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TRAIN ENERGY-EFFICIENT OPERATION WITH STOCHASTIC RESISTANCE COEFFICIENT

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Abstract. *Train energy-efficient operation problem applies the optimal control theory to optimize the speed profile between successive stations such that the tracking energy is minimized. Traditional studies show that the optimal speed profile consists of four phases including maximum acceleration, cruising, coasting and maximum braking. Based on the assumption that the resistance coefficients are random variables as the disturbances arising from the weather and locomotive conditions, this paper proposes a stochastic train energy-efficient operation model, and proves that the coasting phase should be replaced by a quasicoasting phase for the optimal speed profile. An efficient iterative algorithm is designed to solve the optimal switching strategy among different phases, and a numerical example is illustrated to show that the stochastic optimization approach can further save energy by* 3.68% *compared with the traditional studies.*
Keywords: Optimal train control, Energy-efficient operation, Stochastic optimization

1. **Introduction.** Energy and environmental concerns have made energy conservation a hot problem in railway transportation. Generally speaking, train energy conservation technique includes energy-efficient operation (Liu and Golovitcher [20], Howlett et al. [12]), energy-efficient scheduling (Ghoseiri et al. [6], Yang and Li [27], Park et al. [25]), improvement of traction efficiency and reduction on auxiliary power (Bergendorff [3]), while energy-efficient operation is one of the most important approaches, which applies the optimal control theory to optimize the speed profile between successive stations such that the energy consumption for tracking the speed profile is minimized. Literatures on this research may go back to the late 1970s, for example, Kokotović and Singh [17] formulated a nonlinear second-order optimal control model to minimize the electrical energy consumption by controlling the armature current. Milroy [21] firstly proposed the minimization problem of mechanical energy consumption in his PhD thesis. He showed that an energy-efficient speed profile for short trip has three phases including maximum acceleration, coasting and maximum braking. Furthermore, Lee et al. [18] discovered cruising as the fourth phase for longer trip. Howlett [8] produced the first theoretical confirmation that an optimal speed profile should use an maximum acceleration-cruising-coasting-maximum braking phase sequence, which reformulates the problem to optimize

the switching strategy among different phases. Although these researches have very restricting assumptions on line gradient, speed limit and traction efficiency, their theoretical results lay the foundation for modern train control theory (see Asnis et al. [1] and Howlett [8]). A research with variable gradient was carried out by Golovitcher [7], which proved that the cruising phase must be interrupted by an acceleration phase for each steep uphill section and by a coasting phase for each steep downhill section. A study on variable traction efficiency was given by Cheng and Howlett [5], and a case with different gradients and speed limits segments was investigated by Baranov [2]. Recently, Khmelnitsky [15] presented a complete study on train energy-efficient operation problem, in which variable gradient, variable traction efficiency and arbitrary speed limit were all considered.

All above researches were based on the assumption that the traction force is a continuous variable, i.e., we can choose any value between zero and the maximum. However, for some typical diesel-electric locomotive (see Howlett [10]), the traction force is proportional to the rate of fuel supply, and the rate is determined by a limited number of discrete settings, then only a finite number of traction forces are available. A systematic study on discrete energy-efficient operation problem was given by the Scheduling and Control Group at the University of South Australia (see Howlett [11] and Vu [26]). For example, Howlett [11] considered an energy-efficient operation problem with a generalized motion equation. The author considered both the continuous control problem and the discrete control problem. For the continuous problem, the Pontryagin maximum principle was used to find the necessary conditions of the optimal strategy. For the discrete problem, the Kuhn-Tucker equation was used to find the key equations that determine the optimal switching strategy. Note that the modern traction systems may produce any traction force within power and adhesion restrictions; therefore, the continuous control model is practical and sufficient enough for most applications.

