rate of return is equal to the average rate of return for the asset held indefinitely.

Theorem 4.4. *Let $r = E(\xi)$, $D^2 = D(\xi)$, for any positive short-term rate of return $r > 0$, a corresponding value of variance D^2 can be identified and this variance D^2 satisfies the following equation*

$$D^2 = (1+r)^2\left((1+r)^2 - 1\right) \tag{19}$$

Correspondingly, the values of the lognormal distribution parameters taken are $\mu = 0$, $\sigma^2 = 2\ln(1+r)$, and then the average rate of return of the indefinitely held risky asset is 0, i.e., $r_\infty = 0$.

Proof: According to expression (6) we have $r_\infty = e^\mu - 1$. Let $r_\infty = 0$, so that $e^\mu = 1$, so we have $\mu = 0$. According to expression (9)

$$r = E(\xi) = e^{\mu+\frac{\sigma^2}{2}} - 1 = e^{\frac{\sigma^2}{2}} - 1 > 0$$

Therefore, $e^{\frac{\sigma^2}{2}} = 1 + r$ and equivalently $\sigma^2 = 2\ln(1 + r)$ which is the formula identifying the parameter σ using the given value $r > 0$. Then according to expression (10), we have $D^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$. Taking $\mu = 0$, $\sigma^2 = 2\ln(1 + r)$, thus

$$D^2 = (1 + r)^2\left((1 + r)^2 - 1\right)$$

According to the derivation above, we have $r_\infty = 0$ and the theorem gets proved.

Theorem 4.5. *Given any positive short-term rate of return $E(\xi) = r > 0$, the variance of the short-term rate of return is $D(\xi) = D^2$, then if $D^2 < (1 + r)^2((1 + r)^2 - 1)$, then $r_\infty > 0$; if $D^2 > (1 + r)^2((1 + r)^2 - 1)$, then $r_\infty < 0$; if $D^2 = (1 + r)^2((1 + r)^2 - 1)$, then $r_\infty = 0$.*

Proof: The formulae of the expectation r and variance D^2 of the log-normally distributed short-term rate of return are as follows:

$$r = e^{\mu+\frac{\sigma^2}{2}} - 1, \quad D^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

Since $r = e^{\mu+\frac{\sigma^2}{2}} - 1$, there is $e^{\sigma^2} = e^{-2\mu}(1 + r)^2$, so $e^{\sigma^2+2\mu} = (1 + r)^2$, substitute in the expression of D^2, and we obtain $D^2 = (1+r)^2(e^{-2\mu}(1 + r)^2 - 1)$. According to expression (16) we have $r_\infty = e^\mu - 1$.

If $D^2 < (1 + r)^2((1 + r)^2 - 1)$, it must hold that $\mu > 0$, and there is $r_\infty > 0$.
If $D^2 > (1 + r)^2((1 + r)^2 - 1)$, it must hold that $\mu < 0$, and there is $r_\infty < 0$.
If $D^2 = (1 + r)^2((1 + r)^2 - 1)$, it must hold that $\mu = 0$, and there is $r_\infty = 0$.
The theorem gets approved.

Through Theorem 4.4 we have obtained an important functional relationship $D^2 = (1+r)^2((1 + r)^2 - 1)$. For given $r > 0$ as the expectation of the short-term rate of return, if the variance takes the value of D^2 determined by Equation (19), and then it can be guaranteed that $r_\infty = 0$. For example, if the expectation of the short-term rate of return takes the value $r = 0.1$, which is comparatively high, we shall obtain $D^2 = 0.254$ based on Equation (19) which indicates a risk level actually not too high. The conclusion of Theorem 4.4 also shows that when the variance of the short-term rate of return is larger than 0.254, we must end with negative rate of return holding the asset permanently. Combined with Theorem 4.5, it can be shown that equation $D^2 = (1 + r)^2((1 + r)^2 - 1)$ identifies a critical curve related to the value that r_∞ takes. In the coordinate system determined by (r, D), the curve $D^2 = (1+r)^2((1 + r)^2 - 1)$ divides the first quadrant into two regions, the upper region corresponds to $r_\infty < 0$, and the lower region corresponds to $r_\infty > 0$.

Theorem 4.6. *If $-\frac{\sigma^2}{2} < \mu < 0$, then $E(\xi) > 0$, and $r_\infty < 0$; if $-\frac{\sigma^2}{2} < \mu < -\frac{\sigma^2}{2n}$, then $E(\xi) > 0$, and $r_n < 0$; r_n is the average annual rate of return with the holding period n determined by expression (14).*

Proof: According to expression (16), $r_\infty = e^\mu - 1$, $r_\infty < 0$ thus $\mu < 0$. According to expression (9), $E(\xi) = e^{\mu+\frac{\sigma^2}{2}} - 1$, $E(\xi) > 0$; therefore, $\mu + \frac{\sigma^2}{2} > 0$ which implies $\mu > -\frac{\sigma^2}{2}$. Therefore, when $-\frac{\sigma^2}{2} < \mu < 0$, $E(\xi) > 0$, and $r_\infty < 0$.

