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NEW RESULTS ON STABILITY MARGINS OF NONLINEAR DISCRETE-TIME RECEDING HORIZON $\mathcal{H}_\infty$ CONTROL

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Abstract. *In this paper, we present some new results on stability margins of receding horizon $\mathcal{H}_\infty$ control for nonlinear discrete-time systems with disturbance. First, we propose a nonlinear inequality condition on the terminal cost for nonlinear discrete-time systems with disturbance. Under this condition, noninceasing monotonicity of the saddle point value function of the finite horizon dynamic game is shown to be guaranteed. We show that the derived condition on the terminal cost ensures closed-loop internal stability. The proposed receding horizon $\mathcal{H}_\infty$ control guarantees the infinite horizon $\mathcal{H}_\infty$ norm bound of the closed-loop system. Under additional conditions, the global result and the input-to-state stable (ISS) property of the proposed receding horizon controller are also given. Finally, we derive new $\mathcal{H}_\infty$ stability and ISS margins for the proposed receding horizon controller. The proposed result has a larger stability region than the existing inverse optimality based results.*

Keywords: Receding horizon control (RHC), Nonlinear systems, Discrete-time, $\mathcal{H}_\infty$ control, Input-to-state stability (ISS), Stability margin

1. **Introduction.** Receding horizon control (RHC) has been widely investigated as a successful feedback strategy [1, 2, 3, 4, 5, 6, 7, 8]. The basic concept of the receding horizon control is to solve an optimization problem for a finite future horizon at the current time and implement the first solution as a current control law. For the closed-loop stability of the RHC, one approach is to impose infinite terminal weighting which is equivalent to setting a zero terminal weighting matrix for the inverse Riccati equation [1, 2]. This is referred to as the terminal equality condition. Since imposing infinite terminal weighting is demanding, use of finite terminal weighting matrices has been investigated [5, 6]. As an alternative approach to finite horizons, an infinite horizon formulation has been explored [3, 4]. However, infinite horizon formulations can also be treated and approximated as finite horizon formulations with appropriate finite terminal weighting matrices as shown in [6] and [9] for linear systems and nonlinear systems, respectively.

This RHC has been applied to $\mathcal{H}_\infty$ problems [7, 10, 11] in order to combine the practical advantage of the RHC with the robustness of the $\mathcal{H}_\infty$ control. For the closed-loop stability of linear continuous-time systems, the terminal equality condition was proposed in [12] and the terminal inequality condition was presented in [13] for the monotonicity of the Riccati equation. A linear discrete-time result was also presented in [14]. In addition, the terminal inequality condition for the non-deceasing monotonicity was presented to include the terminal equality condition [12] for the linear continuous-time case [15] and for the linear discrete-time case [16]. Receding horizon $\mathcal{H}_\infty$ control schemes were proposed for continuous-time nonlinear systems in [17, 18]. In these results, the solution to the nonlinear receding horizon $\mathcal{H}_\infty$ control problem was based on the linearization around the origin. The set of admissible disturbances was of the special form $\{w_k \in R^p | \ \|w_k\|^2 \leq \frac{1}{\gamma^2}\|z_k\|^2\}$. For the nonlinear discrete-time case, the result of [19] presented the extended domain of attraction, while it had the assumption that the $\mathcal{H}_\infty$ control problem was solvable for the linearized system and the admissible set of disturbance form was taken to be the set of the form as $\|w_k\|^2 \leq \frac{1}{\gamma^2}\|z_k\|^2$. In [20], a somewhat generalization of [19] was presented. For stability margins, some results for nonlinear continuous-time systems were proposed in [18] via the inverse optimality approach. However, to date there have been no published results on stability margins for nonlinear discrete-time systems.

In this paper, we present some new results on the cost monotonicity-based receding horizon $\mathcal{H}_\infty$ controller for nonlinear discrete-time systems with disturbance. We propose a cost monotonicity condition on the terminal cost under which the closed-loop internal stability and the infinite horizon $\mathcal{H}_\infty$ norm bound are guaranteed. Under additional conditions, the global result and the input-to-state stable (ISS) property of the proposed receding horizon control are also given. Finally, we derive new $\mathcal{H}_\infty$ stability and ISS margins for the cost monotonicity-based receding horizon $\mathcal{H}_\infty$ controller. Our result ensures a larger stability region than the inverse optimality-based receding horizon $\mathcal{H}_\infty$ controller developed in continuous-time framework [18].

