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A STABLE SELF-ORGANIZING FUZZY PD CONTROL FOR ROBOT MANIPULATORS

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Abstract. *In this paper we propose a Self-Organizing Fuzzy Proportional Derivative (SOF-PD) tracking controller for robot manipulators, which exploits the simplicity and robustness of the simple PD control and enhances its benefits. This proposed controller has a gain-scheduling structure, in which, based on the position error, a SOF system performs the gains tuning of a simple PD controller in the feedback loop. The SOF system is a fuzzy system in which the inference rules are continuously updated according to two performance index tables designed for adjusting – in a separated way – the P and D gains. The tuning of the gains is performed independently for each joint. By using the Lyapunov theory, it is shown that, for an arbitrary bounded desired trajectory, uniform ultimate boundedness of the tracking errors is guaranteed by selecting suitable maximum and minimum allowed PD gain values. This stability result can be generalized to others varying gains schemes for PD tracking control, in which the gains are bounded functions of the tracking errors. Experimental results in a two degrees of freedom robotic arm show the superiority of the proposed approach over the classic PD control, in terms of the position errors.*

Keywords: Self-organizing fuzzy control, PD control, Robot control, Uniform ultimate boundedness

1. **Introduction.** The Proportional Derivative (PD) control is one of the most widely employed controllers for robot manipulators [1], probably due to its simplicity and ease of implementation, since it only depends on two gains, and the fact that it is not a requisite to know the dynamic model of the robot. However, when the control reference is continually and rapidly changing, as in trajectory tracking control of some robotics applications, or when there are dynamic parameters variations, a PD controller with constant gains may not produce adequate results. In order to improve the performance of this controller, several self-tuning mechanisms have been proposed. One of the most successfully employed is the fuzzy self-tuning scheme, which has been applied to the tuning of PD controllers plus dynamic model compensation, as well as other conventional controllers for robot control [2, 3, 4], process control [5, 6, 7], maritime vehicles [8] and other plants [9, 10, 11].

The fuzzy logic controllers can be roughly classified in direct action controllers, if the fuzzy inference system is placed in the forward trajectory of the control loop, or gain-scheduling controllers, if the fuzzy inference system computes the gains of the controller in the forward trajectory of the control loop, this is, in a supervisory level [2]. In a fuzzy gain-scheduling PD controller, the PD gains are continuously computed by a fuzzy inference system, based on the error or the performance evaluation, which are processed according to predefined fuzzy inference rules. An important concern in this design is the formulation of the rule base. Conventionally, the process of formulation of the rule base requires an expert knowledge of the plant input-output operation. This dependence may represent a drawback if the expert knowledge of the plant is partial or is not available. Moreover, in general the formulated rule base cannot be modified on-line, which could limit the fuzzy controller ability to respond to parameter variation or uncertainty. One of the proposed approaches in the literature to cope with these issues is the Self-Organizing Fuzzy (SOF) controller, proposed by Procyk and Mamdani [12]. In this controller, a set of initial inference rules is first formulated, then a learning mechanism is started to evaluate the performance of each rule applied and to generate on-line corrective actions for the rule, if needed. Since its early applications, several modifications have been proposed to improve some features of the SOF controller, such as improving the simplicity and computing efficiency [13, 14], improving the learning strategy by using a performance index table [14, 15], or using the closed-loop errors directly to modify the linguistic fuzzy rule table [16, 17]. In [18] the output of a reference model is used as a reference for rule modification, while in [19] an estimation model is used to calculate the correction for each fuzzy rule.

Recent applications of the SOF controller in several nonlinear plants include applications to control of robot manipulators [20, 21], active suspension systems [17], and biomedical processes [22]. In these works many improvements have been added to the SOF controller, such as automatic selection of parameters and optimization of the rules of the performance index table. Since some parameters of the SOF controller, e.g., the scaling factors, weight distribution or learning rate, are usually hard to adjust by the trial and error method, there are some alternatives proposed in the literature to achieve this task. One of these approaches is proposed in [23], where the learning rate and the weighting distribution are appropriately determined by two additional fuzzy logic controllers. In [20, 21] a self-organizing learning mechanism is employed to modify the rules of a fuzzy sliding-mode controller, and the scaling factors are adjusted by the so-called model-matching technique. In [22] a new SOF architecture is proposed, with the inclusion of a genetic algorithm for the optimization of the rules of the performance index table, which is dynamic instead of fixed, resulting in robustness with respect to the selection of scaling factors and parameters variation.

