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STOCHASTIC MODELING AND TRACKING CONTROL FOR A TWO-LINK PLANAR RIGID ROBOT MANIPULATOR

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Abstract. *In this paper, the problem of stochastic modeling and tracking control of a two-link planar rigid robot manipulator is considered. A stochastic Lagrangian model is constructed to describe the motion of the manipulator in random vibration environment. Based on the constructed model, a state feedback controller is designed such that the error system is 4-th moment exponentially practically stable. The simulation result demonstrates the efficiency of the proposed scheme.*

Keywords: Two-link planar rigid manipulator, Stochastic Lagrangian model, Tracking control

1. **Introduction.** In recent several decades, the robot control and its application are very popular research topics in control field. It is well known that robot manipulators are often subjected to random vibration, which affects the performance of manipulators and even damages the structure. In this paper, we consider a well-used robotic system, the two-link planar rigid manipulator, in the random vibration environment as shown in Figure 1, which is connected to O on the floor by a freely moveable joint. For $i = 1, 2$, m_i denotes the mass of i-th link, l_i denotes the length of i-th link, l_{ci} denotes the distance from the previous joint to the center of mass of i-th link, q_i denotes the i-th joint angle and u_i denotes the torque with respect to i-th joint, whose units are kg, m, rad and N·m, respectively. As in Section 8.2.4 of [1], let ξ_1, ξ_2 denote the acceleration of the point O in horizontal and vertical directions, which can be seen as independent white noises.

The reasonable dynamic modeling is a necessary premise to design efficient control strategies. For the deterministic case, i.e., $\xi_i = 0$, various modeling methods can be found in many monographes such as [2-4]. A common dynamic model is derived from the general Lagrangian equation method to describe the relationship between force and motion, i.e.,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + h(q) = G(q)u, \tag{1}$$

where $q = (q_1, q_2)^T$, $u = (u_1, u_2)^T$, the symmetric and positive definite matrix $M(q) \in \mathbb{R}^{2\times 2}$ is the inertia matrix (generalized mass), $C(q, \dot{q})\dot{q}$ is the vector of centrifugal and Coriolis force, $h(q) = \frac{\partial P}{\partial q}$ with P is the potential energy of the system, and $G(q)u$ is the generalized force caused by the control u. However, for the stochastic case, the deterministic model (1) cannot well describe the motion of the manipulator. One objective

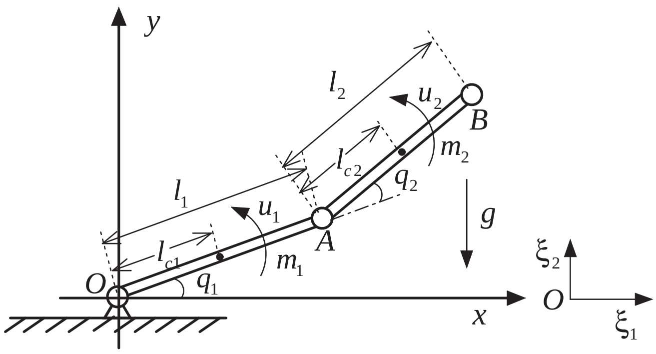

FIGURE 1. Two-link planar rigid manipulator subjected to random vibration

of this paper is to construct a reasonable model to describe the motion of the manipulator in the random vibration environment.

A basic problem in controlling robots [5-8] is to make the manipulator follow a desired trajectory $q_r(t) \in \mathcal{C}^2(\mathbb{R}^2)$, i.e., design a controller u such that the tracking error

$$e_1(t) = q(t) - q_r(t) \tag{2}$$

tends to zero. For the deterministic case, based on Lagrangian model (1), great efforts have been made in developing desirable schemes to achieve the tracking control objectives. In many references such as [9-11], some effective control strategies such as PID control, computed torque control, robust control, adaptive control were considered. However, for the stochastic case, to the best of the authors' knowledge, there is no result. Hence, another objective of this paper is to design a tracking controller for the manipulator in the random vibration environment based on the constructed model.

The main work consists of the following two aspects:

(i) Since the manipulator is subjected to random vibration, the main difficulty for stochastic dynamic modeling is how to transform the random vibration in environment to the mass points along the links between them. By using the Lagrangian mechanics and the equivalence principle of mechanics, a stochastic model is established to describe the motion of the two-link planar rigid robot manipulator in random vibration environment.

(ii) Based on the constructed model, a state feedback controller is designed such that the tracking error system is 4-th moment exponentially practically stable, the mean square of tracking error tends to an arbitrarily small neighborhood of zero by tuning design parameters, and so does its derivative. The simulation result demonstrates the efficiency of the proposed scheme.

This paper is organized as follows: The mathematical preliminaries are given in Section 2. Section 3 constructs a stochastic model. In Section 4, controller design and stability analysis are addressed, following a simulation result in Section 5. The paper is concluded in Section 6.

Notations: The following notations are used throughout the paper. For a vector x, x^T denotes its transpose and $|x|$ denotes its usual Euclidean norm. For a matrix X, X^{-1} denotes its inverse and $\|X\|_F$ denotes its Frobenius norm which is defined by $\|X\|_F = (\mathrm{Tr}\{XX^T\})^{1/2}$, where $\mathrm{Tr}(\cdot)$ denotes the square matrix trace; $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n \times r$ matrix space; $\mathcal{C}^i(\mathbb{R}^n)$ denotes the set of all functions with continuous i-th partial derivative on $\mathbb{R}^n$, $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t and $\mathcal{K}$ denotes the set of all functions:

$\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero. For simplicity, sometimes the arguments of functions are omitted.