The class of nonlinear systems and the basic problem formulation considered in this paper are described in Section 2. In Section 3, some results on the cost monotonicity-based receding horizon $\mathcal{H}_\infty$ controller for nonlinear discrete-time systems are presented. In Section 4, we derive new stability margins for the proposed receding horizon $\mathcal{H}_\infty$ controller. The conclusion is given in Section 5.

2. **Problem Formulation.** We consider the following nonlinear discrete-time system:

$$x_{k+1} = a(x_k) + b(x_k)u_k + g(x_k)w_k = f(x_k, u_k, w_k), \tag{1}$$

$$z_k = [h(x_k) \ u_k]^T, \tag{2}$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ is the control input, and $w_k \in R^p$ is the external disturbance. $a(\cdot)$, $b(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are sufficiently smooth nonlinear functions with $a(0) = 0$ and $h(0) = 0$. The solution to the dynamic game will be in sets of time-varying feedback-type functions. These spaces are the strategy spaces that we shall call κ and ν. The finite horizon dynamic game at time k consists of minimization with respect to $\kappa(i-k, x_k)$ and maximization with respect to $\nu(i-k, x_k)$ of the cost function

$$J(x_k, \kappa, \nu, N) = \sum_{i=k}^{k+N-1} (\|z_i\|^2 - \gamma^2\|w_i\|^2) + V_f(x_{k+N}), \tag{3}$$

where $V_f(\cdot)$ is a smooth nonnegative function with $V_f(0) = 0$ and γ is a positive constant that stands for the disturbance attenuation level. For any state x_k, if a feedback saddle-point solution exists, we denote the solution as $\kappa^*(i-k, x_k)$ and $\nu^*(i-k, x_k)$, where $k \leq i \leq k+N-1$. The optimal value of the finite horizon dynamic game will be denoted by $V(x_k, N) = J(x_k, \kappa^*, \nu^*, N)$. In receding horizon control, at each time k the resulting feedback control $\kappa^{RH}(x_k)$ at state x_k is obtained by solving the finite horizon dynamic game (FHDG) and setting

$$\kappa^{RH}(x_k) = \kappa^*(0, x_k). \tag{4}$$

According to standard results of the dynamic game, we have

$$\begin{aligned} V(x_i) =& \min_u \max_w \Big[h(x_i)^T h(x_i) + u_i^T u_i - \gamma^2 w_i^T w_i + V(x_{i+1}) \Big] \\ =& h(x_i)^T h(x_i) + \kappa^{*T}(i-k, x_k)\kappa^*(i-k, x_k) - \gamma^2 \nu^{T*}(i-k, x_k)\nu^*(i-k, x_k) \\ &+ V(a(x_i) + b(x_i)\kappa^*(i-k, x_k) + g(x_i)\nu^*(i-k, x_k)), \end{aligned} \tag{5}$$

where $k \le i \le k+N-1$ and

$$V(x_{k+N}) = V_f(x_{k+N}). \tag{6}$$

If there exists a function $V(x_i)$ satisfying (5) and (6), the pair $\{\kappa^*(i-k, x_k), \nu^*(i-k, x_k)\}$ is a feedback saddle point solution to the FHDG [21].

3. Cost Monotonicity-Based Receding Horizon $\mathcal{H}_\infty$ Control.

3.1. Cost monotonicity and $\mathcal{H}_\infty$ performance with internal stability.

In the following theorem, a monotonicity condition for the saddle point value is investigated.

Theorem 3.1. *If the terminal cost function $V_f(x_k)$ satisfies the following inequality for some $\rho(x_k)$ and $\nu(x_k)$ for all $x_k \neq 0$:*

$$\begin{aligned} & h(x_k)^T h(x_k) + \rho^T(x_k)\rho(x_k) - \gamma^2 \nu^T(x_k)\nu(x_k) \\ & + V_f(a(x_k) + b(x_k)\rho(x_k) + g(x_k)\nu(x_k)) - V_f(x_k) \le 0, \end{aligned} \tag{7}$$

then the saddle point value $V(x_k, N)$ decreases monotonically as follows:

$$V(x_k, N+1) - V(x_k, N) \le 0, \tag{8}$$

for all positive integer N.