The employment of the SOF controller in a gain-scheduling scheme is proposed in [24] as a PID controller with gains tuned by a SOF algorithm (SOF-PID controller). In this controller, the P gain is directly tuned by the SOF algorithm, while the I and D gains are computed based in the Ziegler Nichols method for PID tuning. Good results in set-point control of a 2 d.o.f. robot were obtained. This scheme was later applied to trajectory tracking control of robots [25]. Subsequently, the SOF algorithm was proposed to directly tune all the gains of the PID controller [26, 27]. However, in [24, 25, 26] the performance index table is the same for the SOF tuners for each of the PID gains. In [27], two performance index tables are employed, one for the P gain and the other for the D gain, and it is proposed a third table for the I gain; however, the table for the D gain needs the last computed P gain correction as an input, which results in the dependence of the D gain on the P gain.

On the other hand, an important feature of any control system is its stability. The stability analysis of SOF controllers has been addressed in several ways. One of these approaches, previously proposed in [28] but extended to MIMO systems, is employed in the study of a SOF control for robots [29]. In this approach, the relative stability and robustness of the equilibrium point at the origin is investigated by employing two stability indices. In [20, 21], as well as in [30], in which the SOF controllers are used in hybrid schemes with sliding-mode controllers, the stability of the system is proven using the Lyapunov theory. In [2, 31] the studied controllers are of the gain-scheduling fuzzy type, and the stability analysis is addressed by analyzing the Lyapunov-based stability of the controllers in the feedback loop. The stability of a simple tracking PD control with constant gains has been analyzed in [32], where the uniform ultimate boundedness of the solutions of the closed-loop system is proved, but equal gains are considered for all the joints, which can be impractical when working with real robots. In [33] the exponential convergence of the tracking errors to closed spheres is demonstrated, while in [34] a PD-type output feedback controller is studied, resulting in the boundedness of the tracking errors.

As it was mentioned above, the PD control is one of the most employed controllers for robot manipulators. Its parameters are only the proportional and derivative gains, and in its simplest form, it does not need to compute any dynamic model compensation. These excellent features motivate us to enhance even more its benefits. The addition of a SOF controller in a gain-scheduling scheme would give the PD controller better capabilities to track fast motion trajectories and to be more robust to the variations of the model parameters. The contribution of this work is threefold: 1) to develop a SOF controller for robot manipulators based on the PD structure, but now with variable gains depending on the position errors, which takes advantage of the simplicity and robustness features of such a controller, but improving significantly its performance; 2) to prove, by using the Lyapunov theory, that the tracking errors obtained with this class of gain-scheduling controllers are uniformly ultimately bounded; and 3) to validate these improvements through real-time experiments.

In this paper, based on the commented previous results, we propose a SOF-PD controller, in which the P and D gains are tuned by a SOF algorithm. One SOF controller is employed to tune each single gain, resulting in an independent adjustment of these gains for the n control loops. Moreover, since the P and D gains adjustment should follow different criteria, we propose to employ two different performance index tables for P and D gains, unlike [24, 25, 26]. Since it is based on a PD structure, the proposed approach has the advantages of its simplicity and its good performance in robotic applications, over the hybrid SOF schemes [20, 21, 23, 30], which have a more complex structure. The stability analysis of the closed-loop system, an issue not addressed before for SOF gain-scheduling controllers [24, 25, 26, 27], is carried out by applying the theory of perturbed systems. As a result, it is proven that, under an appropriate selection of the bounds of the PD variable gains, the tracking errors are uniformly ultimately bounded, rather than only exponential convergence to closed spheres with radiuses depending on the PD gains as in [33]. To the best knowledge of the authors, the presented results have not been reported before. Furthermore, this is a general result which can be applied to other gain-scheduling PD tracking controllers, regardless of the tuning mechanism, provided that the gains are bounded. A procedure to select lower and upper bounds on the variable gains that meet the error boundedness criterion is proposed. Finally, the better performance of this controller over a classic fixed gains PD controller is shown by performing real-time experiments on a 2 d.o.f. robot arm.