Automatic train operation (ATO) technique basically consists of high-level control and low-level control: the former optimizes the reference speed profile for the purpose of energy conservation, comfortability and punctuality, while the latter studies the control technique for tracking the reference speed profile precisely. Energy-efficient operation is a kind of high-level control problem, which mainly concerns the optimization on speed profile. As the development of ATO technique, recent researches on energy-efficient operation mainly focus on the reduction of computation time for satisfying the requirement of on-board control. For example, Liu and Golovitcher [20] considered a continuous control problem with constant traction efficiency and arbitrary speed limit. The authors developed an analytical solution method that gave the sequence of optimal controls and equations to find the switching points. Ke and Chen [13] designed a genetic algorithm to determine the energy-efficient speed profile with restrictions on gradient, speed limit and minimum headway. The application of genetic algorithm on train control problem was also studied by Chang and Sim [4] and Yang et al. [28]. Howlett [12] provided another analytical method for solving continuous control problem with more than one steep slopes, which first divided the route into small parts such that each part contains at most one steep slope, then solved the precise switching points for each part by using a local energy minimization principle. Li et al. [19] designed a bisection algorithm based on the analytical solution approach for train energy-constraint operation problem, which minimizes the trip time under certain energy consumption constraint. In addition, dynamic programming algorithm (Ko et al. [16]), sequential quadratic programming algorithm (Miyatake and Matsuda [22] and Miyatake and Ko [23]) and ant colony optimization algorithm (Ke et al. [14]) were also designed to solve the energy-efficient speed profile.

Since the operation strategy will be used by many trains over a long period of time, during which the resistance coefficients will vary due to the disturbances of weather, route and locomotive conditions, it is impossible to predict their values precisely when we formulate the energy-efficient operation model. For example, Morten and Spencer [24] pointed out that an old British diesel multiple unit train has aerodynamic resistance coefficient 0.000044, which is significantly greater than 0.000020 for a new train. However, the existing literatures and approaches usually ignore these uncertain disturbances and formulate deterministic models with standard parameter value representing good conditions, which may increase the energy consumption when the resistance coefficient deviates the standard value. In this paper, we propose an expected energy consumption minimization model with stochastic resistance coefficient for overcoming the uncertain disturbances.

The rest of this paper is organized as follows. In Section 2, we review the traditional optimal control model for train energy-efficient operation problem. In Section 3, a stochastic model is proposed by minimizing the expected energy consumption. We analyze the optimality conditions by applying the Pontryagin maximum principle, which tells us that the optimal speed profile consists of maximum acceleration, cruising, quasicoasting and maximum braking. In Section 4, we design a numerical algorithm to solve the optimal switching strategy. In Section 5, we present a numerical example to compare the stochastic model with the deterministic model on energy consumption, which implies that the stochastic model can further save energy by 3.68% in average. In Section 6, we conclude the paper.

2. **Train Energy-Efficient Operation.** Suppose that a train runs on a section with no line gradients and speed limits. Since Howlett and Pudney [9] has proved that the motion of a distributed mass train can be reduced to the motion of a point mass train, the motion equation can be written as

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{f(s) - (a + cv^2(s))}{v(s)}, \quad \forall\ 0 \le s \le S \tag{1}$$

where s denotes the train position, S denotes the length of the section, v denotes the train speed, $a + cv^2$ denotes the resistance force per unit mass, and f denotes the external force per unit mass. Note that the external force applied to the train includes both the traction force and the braking force, which are denoted as its positive part and negative part, respectively.

For any feasible speed profile v satisfying the motion Equation (1), it is easy to prove that the total trip time taken for tracking it is

$$\int_0^S 1/v(s)\mathrm{d}s. \tag{2}$$

Now, we consider the speed profile with the minimum trip time. Let v_1 be the solution for $s \ge 0$ to differential equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \left(F(v(s)) - (a + cv^2(s))\right)/v(s)$$

with $v(0) = 0$, and let v_2 be the solution for $s \le S$ to differential equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = -\left(B(v(s)) + (a + cv^2(s))\right)/v(s)$$

with $v(S) = 0$. Then the minimum time speed profile is proved to be $v(s) = \min\{v_1(s), v_2(s)\}$, and the minimum trip time can be calculated by Equation (2). Generally speaking, the given trip time T is larger than the minimum value $T_{\min}$. Therefore, it is possible that there are more than one feasible speed profile satisfying the time constraint. Among all these speed profiles, the one which minimizes the energy consumption

$$\int_0^S \max\{f(s), 0\}\mathrm{d}s \tag{3}$$

is called the energy-efficient speed profile. The train energy-efficient operation model is formulated as follows:

$$\begin{cases} \min \quad \displaystyle\int_0^S \max\{f(s), 0\}\mathrm{d}s \\ \text{s.t.} \quad \displaystyle\int_0^S 1/v(s)\mathrm{d}s = T \\ \qquad \dfrac{\mathrm{d}v(s)}{\mathrm{d}s} = (f(s) - (a + cv^2(s)))/v(s),\ 0 \le s \le S \\ \qquad -B(v(s)) \le f(s) \le F(v(s)),\ 0 \le s \le S \\ \qquad v(s) \ge 0,\ 0 \le s \le S,\ v(0) = v(S) = 0, \end{cases} \tag{4}$$

where the third constraint denotes the traction and braking capacity, and the last one denotes the boundary speed conditions. In this model, the control variable is $f(s)$ and the state variable is $v(s)$.

According to the Pontryagin maximum principle, it has been proved that the energy-efficient speed profile essentially consists of four phases including maximum acceleration, cruising, coasting and maximum deceleration (see Howlett et al. [12]). Then the optimal train control problem is transformed into a nonlinear optimization problem which determines the switching strategy.

3. **Stochastic Model.** Generally speaking, it is impossible to predict the resistance coefficients precisely due to the disturbance of weather, route and locomotive conditions. However, the existing literatures and approaches usually ignore these uncertain factors and formulate deterministic model with standard coefficient values representing good conditions, which may increase the energy consumption when the resistance coefficients deviate the standard values. For example, in Morten and Spencer [24], the authors pointed out that an old British diesel multiple unit train has aerodynamic resistance coefficient 0.000044, which is significantly greater than 0.00002 for a new train. For an old train, if we take $a = 16.6$, $S = 40$ km, $T = 700$ s, $F = 0.8/(1+0.005v)$ N and $B = 0.4/(1+0.003v)$ N, the energy consumption for tracking the energy-efficient speed profile is 9223 kJ. However, if we treat it as a new train by taking $c = 0.00002$, then the tracking energy increases to 11979 kJ.

For dealing with the uncertainty on resistance coefficient, we estimate it as a random variable ξ with density function $\phi(x) = (x - l)/(u - l)$ for all $x \in [l, u]$. A speed profile is said to be feasible if and only if for each $x \in [l, u]$, there is a feasible value $f(s, x)$ satisfying the motion equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = (f(s, x) - (a + xv^2(s)))/v(s),\ \forall\, 0 \le s \le S.$$

Denote $e = (l + u)/2$. In order to make it possible to track a common speed profile for all $x \in [l, u]$, it is clear that $f(s, x)$ and $f(s, e)$ should satisfy the following equation

$$f(s, x) - (a + xv^2(s)) = f(s, e) - (a + ev^2(s)), \tag{5}$$

which implies that $f(s,x) = f(s,e) + (x-e)v^2(s)$. Take x to be the minimum value l, it follows from the braking capacity constraint $f(s,l) \geq -B(v(s))$ that

$$f(s,e) \geq -B(v(s)) + (u-l)v^2(s)/2. \tag{6}$$

On the other hand, take x to be the maximum value u. It follows from the capacity constraint $f(s,u) \leq F(v(s))$ that

$$f(s,e) \leq F(v(s)) - (u-l)v^2(s)/2. \tag{7}$$

Now, let us consider the expected energy consumption. First, it is clear that the energy consumption for tracking the feasible speed profile $v(s)$ with resistance coefficient x is

$$E(v,x) = \int_0^S \max\{f(s,e) + (x-e)v^2(s), 0\}\mathrm{d}s.$$

Since x varies randomly, $E(v,x)$ is also a random variable. We take its expected value to measure the total energy consumption. It follows from the Fubini's theorem that the expected energy consumption is

$$\begin{aligned}
&\int_l^u \int_0^S \max\{f(s,e) + (x-e)v^2(s), 0\}\phi(x)\mathrm{d}s\mathrm{d}x \\
=& \int_0^S \int_l^u \max\{f(s,e) + (x-e)v^2(s), 0\}\phi(x)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Based on above analysis, the stochastic energy-efficient operation model can be formulated as follows:

$$\begin{cases}
\min & \displaystyle\int_0^S \int_l^u \max\{f(s,e) + (x-e)v^2(s), 0\}\phi(x)\mathrm{d}x\mathrm{d}s \\
\text{s.t.} & \displaystyle\int_0^S 1/v(s)\mathrm{d}s = T \\
& \dfrac{\mathrm{d}v(s)}{\mathrm{d}s} = \dfrac{f(s,x) - (a + xv^2(s))}{v(s)}, \ 0 \leq s \leq S \\
& f(s,e) \leq F(v(s)) - (u-l)v^2(s)/2, \ 0 \leq s \leq S \\
& f(s,e) \geq -B(v(s)) + (u-l)v^2(s)/2, \ 0 \leq s \leq S \\
& v(s) \geq 0, \ 0 \leq s \leq S, \ v(0) = v(S) = 0
\end{cases} \tag{8}$$

where the control variable is $f(s,e)$, and the state variable is $v(s)$.

In what follows, we analyze the optimality conditions of the stochastic model. First, we discuss the integral

$$\int_l^u \max\{f + (x-e)v^2, 0\}\phi(x)\mathrm{d}x. \tag{9}$$

For simplicity, we denote $z = (u-l)v^2/2$. If $f < -z$, it is easy to prove that $f+(x-e)v^2 \leq 0$ for all $x \in [l,u]$ and the integral is 0. If $f > z$, it is easy to prove that $f + (x-e)v^2 \geq 0$ for all $x \in [l,u]$ and the integral is f. Otherwise, we have $u \geq e - f/v^2 \geq l$ and the integral is

$$\int_{e-f/v^2}^u (f + (x-e)v^2)/(u-l)\mathrm{d}x = (f+z)^2/4z.$$

In general, it is easy to conclude the following equation

$$\begin{aligned}&\int_l^u \max\{f+(x-e)v^2,0\}\phi(x)\mathrm{d}x\\&=\left(\max\{2f,-(u-l)v^2\}+\max\{2f,(u-l)v^2\}\right)^2/8\max\{2f,(u-l)v^2\}.\end{aligned}\tag{10}$$

Furthermore, according to the Pontryagin maximum principle, for each $0\le s\le S$, the optimal control strategy should maximize the following Hamiltonian function

$$\begin{aligned}H=&\frac{p}{v}\times\left(f-(a+cv^2)\right)-\big(\max\{2f,-(u-l)v^2\}\\&+\max\{2f,(u-l)v^2\}\big)^2/8\max\{2f,(u-l)v^2\}\end{aligned}$$

where $p(s)$ is the conjugate function satisfying equations

$$\frac{\mathrm{d}p}{\mathrm{d}s}=-\frac{\mathrm{d}H}{\mathrm{d}v},\quad \frac{\mathrm{d}v}{\mathrm{d}s}=\frac{\mathrm{d}H}{\mathrm{d}p}.$$

The argument breaks down into five cases.

Case 1. $p>v$. It is easy to prove that the Hamilton function H is increasing with respect to f, which implies that it attains the maximum value when $f=F(v)-(u-l)v^2/2$.

Case 2. $p=v$. In this case, it is easy to prove that H attains its maximum value if and only if $f\ge(u-l)v^2/2$. Since the Hamiltonian function has no relation with respect to p, we have $\mathrm{d}H/\mathrm{d}p=0$. It follows from the conjugate condition that $\mathrm{d}v/\mathrm{d}s=0$, which implies that the train runs with a constant speed. Therefore, we have $f=(l+u)v^2/2$.

Case 3. $0<p<v$. In this case, it is easy to prove that H is a unimodal function. Take $\mathrm{d}H/\mathrm{d}f=0$, it is solved that $f=(p/v-1/2)(u-l)v^2$. Furthermore, according to the conjugate conditions $\mathrm{d}p/\mathrm{d}s=up-rp/v^2$ and $\mathrm{d}v/\mathrm{d}s=(u-l)p-r/v-uv$, we have

$$\frac{\mathrm{d}(p/v)}{\mathrm{d}s}=2u(p/v)-(u-l)(p/v)^2.$$

Then it is solved that

$$\begin{aligned}&p/v=2u/(u-l+\alpha\exp(-2us)),\\&f=\left(2u/(u-l+\alpha\exp(-2us))-0.5\right)(u-l)v^2\end{aligned}$$

where α is a positive real parameter. It follows from $p<v$ that $\alpha>(l+u)\exp(2us)$.