Proof: Denote the optimal solutions for $V(x_k, N+1)$ and $V(x_k, N)$ by indices 1 and 2, respectively. Subtraction $V(x_k, N)$ from $V(x_k, N+1)$ yields

$$\begin{aligned} & V(x_k, N+1) - V(x_k, N) \\ =& J(x_k, \kappa^*, \nu^*, N+1) - J(x_k, \kappa^*, \nu^*, N) \\ =& \sum_{i=k}^{k+N} (||z_{1i}||^2 - \gamma^2 ||w_{1i}||^2) + V_f(x_{1,k+N+1}) - \sum_{i=k}^{k+N-1} (||z_{2i}||_i^2 - \gamma^2 ||w_{2i}||^2) - V_f(x_{2,k+N}). \end{aligned}$$

Replacing u_{1i} and w_{2i} with u_{2i} and w_{1i} on $i \in [k,\ k+N-1]$ and using arbitrary $u_{1,k+N} = \rho(x_{k+N})$ and $w_{1,k+N} = \nu(x_{k+N})$, we have

$$\begin{aligned} & V(x_k, N+1) - V(x_k, N) \\ & \le ||z_{k+N}||^2 - \gamma^2 ||w_{k+N}||^2 + V_f(x_{k+N+1}) - V_f(x_{k+N}) \\ & = h(x_{k+N})^T h(x_{k+N}) + \rho^T(x_{k+N})\rho(x_{k+N}) - \gamma^2 \nu^T(x_{k+N})\nu(x_{k+N}) \\ & + V_f(a(x_{k+N}) + b(x_{k+N})\rho(x_{k+N}) + g(x_{k+N})\nu(x_{k+N})) - V_f(x_{k+N}) \le 0. \end{aligned}$$

This completes the proof. ∎

Without the external disturbance ($w_k = 0$), the asymptotic stability of the proposed receding horizon control law is obtained in the following theorem:

Theorem 3.2. *Without the external disturbance, if the monotonicity condition of the saddle point value (7) is satisfied, the closed-loop system controlled by the proposed receding horizon control law is asymptotically stable.*

Proof: From the definitions of the finite horizon cost and the saddle point value, we have

$$V(x_k, N) = h^T(x_k)h(x_k) + u_k^T u_k + V(x_{k+1}, N-1).$$

Using results in Theorem 3.1, we have

$$V(x_k, N) \geq h^T(x_k)h(x_k) + u_k^T u_k + V(x_{k+1}, N).$$

Since $V(x_k, N)$ is bounded below as $V(x_k, N) = \sum_{i=k}^{k+N-1} \|z_i\|^2 + V_f(x_{k+N}) \geq 0$, $V(x_k, N)$ approaches the constant number. Thus, $h^T(x_k)h(x_k) + u_k^T u_k \longrightarrow 0$ as $k \longrightarrow 0$, which implies that x_k goes to zero. This completes the proof. ∎

The next two results are used later to show that the proposed receding horizon controller can guarantee the infinite horizon $\mathcal{H}_\infty$ norm bound.

Lemma 3.1. *The saddle point value of the finite horizon dynamic game satisfies* $V(x_k, N) \geq 0$ *for all nonnegative integer* N.

Proof: Given $\nu(i-k, x_k) = 0$, $i \in [k, k+N]$, for every $\kappa^*(i-k, x_k)$, we have $J(x_k, \kappa^*, \nu, N) \geq 0$, and then $V(x_k, N) = J(x_k, \kappa^*, \nu^*, N) \geq J(x_k, \kappa^*, \nu, N) \geq 0$. This completes the proof. ∎

Lemma 3.2. *Under the monotonicity condition of the saddle point value (7), the saddle point value satisfies* $V(0, N) = 0$ *for all nonnegative integer* N.

Proof: If $x_0 = 0$, due to Theorem 3.1 and Lemma 3.1, then we have $0 \leq V(0, N) \leq V(0, N-1) \leq \cdots \leq V(0, 0) = V_f(0) = 0$. This completes the proof. ∎

The following result shows that the proposed receding horizon control law can guarantee the infinite horizon $\mathcal{H}_\infty$ norm bound.

Theorem 3.3. *Assume that the terminal cost* $V_f(\cdot)$ *satisfies the monotonicity condition of the saddle point value (7), the closed-loop system controlled by the proposed receding horizon controller guarantees the infinite horizon* $\mathcal{H}_\infty$ *norm bound with the initial state* $x_0 = 0$.