2. **Mathematical Preliminaries.**

Consider the following stochastic nonlinear system

$$dx(t) = f(x(t), t)dt + g(x(t), t)dW(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state of system, $W(t)$ is an r-dimensional independent standard Wiener process (or Brownian motion), and the underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Both functions $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times r}$ are locally Lipschitz in $x \in \mathbb{R}^n$ and piecewise continuous in t, namely, for any $R > 0$, there exists a constant $C_R \geq 0$ such that

$$|f(x_1, t) - f(x_2, t)| + \|g(x_1, t) - g(x_2, t)\|_F \leq C_R|x_1 - x_2|, \quad \forall t \in \mathbb{R}_+, \tag{4}$$

for any $x_1, x_2 \in U_R = \{\xi : |\xi| \leq R\}$. Moreover, $f(0, t)$ and $g(0, t)$ are bounded a.s..

For $V(x, t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$, the infinitesimal generator is defined by

$$\mathcal{L}V(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \tfrac{1}{2}\mathrm{Tr}\left\{g^T(x, t)V_{xx}(x, t)g(x, t)\right\}, \tag{5}$$

where $V_t = \frac{\partial V}{\partial t}, V_x = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right)$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}$.

For stability analysis, in view of the concept of p-th moment exponential stability in [12,13], p-th moment exponential practical stability is reasonably introduced in the following.

Definition 2.1. *The system* (3) *is said to be p-th moment exponentially practically stable if there exist positive constants λ, d and function $\kappa \in \mathcal{K}$ such that*

$$E|x(t)|^p \leq \kappa(|x_0|)e^{-\lambda(t-t_0)} + d, \ t \geq t_0, \ x_0 \in \mathbb{R}^n. \tag{6}$$

When $p = 2$, we also say that it is exponentially practically stable in mean square.

The criterion for p-th moment exponential practical stability is given as follows.

Lemma 2.1. *For system* (3)*, assume that there exist a function $V(x, t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$, positive constants k_i, k_i', p_i, p_i', c and d_c such that*

$$\textstyle\sum_{i=1}^n k_i|x_i|^{p_i} \leq V(x, t) \leq \sum_{i=1}^n k_i'|x_i|^{p_i'}, \tag{7}$$

$$\mathcal{L}V(x, t) \leq -cV(x, t) + d_c. \tag{8}$$

Then, there exists a unique strong solution $x(t) = x(t; x_0, t_0)$ of system (3) *for each $x(t_0) = x_0 \in \mathbb{R}^n$ and system* (3) *is p-th moment exponentially practically stable where $p = \min\{p_1, \cdots, p_n\}$.*

Proof: For any $k > 0$, define the first exit time $\eta_k = \inf\{t : t \geq t_0, |x(t)| > k\}$ with the special case $\inf \emptyset = \infty$, and set $t_k = \eta_k \wedge t$ for any $t \geq t_0$. According to Theorem 4.1 of [14], there exists a unique strong solution $x(t) = x(t; x_0, t_0)$ of system (3) for each $x(t_0) = x_0 \in \mathbb{R}^n$, that is, $\eta_k \to \infty$ a.s. as $k \to \infty$.

In view of (7), the infinitesimal generator of the function $e^{ct}V(x, t)$ is

$$\mathcal{L}(e^{ct}V(x, t)) = e^{ct}(\mathcal{L}V(x, t) + cV(x, t)) \leq e^{ct}d_c. \tag{9}$$

Hence, by Lemma 3.3.1 in [12], for t_k, we have

$$E(e^{ct_k}V(x(t_k), t_k)) \leq e^{ct_0}V(x_0, t_0) + E\textstyle\int_{t_0}^{t_k} e^{cs}d_c ds. \tag{10}$$

Since $t_k \leq t$, according to the dominated convergence theorem, letting $k \to \infty$, one gets

$$EV(x(t), t) \leq V(x_0, t_0)e^{-c(t-t_0)} + \frac{d_c}{c}, \quad \forall\ t \geq t_0. \tag{11}$$

From (7), for $i = 1, 2, \cdots, n$, this estimate yields

$$E|x_i(t)|^{p_i} \le \frac{1}{k_i} \sum_{j=1}^{n} k'_j |x_0|^{p'_j} e^{-c(t-t_0)} + \frac{d_c}{k_i c}. \tag{12}$$