Case 4. $p=0$. In this case, we have $p/v=0$, and H attains its maximum value 0 if and only if $f=-(u-l)v^2/2$. On the other hand, it follows from the conjugate condition that $\mathrm{d}v/\mathrm{d}s=0$. Hence, there is another positive force keeping the train a constant speed, which implies that this phase exists at steep downhill segment only.

Case 5. $p<0$. It is easy to prove that H is a decreasing function with respect to f, which implies that it is maximized if and only if f reaches its minimum value $-B(v)+(u-l)v^2/2$.

Based on the assumption that the track is flat, the optimal speed profile consists of four phases. For simplicity, we respectively name them as maximum acceleration, cruising, α-quasicoasting and maximum braking. Note that the third phase is named as quasicoasting since it coincides to the traditional coasting phase when $l=u$. Let v be the optimal speed profile with cruising point s_1, α-quasicoasting point s_2 and braking point s_3. Then the traction strategy should be

$$f(s,e)=\begin{cases} F(v(s))-(u-l)v^2(s)/2, & \text{if } 0<s\le s_1\\ a+(l+u)v^2(s)/2, & \text{if } s_1<s\le s_2\\ (2u/(u-l+\alpha\exp(-2us))-0.5)(u-l)v^2(s), & \text{if } s_2<s\le s_3\\ -B(v(s))+(u-l)v^2(s)/2, & \text{if } s_3<s\le S, \end{cases} \tag{11}$$

and the speed profile v should satisfy the following differential equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s}=\begin{cases} (F(v(s))-a-uv^2(s))/v(s), & \text{if } 0<s\le s_1\\ 0, & \text{if } s_1<s\le s_2\\ u(2(u-l)/(u-l+\alpha\exp(-2us))-1)v(s), & \text{if } s_2<s\le s_3\\ -(B(v(s))+a+lv^2(s))/v(s), & \text{if } s_3<s\le S. \end{cases} \tag{12}$$

Note that if the resistance coefficient is fixed to be a crisp number, that is, $l=u$, then Equation (12) coincides to the traditional equation for energy-efficient speed profile.

4. **Algorithm.** In this section, we design a numerical algorithm to solve the optimal switching strategy. First, we uniformly divide the section into N parts. Denote $\triangle s=S/N$ and $p_i=i\times\triangle s$ for all $i=0,1,2,\cdots,N$.

Trip time. For each feasible switching strategy (s_1,s_2,s_3) with $s_j\in\{p_1,p_2,\cdots,p_N\}$, according to Equation (12) and the boundary condition $v_0=v_N=0$, it is easy to solve the values $\{v_i,i=1,2,\cdots,N\}$. Then the trip time may be approximated as

$$\sum_{i=1}^{N-1}\triangle s/v_i\to\int_0^S 1/v(s)\mathrm{d}s. \tag{13}$$

Energy consumption. Denote $n_i=s_i/\Delta s$ for $i=1,2,3$. According to Equation (5), the energy consumption during the maximum acceleration, cruising and quasicoasting phases can be respectively approximated as

$$\begin{cases} \displaystyle\sum_{i=1}^{n_1-1}\left(F(v_i)-(u-l)v_i^2/2\right)\Delta s\to\int_0^{s_1}F(v(s))-(u-l)v^2(s)/2\mathrm{d}s\\ \displaystyle\sum_{i=n_1}^{n_2-1}\left(a+(l+u)v_i^2/2\right)\Delta s\to\int_{s_1}^{s_2}a+(l+u)v^2(s)/2\mathrm{d}s\\ \displaystyle\sum_{i=n_2}^{n_3-1}\max\{2u/(u-l+\alpha\exp(-2up_i))-0.5,0\}(u-l)v_i^2\Delta s\to\\ \qquad\displaystyle\int_{s_2}^{s_3}\max\{2u/(u-l+\alpha\exp(-2us))-0.5,0\}(u-l)v^2(s)\mathrm{d}s \end{cases}$$

Braking point. For any given cruising point s_1 and quasicoasting point s_2, since both the braking curve and the quasicoasting curve are strictly decreasing but have different gradients for all position s, the intersection point is unique which is just the braking point (see Figure 1).