Proof: First, under the assumption of (7), we show that the proposed scheme satisfies the dissipative property. Denote the optimal solutions for $V(x_{k+1}, N)$ and $V(x_k, N)$ by indices 1 and 2, respectively. Then, we have

$$\begin{aligned}
&V(x_{k+1}, N) - V(x_k, N) \\
&= \sum_{i=k+1}^{k+N} [h^T(x_{1i})h(x_{1i}) + u_{1i}^T u_{1i} - \gamma^2 w_{1i}^T w_{1i}] + V_f(x_{1,k+N+1}) \\
&\quad - \sum_{i=k}^{k+N-1} [h^T(x_{2i})h(x_{2i}) + u_{2i}^T u_{2i} - \gamma^2 w_{2i}^T w_{2i}] - V_f(x_{2,k+N}).
\end{aligned}$$

Replacing u_{1i} and w_{2i} with u_{2i} and w_{1i} on $i \in [k+1,\ k+N-1]$ and using arbitrary $u_{1,k+N} = \rho(x_{k+N})$ and $w_{1,k+N} = \nu(x_{k+N})$, we have

$$\begin{aligned}
&V(x_{k+1}, N) - V(x_k, N) \\
&\leq -h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k + h(x_{k+N})^T h(x_{k+N}) + \rho^T(x_{k+N})\rho(x_{k+N}) \\
&\quad - \gamma^2 \nu^T(x_{k+N})\nu(x_{k+N}) + V_f(a(x_{k+N}) + b(x_{k+N})\rho(x_{k+N}) + g(x_{k+N})\nu(x_{k+N})) - V_f(x_{k+N}) \\
&\leq -h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k. \qquad (9)
\end{aligned}$$

This shows that the closed-loop system controlled by the proposed scheme is dissipative with a storage function $V(x_k, N)$ and the supply rate $-h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k$. Thus, we have

$$V(x_\infty, N) - V(x_0, N) \leq \sum_{k=0}^{\infty} \big[- h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k \big].$$

From Lemmas 3.1 and 3.2, with the initial state $x_0 = 0$, we have $0 \leq \sum_{k=0}^{\infty} \big[-h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k\big]$, which implies $\frac{\sum_{k=0}^{\infty}\left[h^T(x_k)h(x_k)+u_k^T u_k\right]}{\sum_{k=0}^{\infty} w_k^T w_k} \leq \gamma^2$. This completes the proof. ∎

Under a growth condition of the nonlinear function, it is shown that the proposed receding horizon controller can guarantee a global result.

Lemma 3.3. *Assume that there exist $K > 0$ and $r > 0$ such that*

$$||x_k|| \geq r \;\Rightarrow\; ||f(x_k, u_k, 0) - x_k|| \leq K h^T(x_k)h(x_k). \tag{10}$$

Then, the saddle point value $V(x_k, N)$ is radially unbounded.

Proof: Let $\overline{x}_k$ be the trajectory of the system (1) corresponding to $(\kappa^*(i, x_0), 0)$ at $i \in [0, N]$ and $\overline{x}_0 = x_0$. Then, we have

$$\begin{aligned} V(x_0, N) &= J(x_0, \kappa^*, \nu^*, N) \\ &\geq J(x_0, \kappa^*, 0, N) \\ &= \sum_{i=0}^{N-1} \big[h^T(\overline{x}_i)h(\overline{x}_i) + \kappa^{*T}(i, x_0)\kappa^*(i, x_0)\big] + V_f(\overline{x}_N) \\ &\geq \sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i). \end{aligned} \tag{11}$$

If $\|x_0\| > r$, then, because the proposed scheme is asymptotically stable by Theorem 3.2 in the case of $w_k = 0$, there exists a constant $\overline{r} > 0$ such that $\|\overline{x}_N\| \leq \overline{r}$.

- Case 1: $r \leq \|\overline{x}_N\|$

$$\begin{aligned} &\sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i) \\ &\geq \sum_{i=0}^{N-1} \frac{1}{K}||f(\overline{x}_i, \kappa^*(i, x_0), 0) - \overline{x}_i||^2 \\ &= \sum_{i=0}^{N-1} \frac{1}{K}\|\overline{x}_{i+1} - \overline{x}_i\| \geq \frac{1}{K}\left|\left|\sum_{i=0}^{N-1}(\overline{x}_{i+1} - \overline{x}_i)\right|\right| \\ &= \frac{1}{K}\|\overline{x}_N - \overline{x}_0\|. \end{aligned} \tag{12}$$