According to Jensen's inequality[1], from $p_i \ge p$, one has $E|x_i(t)|^p \le (E(|x_i(t)|^p)^{\frac{p_i}{p}})^{\frac{p}{p_i}}$, which, together with (12) and c_r inequality[2], implies that

$$E|x_i(t)|^p \le \left(\tfrac{1}{k_i}\textstyle\sum_{j=1}^{n} k'_j |x_0|^{p'_j}\right)^{\frac{p}{p_i}} e^{-\frac{cp}{p_i}(t-t_0)} + \left(\tfrac{d_c}{k_i c}\right)^{\frac{p}{p_i}}, \tag{13}$$

then

$$\begin{aligned} E|x(t)|^p \le & n^{\frac{p}{2}} \sum_{i=1}^{n} E|x_i(t)|^p \\ \le & n^{\frac{p}{2}} \sum_{i=1}^{n} \left(\frac{1}{k_i} \sum_{j=1}^{n} k'_j |x_0|^{p'_j}\right)^{\frac{p}{p_i}} e^{-\frac{cp}{p_i}(t-t_0)} + n^{\frac{p}{2}} \sum_{i=1}^{n} \left(\frac{d_c}{k_i c}\right)^{\frac{p}{p_i}} \\ \le & \kappa(|x_0|) e^{-\lambda(t-t_0)} + d, \end{aligned} \tag{14}$$

where $\kappa(s) = n^{\frac{p}{2}} \sum_{i=1}^{n} \left(\frac{1}{k_i} \sum_{j=1}^{n} k'_j s^{p'_j}\right)^{\frac{p}{p_i}}$, $\lambda = \frac{cp}{\max_i\{p_i\}}$ and $d = n^{\frac{p}{2}} \sum_{i=1}^{n} \left(\frac{d_c}{k_i c}\right)^{\frac{p}{p_i}}$, that is, system (3) is p-th moment exponentially practically stable.

3. Stochastic Dynamic Modeling for the Manipulator.

In this section, the stochastic model of the manipulator is established by deriving the kinetic and potential energy of the manipulator and then using Lagrangian equation method [3] and the equivalence principle of mechanics [15].

3.1. Kinetic and potential energy.

Consider the system of particles consisting of two links which are regarded as the mass points and select (q_1, q_2) as the generalized coordinate. The base coordinate frame Oxy is shown in Figure 1, then the natural coordinates of particles are

$$\begin{cases} x_1 = l_{c1}\cos q_1 \\ y_1 = l_{c1}\sin q_1, \end{cases} \quad \begin{cases} x_2 = l_1\cos q_1 + l_{c2}\cos(q_1+q_2) \\ y_2 = l_1\sin q_1 + l_{c2}\sin(q_1+q_2). \end{cases} \tag{15}$$

From (15), the total kinetic energy of the system is $K = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where the inertia matrix (generalized mass) $M(q) \in \mathbb{R}^{2\times 2}$ is

$$M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + m_2 l_{c2}^2 + 2m_2 l_1 l_{c2}\cos q_2 & m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos q_2 \\ m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos q_2 & m_2 l_{c2}^2 \end{bmatrix}. \tag{16}$$

Since the (k, j)-th element of the matrix $C(q, \dot{q})$ is defined as $c_{kj} = \sum_{i=1}^{2} c_{ijk}(q)\dot{q}_i$ with the Christoffel symbols $c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)$ (see P.207 of [2]), then

$$C(q, \dot{q}) = \begin{bmatrix} -m_2 l_1 l_{c2}\dot{q}_2 \sin q_2 & -m_2 l_1 l_{c2}\dot{q}_1 \sin q_2 - m_2 l_1 l_{c2}\dot{q}_2 \sin q_2 \\ m_2 l_1 l_{c2}\dot{q}_1 \sin q_2 & 0 \end{bmatrix}. \tag{17}$$

The total potential energy of the system equals to $P = m_1 g y_1 + m_2 g y_2 = (m_1 l_{c1} + m_2 l_1) g \sin q_1 + m_2 g l_{c2} \sin(q_1 + q_2)$, then

$$h(q) = \frac{\partial P}{\partial q} = \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \cos q_1 + m_2 l_{c2} g \cos(q_1+q_2) \\ m_2 l_{c2} g \cos(q_1+q_2) \end{bmatrix}. \tag{18}$$

[1] If φ is a convex function and x is a random variable on Ω, then $\varphi(Ex) \le E\varphi(x)$.

[2] $|a+b|^r \le c_r(|a|^r + |b|^r)$, $r > 0$, where $c_r = \begin{cases} 1, & r \le 1 \\ 2^{r-1}, & r \ge 1 \end{cases}$.

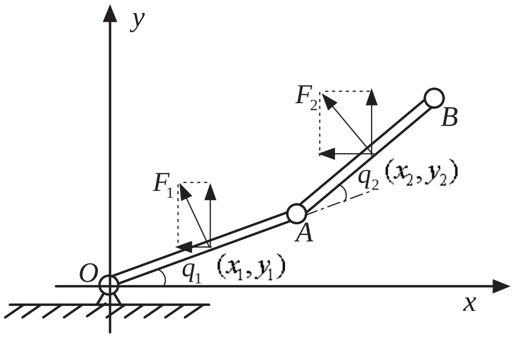

FIGURE 2. The decomposition of forces

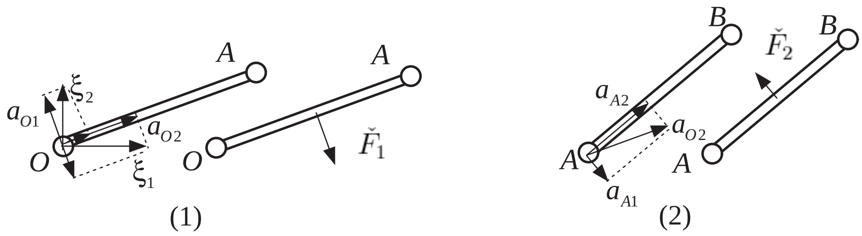

FIGURE 3. The stochastic force

3.2. **The generalized forces.** In order to obtain the control force τ^c, it is supposed that the control u_i, $i = 1, 2$, is the torque F_i acting on the barycenter of i-th link, i.e., $u_i = F_i l_{ci}$, as shown in Figure 2. Based on the generalized forces formula[3] (see P.41 in [16]), the generalized forces corresponding to the control u_1 and u_2 are