The rest of the paper is structured as follows. In Section 2, the SOF-PD controller is described. Next, in Section 3 the procedure to prove the uniform ultimate boundedness of the solutions of the closed-loop system is developed, and the tuning procedure suggested for the bounds of the variable gains is presented. Later, implementation of real-time experiments is described and their results are shown in Section 4. Finally, some conclusions are given in Section 5.

In this work the following notation is used: $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and the maximum eigenvalue of a symmetric positive definite matrix A. The norm of the vector $\boldsymbol{y}$ is defined as $\|\boldsymbol{y}\| = \sqrt{\boldsymbol{y}^T\boldsymbol{y}}$. Vectors are represented with small bold letters.

2. **SOF-PD Controller.** The simplified block diagram of the SOF-PD controller is shown in Figure 1, where $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are the $n \times 1$ joint position and velocity vectors, respectively; $\boldsymbol{q}_d$ and $\dot{\boldsymbol{q}}_d$ are the desired positions and velocities, K_p and K_v are the $n \times n$ diagonal gain matrices, and $\boldsymbol{\tau}$ is the $n \times 1$ applied torque vector. The PD control is in the forward trajectory of the control loop and the SOF mechanism is at the supervisory level. The SOF block comprises the following stages: Input Section, Self-Organizing Fuzzy Tuners and Output Section.

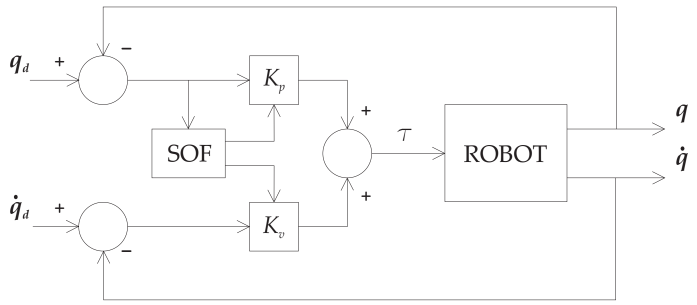

FIGURE 1. SOF-PD controller

2.1. **Input section.** This section is designed to perform the scaling and discretization of the input signals, which are the position error and the change in the position error. The objective of these operations is to normalize the input signals to levels that can be used in the next stages. The block diagram of this section is shown in Figure 2, where $\boldsymbol{e}(t)$ represents the position error vector $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q}$ as a function of time t, and $\Delta\boldsymbol{e}(t)$ is the change in the position error, which is computed as the difference between the error in the current time period and the error in the past time period: $\Delta\boldsymbol{e}(t) = \boldsymbol{e}(t) - \boldsymbol{e}(t-T)$, with T as the sampling period. A single scaling factor is employed for scaling $\boldsymbol{e}(t)$, while two scaling factors are employed for scaling $\Delta\boldsymbol{e}(t)$. For both error and change in error, two different quantizations are performed: Q_I and Q_F. Q_I produces one value from the discrete set $S_I = \{1, 2, 3, 4\}$, for each continuous input value. These values represents the following linguistic values assigned to the current position error and change in the position error: Z, S, M and B, that mean Zero, Small, Medium and Big, respectively. Q_F produces one value from the discrete set $S_F = \{-6.0, -5.5, \ldots, 0, \ldots, 5.5, 6.0\}$, for each continuous input value. For the scaled position error, the outputs of the quantization blocks are $\boldsymbol{e}_Q = [\boldsymbol{e}_{QI} \;\; \boldsymbol{e}_{QF}]$, and for the scaled change in the position error, the outputs are $\Delta\boldsymbol{e}_Q = [\Delta\boldsymbol{e}_{QI} \;\; \Delta\boldsymbol{e}_{QF}]$.

The current PD gains are also scaled and quantized. One scaling factor is used for each gain. Quantization Q_F is also used here, producing one value from the set S_F, named k_Q, for each continuous scaled gain value.