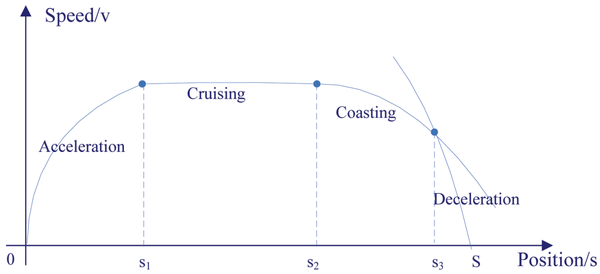

FIGURE 1. Speed profile with given cruising point and quasicoasting point

Cruising point. Now, we consider the feasible values of cruising point. First, it is possible that the cruising point is so small that even through the quasicoasting phase is omitted, the trip time still exceeds the given value. Hence, the lower bound of cruising point s_1^l should be the one satisfying $T(s, s_2, s_2) = T$ where s_2 is the intersection point of the cruising line and the braking curve (see Figure 2). Second, it is possible that the cruising point is so large that even though the cruising phase is omitted, the trip time is still smaller than the given value. Hence, the upper bound of cruising point s_1^u should be the one satisfying $T(s, s, s_3) = T$ where s_3 is the intersection point of the quasicoasting curve and the braking curve.

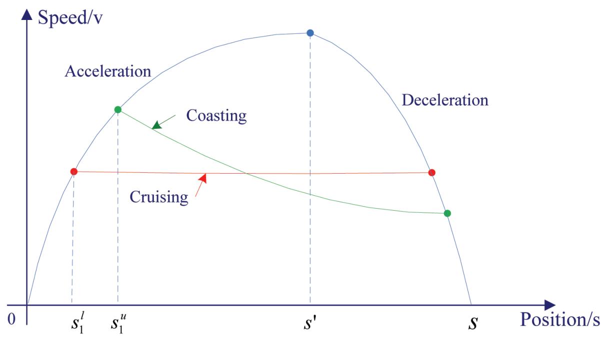

FIGURE 2. The lower and upper bounds for cruising point

Quasicoasting point. For any given cruising point s_1, the quasicoasting point satisfying the time constraint is unique. Suppose that there are two feasible quasicoasting points s_2 and t_2. Without loss of generality, we assume $s_2 < t_2$. If we use s_3 and t_3 to denote the braking points, and use v and v' to denote the speed profiles, then it is easy to prove that $t_3 < s_3$ and $v(s) < v'(s)$ for any $s \in [s_2, s_3]$ (see Figure 3), which contradicts to the trip time constraint. Therefore, the quasicoasting point is unique once the cruising point is fixed, and it can be solved by using the dichotomy algorithm.

Final algorithm. Based on above analysis, we design an iterative algorithm to solve the switching strategy.

Algorithm 4.1. For any $\alpha > 0$, the algorithm for calculating the optimal switching strategy is summarized as follows.

Step 1. Calculate the lower bound s_1^l and upper bound s_1^u for cruising point;
Step 2. Set $i = s_1^l/\Delta s$ and $k = i$;
Step 3. Calculate the quasicoasting point s_2 with cruising point p_i;
Step 4. Calculate the braking point s_3 with cruising point p_i and quasicoasting point s_2;
Step 5. Calculate the expected energy consumption E_i;

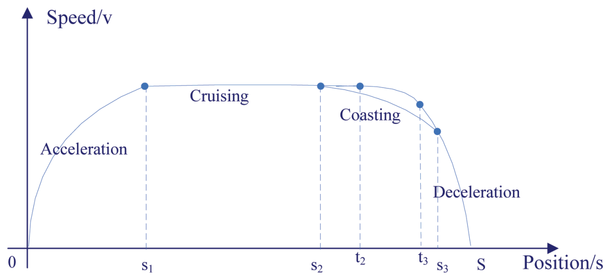

FIGURE 3. Speed profile with given cruising point

Step 6. If $E_i < E_k$, set $k = i$;
Step 7. Set $i = i + 1$. If $p_i \leq s_1^u$, go to Step 3;
Step 8. Calculate the quasicoasting point s_2 and braking point s_3 with cruising point p_k;
Step 9. Return the optimal switching strategy (p_k, s_2, s_3) and the expected energy consumption E_k.

