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CONSTRAINED STOCHASTIC DISTRIBUTION CONTROL FOR NONLINEAR STOCHASTIC SYSTEMS WITH NON-GAUSSIAN NOISES

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Abstract. *In this paper, a stochastic distribution control (SDC) algorithm is presented for nonlinear and non-Gaussian stochastic systems with constraints on control inputs. A generalized entropy optimization criterion is constructed based on the probability density function (PDF) of the tracking error. An optimal control law is then obtained using the penalty function method. Stability analysis for this closed loop system is formulated. Finally, the comparative simulation results are presented to show that the proposed SDC algorithm is superior to PID controller. The contributions of the paper are threefold: 1) the principle of preservation of probability is introduced to deduce the PDF of tracking error under a relaxed assumption on the controlled systems; 2) the mathematical expectation of the squared error is included in the performance index to reduce the tracking error; 3) penalty function method is adopted to solve the SDC problem for nonlinear and non-Gaussian stochastic systems with constraints on control inputs.*

Keywords: Stochastic control, Constrained input, Minimum entropy, Penalty function method

1. **Introduction.** Due to the inevitable randomness in practical industrial processes, stochastic control has been playing important roles in control engineering practice. A number of approaches to controlling stochastic systems have been proposed [1-4]. The minimum variance control proposed in [1] could be one of the most effective methods in the early stage. In addition, other ideas such as predictive stochastic control [2], adaptive nonlinear stochastic control [3] and neural network based control [3] were introduced to deal with stochastic systems. Most of the existing approaches were established under the assumption that the disturbances are Gaussian. However, the noises in most practical systems are not necessarily Gaussian, and the nonlinearity of the systems may lead to non-Gaussian randomness even if the noises obey Gaussian distribution. Stochastic distribution control (SDC) strategies have been developed to deal with such practical control problems in [5-8], in which the controllers were designed so that the probability density function (PDF) of the system output follows a desired distribution as closely as possible. B-Spline neural networks were initially used to approximate the output PDF, and a number of SDC algorithms and practical applications were developed in [9,10]. However, the output PDFs are not measurable and cannot be approximated via B-Spline models in some cases. Some SDC methods based on entropy of the output tracking error were then proposed [6,8,11-13]. In [6], a new recursive optimization control algorithm was presented for general nonlinear and non-Gaussian stochastic systems described by ARMAX

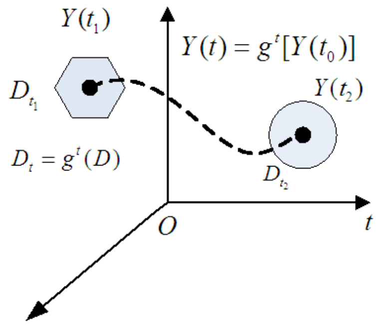


FIGURE 1. Random event description of a stochastic system

models. A novel run-to-run control methodology was presented for semiconductor processes with uncertain metrology delay by incorporating the minimum error entropy with numerical optimization technique [12]. Nevertheless, all the above-mentioned methods do not consider the constraints on control inputs.

Recently, Li and Chen studied the principle of preservation of probability shown as Figure 1, and a generalized density evolution equation was established from the view of the random event description [14,15]. Without loss of generality, let us consider the following transformation:

$$g^t : \ Y(t_0) \to Y(t)$$

As long as the random event does not disappear in the evolution process of the system, the probability of the random event will be invariant, i.e.,

$$\Pr(Y(t_1)) = \Pr(Y(t_2))$$

The principle of preservation can be expressed as [14]

$$\frac{D}{Dt}\int_{D_t}\gamma(y,t)dy = 0$$

where $D(\cdot)/Dt$ is the total derivative. And accordingly the evolution of the PDF of a random variable can be obtained based on the functional relationship among random variables.

In order to establish the performance index of a stochastic distribution control system, it is of great importance to formulate the PDF of the tracking error. Different from the existing formulations of the PDF in [6], this paper derives the PDF of tracking error according to the probability preservation principle. The PDF of tracking error can be obtained without the assumption on the monotonicity of the considered functional relationship. The mathematical expectation of the squared tracking error is included in the performance index so as to minimize the tracking error effectively. Due to the constraints on control inputs, the penalty function method is utilized to solve the SDC algorithm for nonlinear and non-Gaussian stochastic systems.