  From the definition of $\overline{x}_k$, we have $\overline{x}_0 = x_0$. Thus, $V(x_0, N) \geq \frac{1}{K}\|\overline{x}_N - x_0\|$.
- Case 2: $r > \|\overline{x}_N\|$
  In this case, there exists the largest integer M satisfying $\|\overline{x}_M\| \geq r$ and $0 < M < N$.

$$\sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i)$$

$$
\begin{aligned}
&= \sum_{i=0}^{M} h^T(\overline{x}_i)h(\overline{x}_i) + \sum_{j=M+1}^{N-1} h^T(\overline{x}_j)h(\overline{x}_j) \\
&\geq \sum_{i=0}^{M} h^T(\overline{x}_i)h(\overline{x}_i) \\
&\geq \sum_{i=0}^{M} \frac{1}{K}\|f(\overline{x}_i, \kappa^*(i, x_0), 0) - \overline{x}_i\|^2 \\
&= \sum_{i=0}^{M} \frac{1}{K}\|\overline{x}_{i+1} - \overline{x}_i\| \geq \frac{1}{K}\left\| \sum_{i=0}^{M}(\overline{x}_{i+1} - \overline{x}_i)\right\| \\
&= \frac{1}{K}\|\overline{x}_M - \overline{x}_0\| = \frac{1}{K}\|\overline{x}_M - x_0\|.
\end{aligned} \tag{13}
$$

Thus, we have $V(x_0, N) \geq \frac{1}{K}\|\overline{x}_M - x_0\|$.

From Cases 1 and 2, if $x_0 \to \infty$, $V(x_0, N) \to \infty$. This completes the proof. ∎

The next lemma shows that the condition $h(x_k) \in \mathcal{K}_\infty$ can guarantee the radially unboundedness of the receding horizon value function.

Lemma 3.4. *Assume that $h(x_k) \in \mathcal{K}_\infty$. Then, the saddle point value function $V(x_k, N)$ is radially unbounded.*

Proof: Let $\overline{x}_k$ be the trajectory of system (1) corresponding to $(\kappa^*(i, x_0), 0)$ at $i \in [0, N]$ and $\overline{x}_0 = x_0$. Then, we have

$$
\begin{aligned}
V(x_0, N) &= J(x_0, \kappa^*, \nu^*, N) \\
&\geq J(x_0, \kappa^*, 0, N) \\
&= \sum_{i=0}^{N-1} \left[h^T(\overline{x}_i)h(\overline{x}_i) + \kappa^{*T}(i, x_0)\kappa^*(i, x_0)\right] + V_f(\overline{x}_N) \\
&\geq \sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i) \\
&\geq h^T(\overline{x}_0)h(\overline{x}_0) = h^T(x_0)h(x_0).
\end{aligned} \tag{14}
$$

If $x_0 \to \infty$ and $h(x_k) \in \mathcal{K}_\infty$, then $V(x_0, N) \to \infty$. This completes the proof. ∎

Theorem 3.4. *Under the monotonicity condition of the saddle point value function (7), if the nonlinear system (1) satisfies the growth condition (10) or $h(x_k) \in \mathcal{K}_\infty$, then the closed-loop system controlled by the proposed receding horizon controller is globally internally stable and has the global infinite horizon $\mathcal{H}_\infty$ norm bound with the initial state $x_0 = 0$.*

Proof: Under the condition (10) or $h(x_k) \in \mathcal{K}_\infty$, $V(x_k, N)$ can be regarded as a global Lyapunov function. Thus, the global asymptotic stability result with $w_k = 0$ follows from Theorem 3.2 and Lemma 3.3. In addition, the global $\mathcal{H}_\infty$ norm bound is guaranteed form Theorem 3.3 and Lemma 3.3. This completes the proof. ∎

3.2. **ISS property.**

Theorem 3.5. *Under the assumption of (7) and (10), if $h(x_k)$ is radially unbounded, the nonlinear system (1) controlled by the proposed receding horizon control law (4) is input-to-state stable with respect to w_k.*

Proof: From the proof of Theorem 3.3, we have

$$\begin{aligned} V(x_{k+1}, N) - V(x_k, N) &\le -h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k \\ &\le -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k. \end{aligned} \tag{15}$$