$$\begin{aligned}
\tau_1^c =& -F_1 \sin q_1 \frac{\partial x_1}{\partial q_1} + F_1 \cos q_1 \frac{\partial y_1}{\partial q_1} - F_2 \sin(q_1+q_2) \frac{\partial x_2}{\partial q_1} + F_2 \cos(q_1+q_2) \frac{\partial y_2}{\partial q_1} \\
=& F_1 l_{c1} + F_2 l_1 \cos q_2 + F_2 l_{c2} \\
=& u_1 + u_2 \left(1 + \frac{l_1}{l_{c2}} \cos q_2\right), \\
\tau_2^c =& -F_1 \sin q_1 \frac{\partial x_1}{\partial q_2} + F_1 \cos q_1 \frac{\partial y_1}{\partial q_2} - F_2 \sin(q_1+q_2) \frac{\partial x_2}{\partial q_2} + F_2 \cos(q_1+q_2) \frac{\partial y_2}{\partial q_2} \\
=& F_2 l_{c2} = u_2,
\end{aligned} \tag{19}$$

which means that the vector of generalized control forces $\tau^c = G(q)u$ with

$$G(q) = \begin{bmatrix} 1 & 1 + \frac{l_1}{l_{c2}} \cos q_2 \\ 0 & 1 \end{bmatrix}. \tag{20}$$

In order to obtain the random excitation force τ^e, O is regarded as the reference point. As shown in Figure 3(1), decomposing ξ_1 and ξ_2 at the point O, we have

$$\begin{aligned} a_{O1} &= -\xi_1 \sin q_1 + \xi_2 \cos q_1, \\ a_{O2} &= \xi_1 \cos q_1 + \xi_2 \sin q_1. \end{aligned} \tag{21}$$

[3]The generalized force along the j-th generalized coordinates is defined by $Q_j = \sum_{i=1}^N (F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j})$, $j = 1, 2, \cdots, n$, where (F_{ix}, F_{iy}, F_{iz}) is the force acted on the i-th point. The transformation from the natural coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$ to the generalized coordinates $\{q_1, q_2, \cdots, q_n\}$ is represented as $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n, t)$.

Next, decomposing a_{O2} at the point A (see Figure 3(2)), one has

$$\begin{aligned} a_{A1} &= -(\xi_1 \cos q_1 + \xi_2 \sin q_1) \sin q_2, \\ a_{A2} &= (\xi_1 \cos q_1 + \xi_2 \sin q_1) \cos q_2. \end{aligned} \tag{22}$$

According to the principle of relative motion [17], the stochastic force $\check{F}_i$ in the direction of F_i is introduced such that $\check{F}_i/m_i$ describes the stochastic motion of i-th particle, i.e.,

$$\begin{aligned} \check{F}_1 &= -m_1 a_{O1} = m_1 \sin q_1 \xi_1 - m_1 \cos q_1 \xi_2, \\ \check{F}_2 &= -m_2 a_{A1} = m_2 \cos q_1 \sin q_2 \xi_1 + m_2 \sin q_1 \sin q_2 \xi_2. \end{aligned} \tag{23}$$

Based on the generalized forces formula, following the same line as (19), the generalized stochastic forces are

$$\begin{aligned} \tau_1^e &= \Lambda_{11}(q)\xi_1 + \Lambda_{12}(q)\xi_2, \\ \tau_2^e &= \Lambda_{21}(q)\xi_1 + \Lambda_{22}(q)\xi_2, \end{aligned} \tag{24}$$

with $\Lambda_{11}(q) = m_1 l_{c1} \sin q_1 + m_2 l_{c2} \cos q_1 \sin q_2 + \frac{1}{2} m_2 l_1 \cos q_1 \sin 2q_2$, $\Lambda_{12}(q) = -m_1 l_{c1} \cos q_1 + m_2 l_{c2} \sin q_1 \sin q_2 + \frac{1}{2} m_2 l_1 \sin q_1 \sin 2q_2$, $\Lambda_{21}(q) = m_2 l_{c2} \cos q_1 \sin q_2$ and $\Lambda_{22}(q) = m_2 l_{c2} \sin q_1 \times \sin q_2$. This means that the vector of random excitation force $\tau^e = (\tau_1^e, \tau_2^e)^T = \Lambda(q)\xi$ with $\Lambda(q) = (\Lambda_{ij}(q))_{2\times 2}$ and $\xi = (\xi_1, \xi_2)^T$.

The generalized force τ consists of the control force $\tau^c = G(q)u$ caused by the control u acting on the system and the random excitation force $\tau^e = \Lambda(q)\xi$ caused by the independent white noise ξ.