5. Numerical Example.

In order to illustrate the efficiency of the proposed model and algorithm, we present a numerical example in this section which is performed on a personal computer with processor speed 2.4 GHz and memory size 2 GB. The parameters are set as follows:

- traction capacity $F(v) = 0.8/(1 + 0.005v)$ N;
- braking capacity $B(v) = 0.4/(1 + 0.003v)$ N;
- operation distance $S = 40000$ m;
- mechanical resistance coefficient $r = 0.01606$;
- aerodynamic resistance coefficient $\xi \sim \mathrm{U}(0.000020, 0.000044)$.

First, we calculate the minimum trip time by accelerating the train with the maximum traction force and then decelerating the train with the maximum braking force. Let v_1 be the solution for $s \geq 0$ to differential equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \left(0.8/(1 + 0.005v(s)) - (0.01606 + 0.000044v^2(s))\right)/v(s)$$

with $v(0) = 0$, and let v_2 be the solution for $s \leq 200$ to differential equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = -\left(0.4/(1 + 0.003v(s)) + (0.01606 + 0.000020v^2(s))\right)/v(s)$$

with $v(40000) = 0$. Then the speed profile with minimum trip time is $\min\{v_1, v_2\}$, and the minimum trip time is calculated to be $T_{\min} = 598.9$ s.

Now, we compare the stochastic model with the deterministic model on energy consumption with trip time $T \in \{610, 630, 700, 800, 900\}$. First, we perform Algorithm 4.1 to calculate the expected energy consumption with $\alpha \in \{0.0022, 0.003, 0.004, 0.005, 0.01, 0.02, 0.05\}$. Furthermore, we calculate the energy-efficient speed profiles for the deterministic model with fixed aerodynamic resistance coefficient $c \in \{0.000020, 0.000025, 0.000030, 0.000032, 0.000035, 0.000040, 0.000044\}$, and then calculate the expected energy consumption for tracking the energy-efficient speed profile under stochastic environment. The computational results are shown by Table 1. It is concluded that

- Generally speaking, a shorter time makes it necessary to accelerate the train to a higher speed with a larger energy consumption. In Table 1, it is shown that the energy consumption is decreasing with respect to the trip time for both deterministic model and the stochastic model.

Table 1. A comparison between stochastic model and deterministic model

$T = 610$	D-model	$c \times 1000$	0.020	0.025	0.030	0.032	0.035	0.040	0.044
		Energy	16882	14721	13271	12439	11913	11856	11601
	S-model	α	0.0022	0.003	0.004	0.005	0.01	0.02	0.05
		Energy	11314	11380	11445	11484	11560	11598	11620
$T = 630$	D-model	$c \times 1000$	0.020	0.025	0.030	0.032	0.035	0.040	0.044
		Energy	16285	13925	12291	11413	10852	10652	10369
	S-model	α	0.0022	0.003	0.004	0.005	0.01	0.02	0.05
		Energy	10032	10080	10141	10179	10254	10290	10312
$T = 700$	D-model	$c \times 1000$	0.020	0.025	0.030	0.032	0.035	0.040	0.044
		Energy	14407	11590	9812	9021	8416	8012	7667
	S-model	α	0.0022	0.003	0.004	0.005	0.01	0.02	0.05
		Energy	7377	7398	7437	7467	7526	7555	7573
$T = 800$	D-model	$c \times 1000$	0.020	0.025	0.030	0.032	0.035	0.040	0.044
		Energy	12565	9694	7636	7092	6446	5936	5579
	S-model	α	0.0022	0.003	0.004	0.005	0.01	0.02	0.05
		Energy	5350	5358	5381	5403	5447	5469	5481
$T = 900$	D-model	$c \times 1000$	0.020	0.025	0.030	0.032	0.035	0.040	0.044
		Energy	11263	8344	6592	5918	5292	4722	4351
	S-model	α	0.0022	0.003	0.004	0.005	0.01	0.02	0.05
		Energy	4165	4169	4184	4200	4235	4251	4261