If the nonlinear system (1) satisfies the growth condition (10), from Lemma 3.3, $V(x_k, N)$ is radially unbounded. Since $V(x_k, N)$ is positive definite from Lemma 3.1, $V(x_k, N)$ is a class $\mathcal{K}_\infty$ function. Thus, for $V(x_k, N) \in \mathcal{K}_\infty$, we can always find class $\mathcal{K}_\infty$ functions α_1 and α_2 satisfying $\alpha_1(\|x_k\|) \le V(x_k, N) \le \alpha_2(\|x_k\|)$. Since $h^T(x_k)h(x_k) \in \mathcal{K}_\infty$ and $\gamma^2 w_k^T w_k \in \mathcal{K}$, from Definition 3.2 in [22], $V(x_k, N)$ is an ISS Lyapunov function. Thus, by Lemma 3.5 in [22], the closed-loop system is ISS. This completes the proof. ∎

4. **Results on Stability Margins.** In this section, we derive some guaranteed stability margins for the proposed nonlinear receding horizon control law. A sufficient condition that guarantees the disk margin is given in terms of the state and the control. Consider the nonlinear system (NS) (1) with

$$y_k = -\phi(x_k), \tag{16}$$

where $\phi(x_k)$ is such that (1) and (2) have the infinite horizon $\mathcal{H}_\infty$ performance with $u_k = -y_k$. Thus, $\phi(x_k)$ can be regarded as the proposed RHC law (4). Consider the negative feedback interconnection shown in Figure 1 with the uncertainty $\Delta(\cdot)$. In this case, $u_k = -\Delta(y_k)$. Furthermore, we assume that, in the nominal case $\Delta(\cdot) = I$, the nominal closed-loop system has an L_2-gain less than or equal to γ.

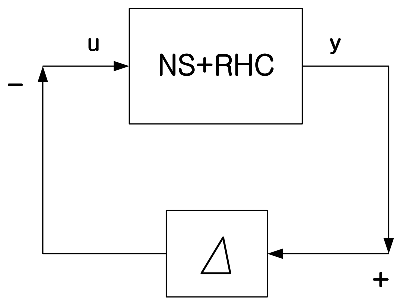

FIGURE 1. Feedback interconnection of $NS + RHC$ and Δ

4.1. **$\mathcal{H}_\infty$ stability margin.**

Definition 4.1. *Let α, $\beta \in R$ be such that $0 \le \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an $\mathcal{H}_\infty$ gain margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \Delta y$ has an L_2-gain less than or equal to γ for all $\Delta = diag\{k_1, k_2, \ldots, k_m\}$, where $k_i \in (\alpha, \beta)$ and $i = 1, \ldots, m$.*

Definition 4.2. *Let α, $\beta \in R$ be such that $0 \le \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an $\mathcal{H}_\infty$ sector margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \sigma(y)$ has an L_2-gain less than or equal to γ for all static nonlinearities $\sigma(\cdot)$ such that $\sigma(0) = 0$, $\sigma(y) = [\sigma_1(y_1), \ldots, \sigma_m(y_m)]^T$, and $\alpha y_i^2 < \sigma(y_i)y_i < \beta y_i^2$ for all $y_i \ne 0$ and $i = 1, \ldots, m$.*

Definition 4.3. *Let α, $\beta \in R$ be such that $0 \le \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an $\mathcal{H}_\infty$ disk margin $D(\alpha)$ if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \sigma(y)$ has an*

L_2-gain less than or equal to γ for all dynamic operator $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state detectable and dissipative with respect to the supply rate as $r(u,y) = u^T y - \rho u^T u$ with a radially unbounded storage function, where $\alpha < \rho \in R$.

Remark 4.1. *It is noted that if the closed-loop system $NS+RHC$ has an $\mathcal{H}_\infty$ disk margin $D(\alpha)$, then $NS+RHC$ has $\mathcal{H}_\infty$ gain and sector margins (α,∞).*

Now we are ready to state that the proposed receding horizon controller can guarantee an $\mathcal{H}_\infty$ disk margin.