The remainder of the paper is organized as follows. The considered system model and error dynamics are presented in Section 2. In Section 3, the PDF of output tracking error is established. In Section 4, a novel performance index including entropy of tracking error, the mean value of squared tracking error and constraints on control input energy is introduced. The SDC algorithm is presented by minimizing the proposed performance index with constrained control inputs. The condition of stability is proposed for nonlinear and non-Gaussian stochastic systems. Comparative simulation results are provided in

Section 5 to illustrate the efficiency of the proposed method. Conclusions are drawn in Section 6.

2. **Plant Description.** Let us consider the following stochastic nonlinear systems [6]

$$y_k = f(y_{k-1}, \cdots, y_{k-n}, u_k, u_{k-1}, \cdots, u_{k-m}, \omega_k) \tag{1}$$

where y_k is the output, u_k the control input, ω_k the external disturbance and $f(\cdot)$ a nonlinear function. Constants n and m are the known orders of the system. Since the external disturbance ω_k is non-Gaussian, y_k determined by (1) is also a non-Gaussian random variable. In order to simplify the controller design procedures, the following assumptions which could be satisfied by many practical cases are required.

Assumption 2.1. $f(\cdot)$ *is bounded and has continuous partial derivative with respect to its arguments. At each sample instant, for a specific* τ *in the definition domain of* y_k*, there exists the solution* t *such that* $\tau = f(y_{k-1}, \cdots, y_{k-n}, u_k, u_{k-1}, \cdots, u_{k-m}, t)$ *and* $\frac{\partial f(\cdot)}{\partial t} \neq 0$ *holds for any* t *in their definition domain.*

Assumption 2.2. *External disturbance* ω_k *is a random variable with known PDF* $\gamma_{\omega_k}(\tau)$ *defined on a bounded interval* $[a, b]$.

Assume that the set point r_k is bounded as well, then, from the nonlinear stochastic system (1), the tracking error e_k can be given by

$$\begin{aligned} e_k = y_k - r_k &= f(y_{k-1}, \cdots, y_{k-n}, u_k, u_{k-1}, \cdots, u_{k-m}, \omega_k) - r_k \\ &= g(y_{k-1}, \cdots, y_{k-n}, u_k, u_{k-1}, \cdots, u_{k-m}, \omega_k, r_k) \\ &= g(\eta_k, u_k, \omega_k) \end{aligned} \tag{2}$$

where $\eta_k = (y_{k-1}, \cdots, y_{k-n}, u_{k-1}, \cdots, u_{k-m}, r_k)^T$, which is known at time k. From Assumption 2.1 and Equation (2), it can be seen that $g(\cdot)$ is also continuous, bounded and first order differentiable with respect to its variables and invertible with respect to ω_k.

The purpose of controller design is to utilize available information of the system input and output to minimize both magnitude and randomness of the non-Gaussian stochastic variable e_k. Therefore, it is necessary to investigate the PDF of tracking error.

3. **PDF of Tracking Error e_k.** Based on the probability theory, the PDF of tracking error e_k can be obtained from Equation (2) and the known PDFs of η_k and ω_k. It can be seen from Equation (2) that the random event $\{e_k < \tau\}$ is equivalent to $\{g(\eta_k, u_k, \omega_k) < \tau\}$. The PDF of tracking error can be obtained no matter whether the function $g(\cdot)$ is monotonic or not.

3.1. **$g(\cdot)$ is a monotonic function.** If $g(\cdot)$ is a monotonic increasing function, it leads to

$$\{g(\eta_k, u_k, \omega_k) < \tau\} = \{\omega_k < g^{-1}(\eta_k, u_k, \tau)\} \tag{3}$$

and

$$\Pr\{g(\eta_k, u_k, \omega_k) < \tau\} = \Pr\{\omega_k < g^{-1}(\eta_k, u_k, \tau)\} \tag{4}$$

It can be further obtained that

$$\int_{-\infty}^{\tau} \gamma_{e_k}(\eta_k, u_k, \tau) d\tau = \int_{-\infty}^{g^{-1}(\eta_k, u_k, \tau)} \gamma_\omega(t) dt \tag{5}$$

Differentiating Equation (5) on both sides with regard to τ yields

$$\gamma_{e_k}(\eta_k, u_k, \tau) = \bar{J}\, \gamma_{\omega_k}(g^{-1}(\eta_k, u_k, \tau)) \tag{6}$$

where $\bar{J} = dg^{-1}(\eta_k, u_k, \tau)/d\tau$.