3.3. **Stochastic model of the manipulator.** On the basis of the above two subsections, the stochastic dynamic equation of the manipulator is obtained

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + h(q) = G(q)u + \Lambda(q)\xi. \tag{25}$$

By replacing ξ with "$\frac{dB}{dt}$" and viewing $(q^T, \dot{q}^T)^T$ as state, the Stratonovich stochastic differential equation (SDE) of (25) can be obtained

$$\begin{aligned} dq &= \dot{q}dt, \\ d\dot{q} &= \big(-M^{-1}(q)(C(q, \dot{q})\dot{q} + h(q)) + M^{-1}(q)G(q)u\big)\, dt \\ &\quad + M^{-1}(q)\Lambda(q) \circ dB, \end{aligned} \tag{26}$$

where B is an r-dimensional independent Wiener process and

$$M^{-1}(q) = \frac{1}{m_1 m_2 l_{c1}^2 l_{c2}^2 + m_2^2 l_1^2 l_{c2}^2 \sin^2 q_2} \times \begin{bmatrix} m_2 l_{c2}^2 & -(m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos q_2) \\ -(m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos q_2) & m_1 l_{c1}^2 + m_2 l_1^2 + m_2 l_{c2}^2 + 2 m_2 l_1 l_{c2} \cos q_2 \end{bmatrix}.$$

Since the diffusion function of $\dot{q}$ subsystem does not depend on $\dot{q}$, the Wong-Zakai correction term (see (6.1.3) in [18]) equals to zero, i.e.,

$$\frac{1}{2} \begin{pmatrix} \Delta_1 \frac{\partial \Delta_1}{\partial q} + \Delta_2 \frac{\partial \Delta_1}{\partial \dot{q}} \\ \Delta_1 \frac{\partial \Delta_2}{\partial q} + \Delta_2 \frac{\partial \Delta_2}{\partial \dot{q}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{27}$$

where $\Delta_1 = 0$ and $\Delta_2 = M^{-1}(q)\Lambda(q)$. The corresponding Itô SDE is

$$\begin{aligned} dq &= \dot{q}dt, \\ d\dot{q} &= \big(-M^{-1}(q)(C(q, \dot{q})\dot{q} + h(q)) + M^{-1}(q)G(q)u\big)\, dt \\ &\quad + M^{-1}(q)\Lambda(q)dB, \end{aligned} \tag{28}$$

i.e., the final stochastic dynamic model of the manipulator is constructed.

Remark 3.1. *By reasonably introducing the random noise ξ to describe the random vibration, and using the Lagrangian mechanics and the equivalence principle of mechanics, the stochastic model* (28) *is established to describe the motion of the two-link planar manipulator in random vibration environment.*

4. Tracking Control via State Feedback.

The configuration q and its velocity $\dot{q}$ can be measured by adding some measuring equipments to the manipulator. It is easy to verify that the functions $M(q)$, $C(q, \dot{q})$, $h(q)$, $G(q)$ and $\Lambda(q)$ are smooth, and $G(q)$ is invertible.

4.1. Tracking error system.

Before the tracking controller is designed, the tracking error system will be developed. Similar to [19], we define the filtered tracking error as

$$e_2(t) = \dot{e}_1(t) + c_1 e_1(t) = \dot{q}(t) - \dot{q}_r(t) + c_1 e_1(t), \tag{29}$$

which is measurable due to the measurability of q and $\dot{q}$, where $c_1 > 0$ is a design parameter. In view of (28), one has

$$\begin{aligned} de_1 &= (-c_1 e_1 + e_2)dt, \\ de_2 &= \big(- M^{-1}(q)(C(q, \dot{q})\dot{q} + h(q)) - \ddot{q}_r - c_1^2 e_1 + c_1 e_2 \\ &\qquad + M^{-1}(q)G(q)u\big)dt + M^{-1}(q)\Lambda(q)dB. \end{aligned} \tag{30}$$

Suppose that the power spectral density (PSD) of white noise ξ equals to $\frac{1}{2\pi}\Sigma$, which is equivalent to the fact $dB = \Sigma dW$, where $\Sigma = (r_{ij})_{2\times 2}$ is a positive matrix and W is a 2-dimensional independent standard Wiener process. Then (30) can be rewritten as

$$\begin{aligned} de_1 &= (-c_1 e_1 + e_2)dt, \\ de_2 &= \big(- M^{-1}(q)(C(q, \dot{q})\dot{q} + h(q)) - \ddot{q}_r - c_1^2 e_1 + c_1 e_2 \\ &\qquad + M^{-1}(q)G(q)u\big)dt + M^{-1}(q)\Lambda(q)\Sigma dW. \end{aligned} \tag{31}$$

Here, the underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions.

4.2. Tracking controller design.

For the Lyapunov function

$$V = \frac{1}{4}(e_1^T e_1)^2 + \frac{1}{4}(e_2^T e_2)^2, \tag{32}$$

the infinitesimal generator of V along (31) satisfies

$$\begin{aligned} \mathcal{L}V = & - c_1 e_1^T e_1 e_1^T e_1 + e_1^T e_1 e_1^T e_2 + e_2^T e_2 e_2^T \big(\psi(q, \dot{q}, \dot{q}_r, \ddot{q}_r) + M^{-1}(q)G(q)u\big) \\ & + \frac{1}{2}\mathrm{Tr}\left\{\Sigma^T \Lambda^T(q) M^{-1}(q)(2e_2 e_2^T + e_2^T e_2 I)M^{-1}(q)\Lambda(q)\Sigma\right\}, \end{aligned} \tag{33}$$

where $\psi(q, \dot{q}, \dot{q}_r, \ddot{q}_r) = -M^{-1}(q)(C(q, \dot{q})\dot{q} + h(q)) - \ddot{q}_r - c_1^2 e_1 + c_1 e_2$. Applying Young's inequality[4] to the second term in (33), one has

$$e_1^T e_1 e_1^T e_2 \leq \frac{c_1}{4}(e_1^T e_1)^2 + \frac{27}{4c_1^3}(e_2^T e_2)^2. \tag{34}$$