2.2. **Self-organizing fuzzy tuners.** In order to tune the PD gains, a tuner for each of these gains is implemented, e.g., if we are controlling the joint position of a 2 d.o.f. robot, four gain tuners are implemented: two for the P gains for both joints, and two for the D gains. In Figure 3 a block diagram of a self-organizing fuzzy tuner is shown. The Performance Index Table is addressed by $\boldsymbol{e}_{QI}$ and $\Delta\boldsymbol{e}_{QI}$ from the Input Section. The value produced in this block is then added to the output of the Past States Buffer, in

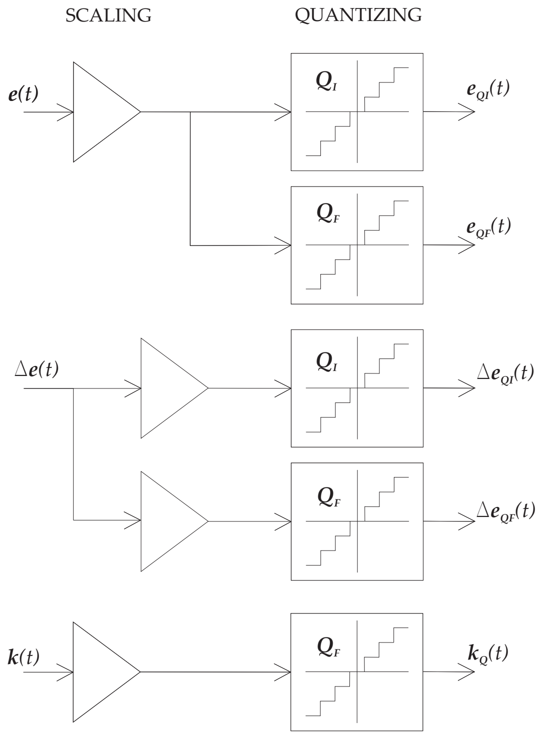

Figure 2. SOF input section

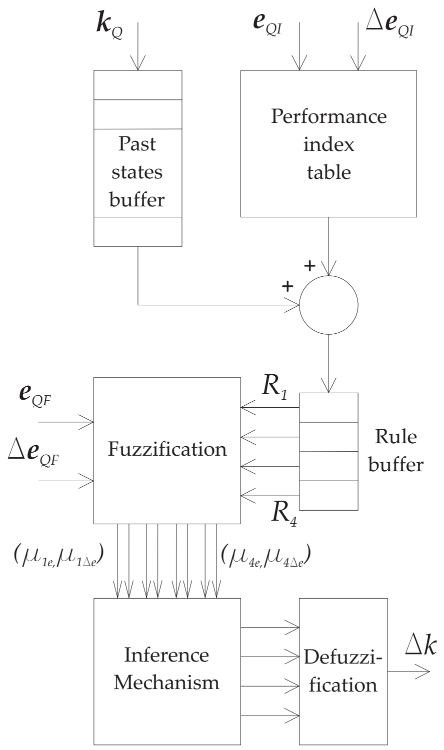

Figure 3. Self-organizing fuzzy tuner

which several past values of the discretized gain k_Q are buffered. The resulting sum is considered a rule, and it is then passed to the Rule Buffer, where four consecutive values or rules, R_1, R_2, R_3 and R_4, are stored. In a strict sense, the so-called four rules are only a single rule computed at four different instants of time.

In the Fuzzification block, four triangular membership functions with constant width are used. The centers of these functions take the values of R_1, R_2, R_3 and R_4; the fuzzy membership grades corresponding to the variables $\boldsymbol{e}_{QF}$ and $\Delta\boldsymbol{e}_{QF}$ are obtained from each one of these functions. These grades are named $\{\mu_{1e}, \mu_{2e}, \mu_{3e}, \mu_{4e}\}$ and $\{\mu_{1\Delta e}, \mu_{2\Delta e}, \mu_{3\Delta e}, \mu_{4\Delta e}\}$, corresponding to $\boldsymbol{e}_{QF}$ and $\Delta\boldsymbol{e}_{QF}$, respectively (see Figure 3).

In the Inference Mechanism block, the min inference on each pair of the obtained membership values from each membership function is performed, that is to say, the output of this block is comprised of the result of the min inference on each pair of values μ_{ie} and $\mu_{i\Delta e}$:

$$G = \{\min(\mu_{1e}, \mu_{1\Delta e}), \min(\mu_{2e}, \mu_{2\Delta e}), \min(\mu_{3e}, \mu_{3\Delta e}), \min(\mu_{4e}, \mu_{4\Delta e})\}\,.$$

The defuzzification is carried out by employing the Mean of Maxima method [35]. This algorithm was chosen due to its simplicity and speed of computation. The result Δk of this algorithm is:

$$\Delta k = \frac{G_u R_u + G_{u-1} R_{u-1}}{2},$$

where G_u and G_{u-1} are the greater two values of G, and R_u and R_{u-1} are the respective rules from which G_u and G_{u-1} are obtained.