If $g(\cdot)$ is a monotonic decreasing function, then

$$\gamma_{e_k}(\eta_k, u_k, \tau) = -\bar{J}\,\gamma_{\omega_k}(g^{-1}(\eta_k, u_k, \tau)) \tag{7}$$

Therefore, the following equality

$$\gamma_{e_k}(\eta_k, u_k, \tau) = \left|\bar{J}\right|\gamma_{\omega_k}(g^{-1}(\eta_k, u_k, \tau)) \tag{8}$$

holds for a monotonic function $g(\cdot)$.

3.2. **$g(\cdot)$ is a non-monotonic function.** If $g(\cdot)$ is a non-monotonic function, the PDF of the tracking error can be derived using the principle of preservation of probability [14,15]. Denote the j^{th} sectionally inverse function as g_j^{-1}, then

$$\gamma_{e_k}(\eta_k, u_k, \tau) = \sum_{j=1}^{\bar{m}} \left|\bar{J}_j\right|\gamma_{\omega_k}(g_j^{-1}(\eta_k, u_k, \tau)) \tag{9}$$

where $\bar{m}$ is the total number of the sectionally inverse functions of $g(\cdot)$.

It should be pointed out that the PDF of tracking error e_k can be obtained using the principle of preservation of probability without the assumption on the monotonic function $g(\cdot)$ in [6].

4. Control Strategy Design.

4.1. **Performance index.** The stochastic distribution controller designed in this paper aims to make the shape of the PDF of tracking error as narrow and sharp as possible, corresponding to a small entropy value [7]. Although the entropy can measure the dispersion of the tracking error, it does not change the mean of the distribution. Therefore, the mean value of the tracking error should be addressed. Moreover, the control energy should also be minimized. For this purpose, the following performance index was used in [6]

$$\begin{aligned} J(u_k) = &- R_1 \int \gamma_{e_k}(\eta_k, u_k, \tau)\ln\gamma_{e_k}(\eta_k, u_k, \tau)d\tau \\ &+ R_2 \int \tau\gamma_{e_k}(\eta_k, u_k, \tau)d\tau + R_3 u_k^2 \end{aligned} \tag{10}$$

where R_1, R_2 and R_3 are weights assigned for entropy, mean value and control input, respectively. In order to simplify its calculation, in this paper, Shannon entropy of the tracking error, $\int \gamma_{e_k}(\eta_k, u_k, \tau)\ln\gamma_{e_k}(\eta_k, u_k, \tau)d\tau$, is replaced by the quadratic Renyi entropy $H_2(e_k)$. In addition, the mean value of the squared tracking error $E(e_k^2)$ is included to minimize the tracking error. Consequently, the performance index is proposed as follows:

$$J(u_k) = R_1 H_2(e_k) + R_2 E(e_k^2) + \frac{1}{2}R_3 u_k^2 \tag{11}$$

where $H_2(e_k) = -\log\int_\alpha^\beta \gamma_{e_k}^2(\eta_k, u_k, \tau)d\tau = -\log V(e_k)$, $E(e_k^2) = \int_\alpha^\beta \tau^2\gamma_{e_k}(\eta_k, u_k, \tau)d\tau$. $V(e_k)$ is the quadratic information potential of the tracking error [16]. Since the quadratic Renyi entropy $H_2(e_k)$ is a monotonic decreasing function of the quadratic information potential $V(e_k)$, minimization of $H_2(e_k)$ is equivalent to the minimization of $-V(e_k)$. Therefore, the performance index can be rewritten as

$$\begin{aligned} J(u_k) &= -R_1 V(e_k) + R_2 E(e_k^2) + \frac{1}{2}R_3 u_k^2 \\ &= \int_\alpha^\beta (-R_1\gamma_{e_k}^2(\eta_k, u_k, \tau) + R_2\tau^2\gamma_{e_k}(\eta_k, u_k, \tau))d\tau + \frac{1}{2}R_3 u_k^2 \end{aligned} \tag{12}$$

4.2. **Constrained optimal controller.** Control inputs are always constrained by physical limitations of actuators in practice. Therefore, it is great significant to study SDC algorithm with following constraint on control input

$$|u_k| \le U_{\max} \tag{13}$$

where $U_{\max}$ is a known upper bound for u_k.