According to expression (14), $r_n = e^{\mu+\frac{\sigma^2}{2n}} - 1$, $r_n < 0$ implies $\mu + \frac{\sigma^2}{2n} < 0$, i.e., $\mu < -\frac{\sigma^2}{2n}$. According to expression (9), $E(\xi) = e^{\mu+\frac{\sigma^2}{2}} - 1$, $E(\xi) > 0$ gives $\mu + \frac{\sigma^2}{2} > 0$, i.e., $\mu > -\frac{\sigma^2}{2}$. Therefore, when $-\frac{\sigma^2}{2} < \mu < -\frac{\sigma^2}{2n}$ it can be concluded that $E(\xi) > 0$ and $r_n < 0$. The theorem gets proved.

The conclusion of Theorem 4.6 shows that $E(\xi) > 0$ does not guarantee that long-term held asset K shall obtain positive annual rate of return, and the combination with larger

σ^2 and smaller μ can cause $r_\infty < 0$ or $r_n < 0$. From expression (14), as the holding period n increases, r_n decreases. When n is relatively minor, the increase of n will cause relatively greater decrease in r_n, when n is large enough, $r_n \approx e^\mu - 1$. According to expression (15), D_n decreases as the holding period n increases. When n is large enough, $D_n \approx 0$, the uncertainty of r_n would be decreasing and r_n converges to r_∞ in probability.

5. **The Theoretical Analysis of the Assets for General Form of Distribution When the Holding Period is Relatively Large.** In this section we shall discuss the related circumstances under which the short-term rate of return follows a general form of distribution. Assume the p.d.f. of the random variable ξ is $p(x)$, the expectation of ξ is r and the variance of ξ is D^2, so

$$r = \int_{-1}^{+\infty} xp(x)dx, \quad D^2 = \int_{-1}^{+\infty} (x-r)^2 p(x)dx$$

Let the p.d.f. of $1+\xi$ be $q(x)$, therefore, $q(x) = p(x-1)$. According to expression (2), the p.d.f. of the random variable $\ln(1+\xi)$ is $d(y)$

$$d(y) = e^y q(e^y) = e^y p(e^y - 1)$$

Let us solve the mathematical expression of η which denotes the expectation of $\ln(1+\xi)$, and according to the definition of mathematical expectation, we have

$$\eta = \int_{-\infty}^{+\infty} ye^y p(e^y - 1)dy$$

Implement the variable substitution $e^y - 1 = x$, thus $y = \ln(1+x)$, and we have

$$\eta = \int_{-\infty}^{+\infty} ye^y p(e^y - 1)dy = \int_{-1}^{+\infty} p(x)\ln(1+x)dx \tag{20}$$

When the short-term rate of return on asset K is log-normally distributed, the parameters of the distribution $p(x)$ are (μ, σ), so

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma(x+1)} e^{-\frac{(\ln(x+1)-\mu)^2}{2\sigma^2}}$$

The expectation of $\ln(1+\xi)$ is η, so

$$\eta = \int_{-1}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma(x+1)} e^{-\frac{(\ln(x+1)-\mu)^2}{2\sigma^2}} \ln(1+x)dx$$

The variance of $\ln(1+\xi)$ is designated as H^2, according to the definition of variance

$$H^2 = \int_{-\infty}^{+\infty} (y-\eta)^2 e^y p(e^y - 1)dy = \int_{-1}^{+\infty} p(x)\left(\ln(1+x)\right)^2 dx - \eta^2 \tag{21}$$

Here consider the expectation and variance of the random variable $\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)$ as well as the approximation of the p.d.f. when n is relatively large. As we know, the expectation of the random variable $\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)$ is still η and the variance of the random variable $\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)$ is $\frac{1}{n}H^2$. According to the central limit theorem, when n is relatively large, the random variable $\frac{1}{n}\sum\limits_{i=1}^{n}\ln(1+\xi_i)$ is approximately normally distributed the corresponding p.d.f. can be denoted as $m_n(y)$, and we have

$$m_n(y) \approx \frac{1}{\sqrt{2\pi}\,(H/\sqrt{n})} e^{-\frac{(y-\eta)^2}{2\left(H^2/n\right)}}$$

According to expression (13), the approximate formula of the p.d.f. $H_n(x)$ of the variable $\eta_n = ((1+\xi_1)(1+\xi_2)\cdots(1+\xi_n))^{1/n} - 1$ is

$$H_n(x) \approx \frac{1}{\sqrt{2\pi}\,(H/\sqrt{n})\,(x+1)} e^{-\frac{(\ln(x+1)-\eta)^2}{2\left(H^2/n\right)}}$$

The above expression is the p.d.f. of a log-normally distributed random variable, according to the properties of the lognormal distribution, and expression (9) and expression (10) can be rewritten as follows

$$E(\eta_n) \approx e^{\eta + \frac{H^2}{2n}} - 1 \tag{22}$$

$$D(\eta_n) \approx e^{2\eta + \frac{H^2}{n}} \left(e^{\frac{H^2}{n}} - 1\right) \tag{23}$$