[4]For any vectors $x, y \in \mathbb{R}^n$ and any scalars $\epsilon > 0$, $p > 1$, there holds $x^T y \leq \frac{\epsilon^p}{p}|x|^p + \frac{1}{q\epsilon^q}|y|^q$, where $q = \frac{p}{p-1}$.

From the definition of $\Lambda(q)$, it is learned that

$$\begin{aligned}(\Lambda_{11}(q)-\Lambda_{11}(q_r))^2 =&\Big(m_1l_{c1}(\sin q_1-\sin q_{r1})+m_2l_{c2}(\cos q_1-\cos q_{r1})\sin q_2\\&+m_2l_{c2}\cos q_{r1}(\sin q_2-\sin q_{r2})\\&+\frac{1}{2}m_2l_1(\cos q_1-\cos q_{r1})\sin 2q_2\\&+\frac{1}{2}m_2l_1\cos q_{r1}(\sin 2q_2-\sin 2q_{r2})\Big)^2\\ \le&\Big(3m_1^2l_{c1}^2+6m_2^2l_{c2}^2(\sin^2 q_2+\cos^2 q_{r1})\\&+\frac{3}{2}m_2^2l_1^2(\sin^2 2q_2+4\cos^2 q_{r1})\Big)|q-q_r|^2.\end{aligned}\tag{35}$$

Similarly, one has

$$\begin{aligned}(\Lambda_{12}(q)-\Lambda_{12}(q_r))^2 \le&\Big(3m_1^2l_{c1}^2+6m_2^2l_{c2}^2(\cos^2 q_2+\sin^2 q_{r1})\\&+\frac{3}{2}m_2^2l_1^2(\cos^2 2q_2+4\sin^2 q_{r1})\Big)|q-q_r|^2,\\(\Lambda_{21}(q)-\Lambda_{21}(q_r))^2 \le& 2m_2^2l_{c2}^2(\sin^2 q_2+\cos^2 q_{r1})|q-q_r|^2,\\(\Lambda_{22}(q)-\Lambda_{22}(q_r))^2 \le& 2m_2^2l_{c2}^2(\cos^2 q_2+\sin^2 q_{r1})|q-q_r|^2.\end{aligned}\tag{36}$$

By the definition of the Frobenius norm, it follows from (35) and (36) that

$$\|\Lambda(q)-\Lambda(q_r)\|_F^2=\sum_{i=1}^{2}\sum_{j=1}^{2}(\Lambda_{ij}(q)-\Lambda_{ij}(q_r))^2\le\delta(q,q_r)|q-q_r|^2,\tag{37}$$

where $\delta(q,q_r)=6m_1^2l_{c1}^2+16m_2^2l_{c2}^2+\frac{15}{2}m_2^2l_1^2$. Then, the Frobenius norm of $\Lambda(q)$ satisfies

$$\|\Lambda(q)\|_F^2\le 2\|\Lambda(q)-\Lambda(q_r)\|_F^2+2\|\Lambda(q_r)\|_F^2\le 2\delta(q,q_r)e_1^2+2\|\Lambda(q_r)\|_F^2,\tag{38}$$

with $\|\Lambda(q_r)\|_F^2=m_1^2l_{c1}^2+2m_2^2l_{c2}^2\sin^2 q_{r2}+\frac{1}{4}m_2^2l_1^2\sin^2 2q_{r2}+m_2^2l_1l_{c2}\sin q_{r2}\sin 2q_{r2}$. According to the definition of Frobenius norm, the norm compatibility and Young's inequality, by (38), the last term of (33) yields

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\left\{\Sigma^T\Lambda^T(q)M^{-1}(q)(2e_2e_2^T+e_2^Te_2I)M^{-1}(q)\Lambda(q)\Sigma\right\}\\&\le 3|e_2|^2(\delta(q,q_r)|e_1|^2+\|\Lambda(q_r)\|_F^2)\|M^{-1}(q)\|_F^2\|\Sigma\|_F^2\\&\le\frac{c_1}{4}(e_1^Te_1)^2+\left(\frac{9}{c_1}\delta^2(q,q_r)+\frac{9}{\epsilon}\|\Lambda(q_r)\|_F^4\right)\|M^{-1}(q)\|_F^4\|\Sigma\|_F^4(e_2^Te_2)^2+\frac{\epsilon}{4},\end{aligned}\tag{39}$$

where $\|M^{-1}(q)\|_F^2=\frac{1}{(m_1m_2l_{c1}^2l_{c2}^2+m_2^2l_1^2l_{c2}^2\sin^2 q_2)^2}\big(m_2^2l_{c2}^4+2((m_2l_{c2}^2+m_2l_1l_{c2}\cos q_2))^2+(m_1l_{c1}^2+m_2l_1^2+m_2l_{c2}^2+2m_2l_1l_{c2}\cos q_2)^2\big)$, $\|\Sigma\|_F^2=r_{11}^2+r_{12}^2+r_{21}^2+r_{22}^2$ and $\epsilon>0$ is a design parameter. Substituting (34) and (39) into (33), one leads to