In order to obtain the optimal control input u_k^*, the following constrained optimization problem needs to be solved:

$$\begin{cases} \min J(u_k) \ = \int_\alpha^\beta (-R_1\gamma_{e_k}^2(\eta_k,u_k,\tau) + R_2\tau^2\gamma_{e_k}(\eta_k,u_k,\tau))d\tau + \frac{1}{2}R_3u_k^2 \\ \text{s.t. } U_{\max} - u_k \ge 0 \\ \qquad u_k + U_{\max} \ge 0 \end{cases} \tag{14}$$

which is a constrained nonlinear programming problem. The following objective function can be formulated using penalty function method

$$J_{new}(u_k, M) = J(u_k) + M[\min(0, U_{\max} - u_k)]^2 + M[\min(0, u_k + U_{\max})]^2 \tag{15}$$

where the penalty factor M is a large positive number. A recursive decreasing gradient method is employed to solve the optimal control input u_k^* according to following steps:

Step 1: Choose the initial control input u_0 and penalty factor M_1, denote $k := 1$, set the magnification factor $\beta > 1$ and accuracy $\varepsilon > 0$;

Step 2: Regard u_{k-1} as the initial point and denote $t := k-1$. Calculate the gradient for u_t, $\nabla J_{new}(u_t) = \frac{\partial J_{new}(u_t)}{\partial u_t}$. If $\nabla J_{new}(u_t) < \varepsilon$, stop and obtain the optimal solution $u_k^* = u_t$. Otherwise, turn to Step 3;

Step 3: Set the step length λ_t and calculate the control input as follows:

$$u_{t+1} = u_t - \lambda_t \nabla J_{new}(u_t) \tag{16}$$

Step 4: Increase t by 1 and return to Step 2. Eventually, the optimal solution u_k^* can be obtained for the unconstrained problem (15).

Step 5: If $M_k\left\{[\min(0, U_{\max} - u_k)]^2 + [\min(0, u_k + U_{\max})]^2\right\} < \varepsilon$, stop and obtain the optimal solution $u_k^* = u_k$. Otherwise, set $M_{k+1} = \beta M_k$, increase k by 1 and go back to Step 2.

4.3. **Stability analysis.** In order to guarantee the boundedness of the closed loop system, the stability of the closed loop system should be studied. A local linearization approach will be used to analyze the stability. For this purpose, (1) is linearized to read

$$\Delta y_k = \sum_{i=1}^{n} \frac{\partial f}{\partial y_{k-i}} \Delta y_{k-i} + \sum_{j=0}^{m} \frac{\partial f}{\partial u_{k-j}} \Delta u_{k-j} + \frac{\partial f}{\partial \omega_k}\Delta\omega_k \tag{17}$$

where $\Delta y_k = y_k - y_{k-1}$, $\Delta u_k = u_k - u_{k-1}$ and $\Delta\omega_k = \omega_k - \omega_{k-1}$. Using the unit backward shift operation z^{-1} to both sides of (17), it leads to

$$\left(1 - \sum_{i=1}^{n} \frac{\partial f}{\partial y_{k-i}} z^{-i}\right)\Delta y_k = \left(\sum_{j=0}^{m} \frac{\partial f}{\partial u_{k-j}} z^{-j}\right)\Delta u_k + \frac{\partial f}{\partial \omega_k}\Delta\omega_k \tag{18}$$

Denote $N(z^{-1},k) = 1 - \sum\limits_{i=1}^{n} \frac{\partial f}{\partial y_{k-i}} z^{-i}$ and $\xi_k = \left(\sum\limits_{j=0}^{m} \frac{\partial f}{\partial u_{k-j}} z^{-j}\right)\Delta u_k + \frac{\partial f}{\partial \omega_k}\Delta\omega_k$, then

$$N(z^{-1},k)\Delta y_k = \xi_k \tag{19}$$

It can be seen from (13) that

$$|\Delta u_k| \le 2U_{\max} \tag{20}$$

Moreover, $\Delta\omega_k$ is bounded according to the Assumption 2.2. Therefore, ξ_k is also bounded.