Here, $\eta = \int_{-1}^{+\infty} p(x)\ln(1+x)dx$, and $H^2 = \int_{-1}^{+\infty} p(x)\left(\ln(1+x)\right)^2 dx - \eta^2$. According to the p.d.f. $p(x)$ of ξ, η and H^2 can be identified; further we can approximate the long-term rate of return $E(\eta_n)$ and variance $D(\eta_n)$ based on expression (22) and expression (23). Letting $n \to +\infty$, from expression (22) we can obtain

$$r_\infty = \lim_{n\to+\infty} E(\eta_n) = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1 \tag{24}$$

$$D_\infty = \lim_{n\to+\infty} D(\eta_n) = 0 \tag{25}$$

Expression (24) was once obtained by implementing a different approach in [8].

The approximate formulae expressions (22) and (23) hold only when n is relatively large, and for general p.d.f. $p(x)$ and relatively small n, the analytic expression of the long-term rate of return $E(\eta_n)$ and variance $D(\eta_n)$ should be derived through the convolution formula. Of course, if $E(\eta_n)$ is being solved for empirical purpose, this can be done through two approaches, one is to use the Monte Carlo simulation, the other one is to deduce the p.d.f. $m_n(y)$ through convolution, thus the approximate solution of the probability function $H_n(x)$ can be obtained, and then the numerical values of the long-term rate of return $E(\eta_n)$ and variance $D(\eta_n)$ are identified. For relatively large n, e.g., $n > 30$, we can use expressions (22) and (23) to approximate the long-term rate of return and the magnitude of risk on risky asset K when the holding period is n.

6. **Conclusions.** The paper theoretically studies some problems related to the long-term rate of return, under the assumption that the short-term rate of return follows lognormal distribution, we have obtained the mathematical formulae of the p.d.f. of the rate of return on long-term for risky asset and the corresponding mean and variance, and we illustrate some properties of the obtained theoretical formulae. Under the situation that the short-term rate of return follows a general form of probability distribution, we deduced the approximation formulae of the mean, variance and the p.d.f. of the long-term rate of return. The theoretical formulae derived in this paper are applicable to assets with relatively smaller risk or situations with comparatively larger n. This paper provided the theoretical proof of the fact that the long-term rate of return on risky asset must be lower than short-term rate of return and demonstrated the mathematical formulae of the difference between the short-term rate of return and long-term rate of return under the situation that the short-term rate of return is log-normally distributed. In addition, we

identified the circumstances that guarantee $r_\infty = 0$, which determine the critical curve related to the mean and variance of short-term rate of return.

Acknowledgment. The authors gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

[1] F. Black and M. Scholes, The pricing of option and corporate liabilities, *Journal of Political Economy*, vol.6, no.81, pp.637-654, 1973.
[2] M. F. Osborne, Brownian motion in the stock market, *Operations Research*, no.7, pp.145-173, 1959.
[3] E. F. Fama, The behavior of stock-market prices, *Journal of Business*, vol.38, no.1, pp.34-72, 1965.
[4] J. Mehra and E. Prescott, The equity premium: A puzzle, *Journal of Monetary Economics*, 1985.
[5] E. Fama and K. French, The equity premium, *Working Paper*, The University of Chicago Graduate School of Business, 2000.
[6] E. Fama and K. French, The cross section of expected stock retunes, *Journal of Finance*, vol.47, pp.427-466, 1992.
[7] S. Kathari and J. Shanken, Stock returns variation and expected dividend, *Journal of Financial Economics*, pp.177-210, 1992.
[8] B. M. Friedman and V. V. Roley, Models of long-term interest rate determination, *The Journal of Portfolio Management*, vol.6, no.3, pp.35-45, 1980.
[9] X. Wang and L. Xu, Study on the long-term rate of return when holding risk asset, *International Journal of Innovative Computing, Information and Control*, vol.6, no.7, pp.3275-3288, 2010.
[10] X. Wang, L. Xu and Q. Fang, The critical curve of the long-term rate of return when holding risk assets, *International Journal of Innovative Computing, Information and Control*, vol.6, no.10, pp.4579-4592, 2010.
[11] X. Wang and B. Sun, New results on the rate of return for the risk assets held indefinitely, *International Journal of Innovative Computing, Information and Control*, vol.10, no.2, pp.823-836, 2014.
[12] J. A. Wachter, A consumption-based model of the term structure of interest rates, *Journal of Financial Economics*, vol.79, no.2, pp.365-399, 2006.
[13] Z. Da and M. Warachka, The disparity between long-term and short-term forecasted earnings growth, *Journal of Financial Economics*, vol.100, no.2, pp.424-442, 2011.
[14] M. Lettau and J. A. Wachter, The term structures of equity and interest rates, *Journal of Financial Economics*, vol.101, no.1, pp.90-113, 2011.