Theorem 4.1. *Consider the closed-loop nonlinear system $NS+RHC$ given by (1) and (16). Assume NS and Δ are zero-state observable. Under the monotonicity condition of the saddle point value function (7), the proposed nonlinear receding horizon control law has an $\mathcal{H}_\infty$ disk margin $D(\frac{1}{4})$ with input w_k and output $h(x_k)$.*

Proof: Assume that Δ is given by

$$\widetilde{x}_{k+1} = \widetilde{a}(\widetilde{x}_k) + \widetilde{b}(\widetilde{x}_k)\widetilde{u}_k, \tag{17}$$

$$\widetilde{y}_k = \widetilde{\phi}(\widetilde{x}_k), \tag{18}$$

with the input $\widetilde{u}_k$, the output $\widetilde{y}_k$, and Δ is dissipative with respect to the following supply rate:

$$r(\widetilde{u}_k,\widetilde{y}_k) = \widetilde{u}_k^T\widetilde{y}_k - \rho\widetilde{u}_k^T\widetilde{u}_k, \tag{19}$$

where $\rho \in R$. Also, assume that Δ has a radially unbounded storage function $\widetilde{V}(\widetilde{x}_k)$ with $\widetilde{V}(0) = 0$. Then, we can construct the following dissipative inequality:

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) \le \widetilde{u}_k^T\widetilde{y}_k - \rho\widetilde{u}_k^T\widetilde{u}_k. \tag{20}$$

By the feedback interconnection in Figure 1, we have $\widetilde{u}_k = y_k$ and $u_k = -\widetilde{y}_k$. Thus, (20) becomes

$$\begin{aligned}\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) &\le -u_k^T y_k - \rho y_k^T y_k &(21)\\ &\le u_k^T u_k + \frac{1}{4}y_k^T y_k - \rho y_k^T y_k &(22)\\ &= u_k^T u_k - (\rho - \frac{1}{4})y_k^T y_k. &(23)\end{aligned}$$

Using (16), we have

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) \le u_k^T u_k - \left(\rho - \frac{1}{4}\right)\phi^T(x_k)\phi(x_k). \tag{24}$$

Under the monotonicity condition of the saddle point value function (7), the dissipative inequality (9) is satisfied. Let a new storage function be $V(x_k,\widetilde{x}_k) = V(x_k,N) + \widetilde{V}(\widetilde{x}_k)$. Adding (9) and (24) yields

$$V(x_{k+1},\widetilde{x}_{k+1}) - V(x_k,\widetilde{x}_k) \le -h^T(x_k)h(x_k) - \left(\rho - \frac{1}{4}\right)\phi^T(x_k)\phi(x_k) + \gamma^2 w_k^T w_k. \tag{25}$$

If we select $\rho \ge \frac{1}{4}$,

$$V(x_{k+1},\widetilde{x}_{k+1}) - V(x_k,\widetilde{x}_k) \le -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k. \tag{26}$$

Thus, we have

$$V(x_\infty,\widetilde{x}_\infty) - V(x_0,\widetilde{x}_0) \le \sum_{k=0}^{\infty}\big[-h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k\big]. \tag{27}$$

With the initial conditions as $x_0 = 0$ and $\widetilde{x}_0 = 0$, we have $0 \leq \sum_{k=0}^{\infty} \big[-h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k \big]$, which implies that the overall feedback interconnected system has infinite horizon $\mathcal{H}_\infty$ performance with input w_k and output $h(x_k)$ as $\frac{\sum_{k=0}^{\infty} h^T(x_k)h(x_k)}{\sum_{k=0}^{\infty} w_k^T w_k} \leq \gamma^2$. From Definition 4.3, we can conclude that the proposed receding horizon controller has an $\mathcal{H}_\infty$ disk margin $D(\frac{1}{4})$. This completes the proof. ∎

Remark 4.2. *The obtained disk margin $D(\frac{1}{4})$ guarantees a larger stability region than the existing result* [18] *developed in the continuous-time framework.*

4.2. **ISS margin.**

Definition 4.4. *Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an ISS gain margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \Delta y$ is ISS for all $\Delta = diag\{k_1, k_2, \ldots, k_m\}$, where $k_i \in (\alpha, \beta)$ and $i = 1, \ldots, m$.*

Definition 4.5. *Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an ISS sector margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \sigma(y)$ is ISS for all static nonlinearities $\sigma(\cdot)$ such that $\sigma(0) = 0$, $\sigma(y) = [\sigma_1(y_1), \ldots, \sigma_m(y_m)]^T$, and $\alpha y_i^2 < \sigma(y_i)y_i < \beta y_i^2$ for all $y_i \neq 0$ and $i = 1, \ldots, m$.*

Definition 4.6. *Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then, the closed-loop nonlinear system $NS + RHC$ given by (1) and (16) is said to have an ISS disk margin $D(\alpha)$ if the negative feedback interconnection of $NS+RHC$ and $\Delta(y) = \sigma(y)$ is ISS for all dynamic operator $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state detectable and dissipative with respect to the supply rate as $r(u, y) = u^T y - \rho u^T u$ with a radially unbounded storage function, where $\alpha < \rho \in R$.*

Remark 4.3. *It is noted that if $NS+RHC$ has an ISS disk margin $D(\alpha)$, then $NS+RHC$ has ISS gain and sector margins (α, ∞).*

In the next theorem, under additional conditions, we show that the proposed nonlinear receding horizon controller has an ISS disk margin $D\left(\frac{1}{4}\right)$.