Let $N(z^{-1},k) = 1 - \sum_{i=1}^{n} \alpha_i(k)z^{-i}$ and $X(k) = [\Delta y_{k-n}\ \Delta y_{k-n+1}\ \cdots\ \Delta y_{k-1}]$, the following state-space representation of Δy_k can then be formulated as

$$X(k+1) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_n(k) & \alpha_{n-1}(k) & \cdots & \alpha_1(k) \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \xi_k \tag{21}$$

Denote

$$A(k) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_n(k) & \alpha_{n-1}(k) & \cdots & \alpha_1(k) \end{bmatrix} \tag{22}$$

If Δy_k is bounded, then the linearized closed-loop system is stable. Consequently, the closed-loop stability condition for the nonlinear and non-Gaussian stochastic system (1) with the constrained input (13) is formulated in the following theorem.

Theorem 4.1. *For a given positive constant $\varepsilon < 1$, the closed-loop stability condition of the system (1) with the constrained input (13) is $\|A_k\| < \varepsilon$.*

5. **A Numerical Example.** To illustrate the use of the proposed control algorithm, let us reconsider the following nonlinear system [6]:

$$y_k = \left(0.5 - 0.2\omega_k \left(1.2 + \tan^{-1} u_k\right)\right) y_{k-1} + 0.2u_k \tag{23}$$

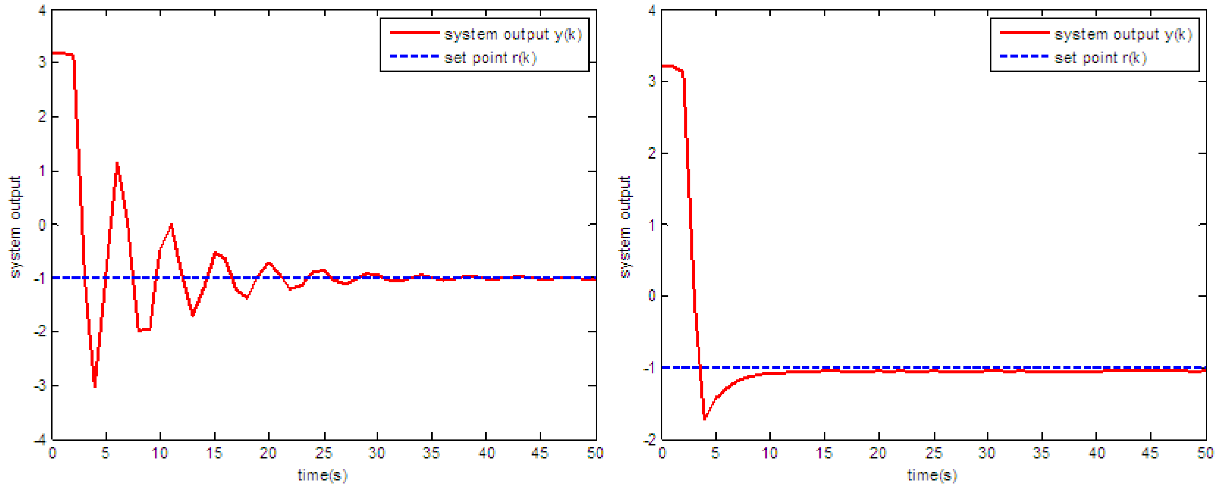
where the noise ω_k has the following PDF:

$$\gamma_\omega(x) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1 - 0.2x^2\right), & |x| \le \sqrt{5} \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

In this example, the bound of the control input is set as $U_{\max} = 10$. The initial values are $u_0 = 8$ and $y_0 = 3.2$. The set-point is set to $r_k = -1$. The weights assigned to the entropy, the square error and the control input are $R_1 = 0.98$, $R_2 = 0.01$ and $R_3 = 0.01$, respectively.