$$\begin{aligned}\mathcal{L}V\le&-\frac{c_1}{2}(e_1^Te_1)^2+e_2^Te_2e_2^T\Bigg(\frac{27}{4c_1^3}e_2+\psi(q,\dot q,\dot q_r,\ddot q_r)+M^{-1}(q)G(q)u\\&+\left(\frac{9}{c_1}\delta^2(q,q_r)+\frac{9}{\epsilon}\|\Lambda(q_r)\|_F^4\right)\|M^{-1}(q)\|_F^4\|\Sigma\|_F^4e_2\Bigg)+\frac{\epsilon}{4}.\end{aligned}\tag{40}$$

Since $G(q)$ is a nonsingular square matrix, the control u is designed as

$$\begin{aligned} u = &G^{-1}(q)M(q)\bigg(-\frac{c_2}{2}e_2 - \frac{27}{4c_1^3}e_2 - \psi(q,\dot{q},\dot{q}_r,\ddot{q}_r) \\ &- \left(\frac{9}{c_1}\delta^2(q,q_r) + \frac{9}{\epsilon}\|\Lambda(q_r)\|_F^4\right)\|M^{-1}(q)\|_F^4\|\Sigma\|_F^4 e_2\bigg) \end{aligned} \tag{41}$$

where $c_2 > 0$ is a design parameter and

$$G^{-1}(q) = \begin{bmatrix} 1 & -1-\frac{l_1}{l_{c2}}\cos q_2 \\ 0 & 1 \end{bmatrix}. \tag{42}$$

Since q and $\dot{q}$ are measurable signals and q_r is the reference signal, the control u can be implemented. Therefore, the infinitesimal generator of V satisfies

$$\mathcal{L}V \le -\frac{c_1}{2}(e_1^T e_1)^2 - \frac{c_2}{2}(e_2^T e_2)^2 + \frac{\epsilon}{4} \le -cV + \frac{\epsilon}{4}, \tag{43}$$

where $c = 2\min\{c_1, c_2\}$. Just for simplicity, c_1 and c_2 are selected as scalars. In fact, they can be chosen as any positive definite constant matrix.

Up to now, from (31) and (41), the following error system is got:

$$\begin{aligned} de_1 =& (-c_1 e_1 + e_2)dt, \\ de_2 =& -\left(\frac{c_2}{2} + \frac{27}{4c_1^3} + \left(\frac{9}{c_1}\delta^2(q,q_r) + \frac{9}{\epsilon}\|\Lambda(q_r)\|_F^4\right)\|M^{-1}(q)\|_F^4\|\Sigma\|_F^4\right)e_2 dt \\ &+ M^{-1}(q)\Lambda(q)\Sigma dW, \end{aligned} \tag{44}$$

based on which, stability analysis will be given.

4.3. Stability analysis.

Theorem 4.1. *For the stochastic model* (25) *and the reference signal q_r, under the state-feedback controller* (41),

(*i*) *Error system* (44) *has a unique strong solution on* $[t_0, \infty)$ *and* 4*-th moment exponentially practically stable for initial values* $e_1(t_0), e_2(t_0) \in \mathbb{R}^2$*;*

(*ii*) *The tracking error* $e_1(t)$ *and* $\dot{e}_1(t)$ *satisfy*

$$\begin{aligned} &\lim_{t\to\infty} E|e_1(t)|^2 \le \left(\tfrac{2\epsilon}{c}\right)^{\frac{1}{2}}, \\ &\lim_{t\to\infty} E|\dot{e}_1(t)|^2 \le 2(1+c_1^2)\left(\tfrac{2\epsilon}{c}\right)^{\frac{1}{2}}, \end{aligned} \tag{45}$$

where the right-hand side can be made small enough by choosing appropriate design parameters.

Proof: Denoting $e(t) = (e_1^T(t), e_2^T(t))^T$, by (32), it is obvious that $V(e) \in \mathcal{C}^2(\mathbb{R}^4)$ and

$$\frac{1}{8}|e(t)|^4 \le V(e) \le \frac{1}{4}|e(t)|^4. \tag{46}$$

Since the functions of the error system (44) satisfy the local Lipschitz condition, by Lemma 2.1, (43) and (46), there exists a unique strong solution of system (44) for each $e(t_0) \in \mathbb{R}^4$. System (44) is 4-th moment exponentially practically stable, and

$$E|e(t)|^4 \le 2e^{-c(t-t_0)}|e(t_0)|^4 + \tfrac{2\epsilon}{c}. \tag{47}$$

In view of $|\dot{e}_1(t)|^2 = (|e_2(t)| + c_1|e_1(t)|)^2 \le 2|e_2(t)|^2 + 2c_1^2|e_1(t)|^2$, (45) holds from (47). Considering $c = 2\min\{c_1, c_2\}$, it is clear that the right-hand side of (45) can be made small enough by choosing c_1, c_2 large enough and ϵ small enough.