- For stochastic model, the expected energy consumption corresponding to the optimal speed profile is strictly increasing with respect to α, which tells us that we should select α as small as possible. On the other hand, its value makes small influence on the expected energy consumption. Take $T = 630$ for example (see Figure 4), if α increases from 0.0022 to 0.05, the variation on energy consumption is only 2.72%.
- For deterministic model, the expected energy consumption is roughly decreasing with respect to the resistance coefficient c, which tells us that the largest resistance coefficient should be recommended to the deterministic model for the purpose of energy saving.
- The stochastic expected value programming model takes a good performance on energy conservation than the deterministic model with the expected resistance coefficient $c = 0.000032$. It follows from Table 2 that the stochastic model with $\alpha = 0.0022$ can further save energy by 18.71% in average.
- Compared with the deterministic model, the stochastic model is efficient on energy conservation. Take $\alpha = 0.0022$ for the stochastic model and $c = 0.000044$ for deterministic model, the energy consumptions are shown by Table 3. It is calculated that the stochastic model can reduce the energy consumption by 3.68% in average.

6. **Conclusions and Future Research.** In this paper, we proposed a stochastic expected value programming model for the optimal train control problem with uniformly distributed resistance coefficient. For solving the model, we first applied the Pontryagin

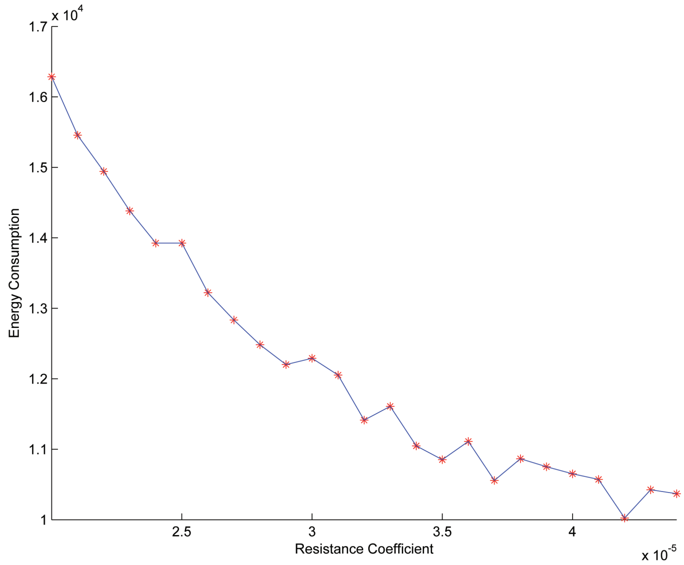

FIGURE 4. The relationship between the energy consumption and the parameter c for deterministic model

TABLE 2. A comparison between stochastic model and deterministic model with expected resistance coefficient

Trip time	610	630	700	800	900
Energy consumption for deterministic model	12439	11413	9021	7092	5918
Energy consumption for stochastic model	11314	10032	7377	5350	4165
Energy reduction	9.04%	12.10%	18.22%	24.56%	29.62%

TABLE 3. A comparison between stochastic model with $\alpha = 0.0022$ and deterministic model with $c = 0.000044$

Trip time	610	630	700	800	900
Energy consumption for deterministic model	11601	10369	7667	5579	4351
Energy consumption for stochastic model	11314	10032	7377	5350	4165
Energy reduction	2.47%	3.25%	4.30%	4.10%	4.27%

maximum principle to analyze the optimality conditions, which tells us that the optimal speed profile basically consists of four phases including maximum acceleration, cruising, quasicoasting and maximum braking. Furthermore, we designed a numerical algorithm to solve the optimal switching strategy. Compared with the deterministic model, we proved that the stochastic model averagely reduces energy consumption by 3.68% in terms of performing numerical example.

Although the stochastic model is proved to be advantageous than the deterministic model on numerical examples, we need to further conduct empirical studies for testing its effectiveness in real railway system. In this work, the track is assumed to be flat and no speed limits are considered. In spirit with the works of Liu and Golovitcher [20] and Howlett [12], we may further study the stochastic model with variable gradients and arbitrary speed limits.

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