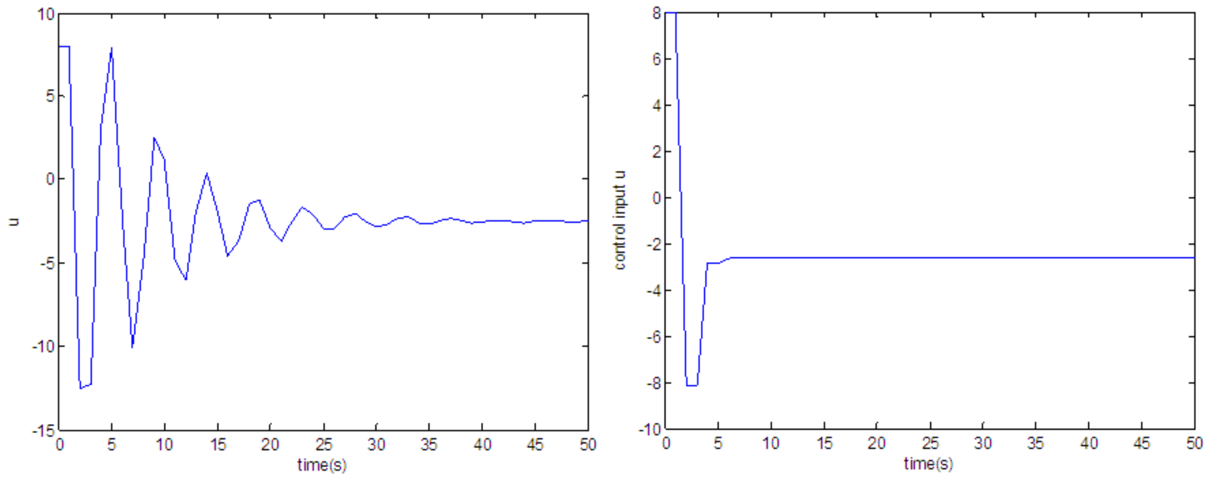
The advantage of the proposed method is shown by comparing with a PID controller whose transfer function is $G_{PID}(s) = k_p + k_i/s + k_d s$. The optimal PID parameters are tuned using the Matlab NCD toolbox: $k_p = 2.5$, $k_i = 1.2$ and $k_d = 1.2$. The comparative results are shown in Figures 2-6.

The responses of the closed loop system under PID control and the proposed SDC strategy are presented in Figure 2. The variations of the control input are shown in Figure 3. It can be seen from Figure 4 that the information potentials V_e of the tracking error under PID control and SDC both decrease along with time, while the performance indices J decrease with time. It is clear from Figure 5 that the shapes of PDFs of the tracking error become narrow and sharp, which can also be verified by the PDFs at typical instants in Figure 6. Both PID controller and the proposed SDC law can drive the system towards a smaller randomness direction, but it is obvious from Figure 5 and Figure 6 that the proposed SDC algorithm has smaller settling time than PID controller. Figure 3(a) shows the control input produced by the PID controller violates its lower limit, which may damage actuator in practice. Therefore, the proposed SDC strategy is more suitable for nonlinear and non-Gaussian systems with constraints on control inputs.