Remark 4.1. *By Chebyshev's inequality and* (45)*, for any* $\varepsilon > 0$ *and* $\varepsilon_0 > 0$*, there exists* $T > 0$ *such that when* $t > T$

$$\begin{aligned} P\{|e_1(t)| > \varepsilon\} &\le \tfrac{1}{\varepsilon^2}\left(\varepsilon_0 + \left(\tfrac{2\epsilon}{c}\right)^{\frac{1}{2}}\right) \le \varepsilon', \\ P\{|\dot{e}_1(t)| > \varepsilon\} &\le \tfrac{1}{\varepsilon^2}\left(\varepsilon_0 + 2(1+c_1^2)\left(\tfrac{2\epsilon}{c}\right)^{\frac{1}{2}}\right) \le \varepsilon', \end{aligned} \tag{48}$$

where ε' *can be made small enough by tuning design parameters. At the expense of large control effort, the asymptotic tracking in probability in some sense can be achieved. Therefore, the effect between the tracking error and allowable control effort must be carefully compromised in the tracking controller design.*

Remark 4.2. *Due to the diffusion term and the nonvanishing signal* q_r*, it is difficult to achieve the asymptotic tracking. The parameter* ϵ *is introduced to deal with the term caused by nonvanishing signal* q_r *in Hessian term. When* $q_r = 0$*,* ϵ *can be avoided, then the stabilization of* q *and* $\dot{q}$ *can be achieved.*

5. **Simulation.** Choose the reference signal $q_r(t) = (0.5 + 0.5\sin t, 0.5\cos t)^T$, whose unit is rad. The state feedback tracking control law is given by (41).

The simulation is performed under the following conditions: the PSD of the white noise $\Sigma = (r_{ij})_{2\times 2}$ with $r_{ii} = 0.1$ and $r_{ij} = 0$ $(i \ne j)$; the system parameters $m_1 = 0.5$kg, $m_2 = 0.25$kg, $l_{c1} = 0.8$m, $l_{c2} = 1$m, $l_1 = 1$m, $l_2 = 1.5$m and $g = 9.8\text{m/s}^2$; the initial values $q_1(0) = 0.4$rad, $q_2(0) = 0.85$rad, $\dot{q}_1(0) = 0$rad/s and $\dot{q}_2(0) = 0$rad/s; the design parameters $c_1 = 8$, $c_2 = 5$ and $\epsilon = 0.2$. The simulation result demonstrates the effectiveness of the control scheme; see Figure 4.

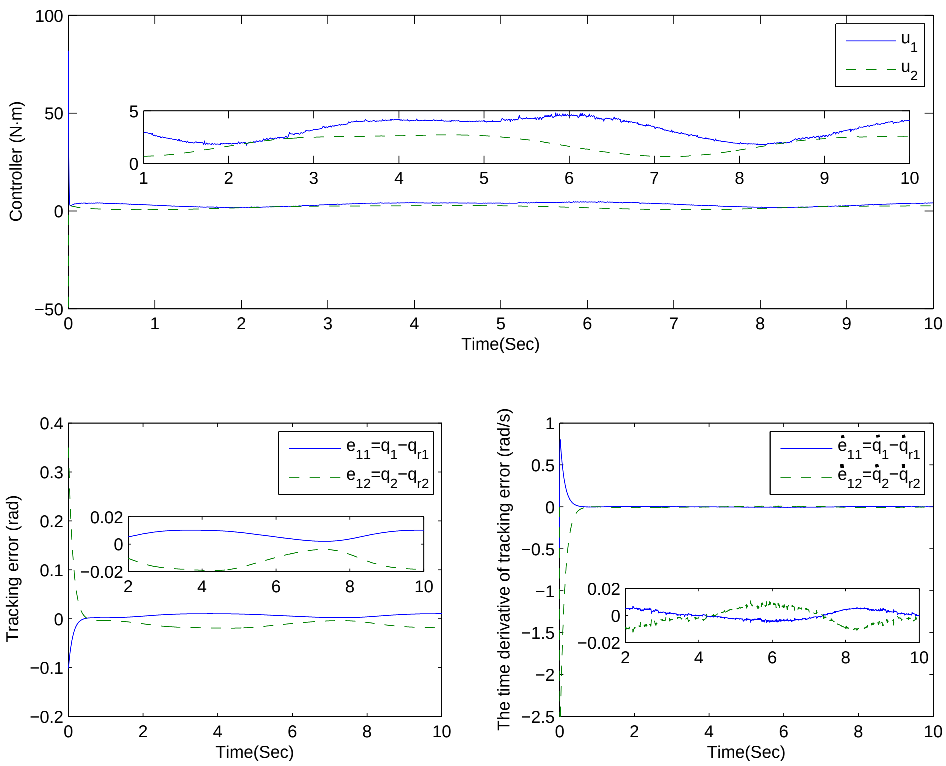

FIGURE 4. The responses of closed-loop system

6. **Conclusions.** For the two-link planar rigid manipulator in the random vibration environment, by analyzing the effect of the random vibration, a stochastic model is reasonably constructed. Based on the model, a state feedback controller is designed such that the mean square of tracking error tends to an arbitrarily small neighborhood of zero.

There are some remaining problems to be investigated: 1) Find other control methods such as [20-23] to solve this problem and compare their effectiveness. 2) Consider the output feedback control, adaptive tracking control and robust control by extending the deterministic control methods ([24,25], etc.) to the stochastic case.

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