Theorem 4.2. *Consider the closed-loop nonlinear system $NS + RHC$ given by (1) and (16). Assume NS and Δ are zero-state observable. Under (7) and (10), if $h(x_k)$ is radially unbounded, the proposed nonlinear receding horizon controller has an ISS disk margin $D(\frac{1}{4})$ with respect to w_k.*

Proof: First, start with assumptions used in the proof of Theorem 4.1. Then, we have

$$V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k) \leq -h^T(x_k)h(x_k) - \left(\rho - \frac{1}{4}\right) \phi^T(x_k)\phi(x_k) + \gamma^2 w_k^T w_k. \quad (28)$$

If we select $\rho \geq \frac{1}{4}$, then

$$V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k) \leq -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k. \quad (29)$$

From Lemma 3.3, if we assume (10), then $V(x_k, \widetilde{x}_k)$ is radially unbounded. Since $V(x_k, \widetilde{x}_k)$ is positive definite, $V(x_k, \widetilde{x}_k) \in \mathcal{K}_\infty$. Thus, we can always find class $\mathcal{K}_\infty$ functions α_1 and α_2 satisfying $\alpha_1(\|x_k\|, \|\widetilde{x}_k\|) \leq V(x_k, \widetilde{x}_k) \leq \alpha_2(\|x_k\|, \|\widetilde{x}_k\|)$. Since $h^T(x_k)h(x_k) \in \mathcal{K}_\infty$ with the assumption that $h(x_k)$ is radially unbounded and $\gamma^2 w_k^T w_k \in \mathcal{K}$, from Definition 3.2 in [22], $V(x_k, \widetilde{x}_k)$ is an ISS Lyapunov function. Thus, by Lemma 3.5 in [22], the closed-loop system is ISS. From Definition 4.6, we can conclude that the closed-loop system controlled by the proposed controller has an ISS disk margin $D(\frac{1}{4})$. This completes the proof. ∎

Remark 4.4. *The proposed results on stability margins of the nonlinear receding horizon $\mathcal{H}_\infty$ control can be used in several control applications. First, we design a nonlinear receding horizon $\mathcal{H}_\infty$ controller for systems without uncertainties. Then, it guarantees the closed-loop $\mathcal{H}_\infty$ stability. If uncertainties appear in the systems, the $\mathcal{H}_\infty$ stability of the controller may not be guaranteed. In this paper, we investigate how robust the controller is even though the controller is designed for systems without uncertainties. We can guarantee the robustness for uncertainties even though the controller is designed for systems without uncertainties. Therefore, from the point of view of control, the proposed results on stability margins are of significance for many applications to systems with uncertainties.*

Remark 4.5. *Most existing results on stability margins of nonlinear receding horizon $\mathcal{H}_\infty$ controls in the literature were restricted to continuous-time systems* [18]. *Unfortunately, with the existing results, it is impossible to investigate stability margins for discrete-time systems. For the first time, this paper proposes some results on stability margins for discrete-time systems. The proposed results in this paper open a new path for application of the nonlinear receding horizon $\mathcal{H}_\infty$ approach to the derivation of stability margins for discrete-time systems.*

5. **Conclusion.** In this paper, we propose some new results on receding horizon $\mathcal{H}_\infty$ control for nonlinear discrete-time systems with disturbance. A cost monotonicity condition on the terminal cost guarantees noninceasing monotonicity of the saddle point value function of the finite horizon dynamic game. Under this condition we can show the closed-loop internal stability and the infinite horizon $\mathcal{H}_\infty$ norm bound. Under additional conditions, the global result and the ISS property of the proposed receding horizon controller are also given. Finally, we derive new $\mathcal{H}_\infty$ stability and ISS margins for the proposed receding horizon controller. These margins guarantee a larger stability region than the inverse optimality-based results developed in the continuous-time framework.

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