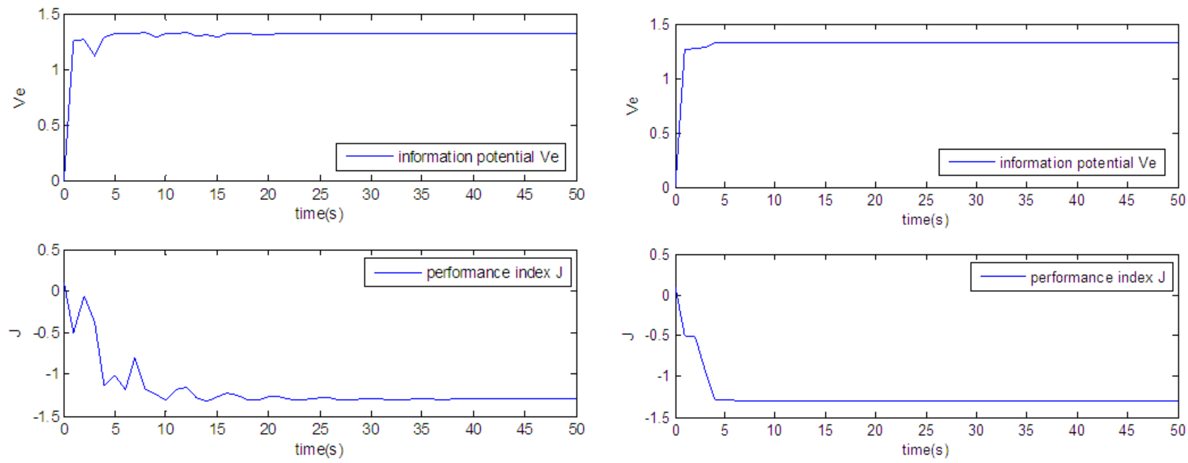
(a) PID (b) SDC

FIGURE 2. Response of system output



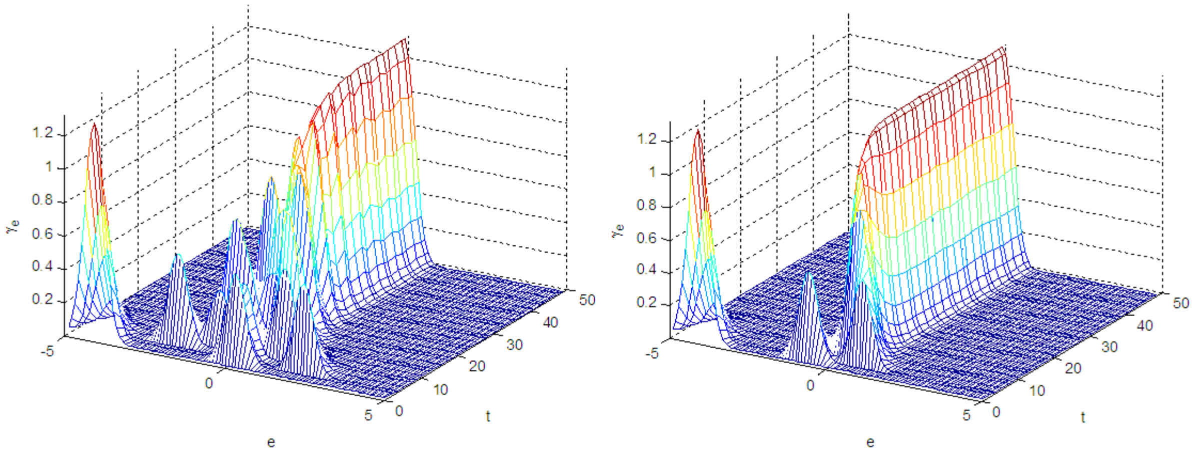
(a) PID (b) SDC

FIGURE 3. Variations of control input



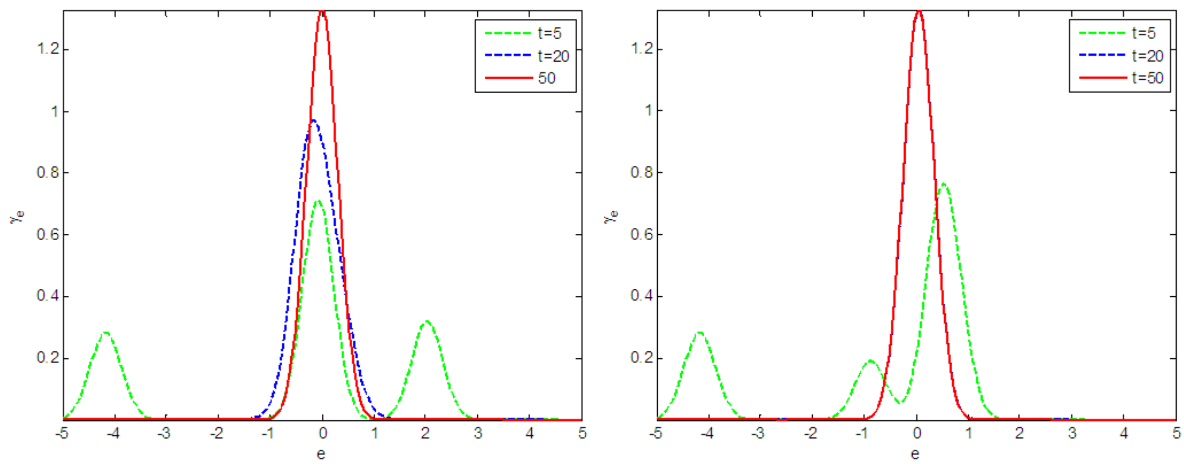
(a) PID (b) SDC

FIGURE 4. Information potentials and performance indices



(a) PID (b) SDC

Figure 5. PDFs of tracking error



(a) PID (b) SDC

Figure 6. PDFs at typical instants

6. **Conclusions.** This paper has presented a new SDC algorithm for nonlinear and non-Gaussian systems with constraints on control inputs. The PDF of tracking error is formulated using the principle of preservation of probability. An improved performance index of closed-loop system is proposed, which includes the information potential of tracking error, mathematic expectation of squared tracking error and constraints on control input. The optimal control algorithm is obtained using penalty function method. Moreover, the local stability condition is derived. The presented SDC strategy and PID control method are both applied to an illustrative nonlinear and non-Gaussian stochastic system with constraint on control input, and the comparative simulation results verify the effectiveness of the given control algorithm.

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