2.2.1. *Performance index tables.* We propose to employ two distinct fuzzy rule bases, which are subject to be modified by two different sets of performance index values, one for proportional gains tuning, and the other for derivative gains tuning. The purpose of using these two tables is that the P and D gains can be tuned in an independent way. These performance index values are shown in Tables 1 and 2, respectively, and were selected according to the following criteria. Concerning to the P gain, if $\boldsymbol{e}_{QI}$ or $\Delta\boldsymbol{e}_{QI}$ are both zero, the rule does not need to be changed, or if $\boldsymbol{e}_{QI}$ is big and $\Delta\boldsymbol{e}_{QI}$ is small, we can infer that the current position is approaching to the desired position, and the rules does not need to be changed, thus the performance index table output is zero. If $\boldsymbol{e}_{QI}$ is small, medium or big, the rule needs to be modified either upwards or downwards, so that an appropriate value can be produced by the algorithm to modify the current P gain. Concerning to the D gain, if $\boldsymbol{e}_{QI}$ or $\Delta\boldsymbol{e}_{QI}$ are zero, the rule does not need to be corrected, or if $\boldsymbol{e}_{QI}$ is big, the D gain is not going to contribute in reducing this error. If $\boldsymbol{e}_{QI}$ is small or medium, the rule is modified either upwards or downwards so the algorithm can produce a value to modify the current D gain.

2.3. **Output section.** In this section, updated values for PD gains are obtained. The output of the defuzzification process is multiplied by a reverse scaling (or descaling) factor, and the result is added to the past value of the respective PD gain:

$$k(t) = k(t - T) + \Delta k * d_k, \tag{1}$$

where d_k is the reverse scaling factor, $k(t - T)$ is the past value of the gain and $k(t)$ is the new value. The factor d_k is implemented as a means to keep the gain increment or decrement within a predetermined range.

Table 1. Performance index for P gains

$\boldsymbol{e}_{QI}\backslash\Delta\boldsymbol{e}_{QI}$	Z	S	M	B
Z	0	0	0	0
S	0	–2	–2	–2
M	0	4	4	4
B	0	0	2	2

TABLE 2. Performance index for D gains

$\boldsymbol{e}_{QI} \backslash \Delta \boldsymbol{e}_{QI}$	Z	S	M	B
Z	0	0	0	0
S	0	–2	–2	–2
M	0	2	2	2
B	0	0	0	0

3. **Stability Analysis.** We now present the stability analysis based on the Lyapunov theory. For this purpose, the SOF-PD controller can be simply considered as a PD control with variable gains which are functions of the position error. In this section is shown that, in order to assure the uniform ultimate boundedness of the solutions of the closed-loop system, these functions must be upper and lower bounded functions. We begin this analysis by introducing the dynamic model of the robot.

3.1. **Robot dynamics.** The model of a revolute joints rigid robot, in absence of friction, is [1, 36]

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{2}$$

where $\ddot{\boldsymbol{q}}$ is the $n \times 1$ joint acceleration vector, $M(\boldsymbol{q})$ is the $n \times n$ symmetric positive definite inertia matrix, $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the $n \times n$ Coriolis and centripetal forces matrix, and $\boldsymbol{g}(\boldsymbol{q})$ is the $n \times 1$ gravitational forces vector. Next, some definitions and properties, which are useful for this analysis, are presented.

Definition 3.1. *(Based on [2]). The gain matricial functions $K_p(\tilde{\boldsymbol{q}}) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and $K_v(\tilde{\boldsymbol{q}}) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are defined as follows:*

$$K_p(\tilde{\boldsymbol{q}}) = \begin{bmatrix} k_{p_1}(\tilde{q}_1) & 0 & \cdots & 0 \\ 0 & k_{p_2}(\tilde{q}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{p_n}(\tilde{q}_n) \end{bmatrix},$$

$$K_v(\tilde{\boldsymbol{q}}) = \begin{bmatrix} k_{v_1}(\tilde{q}_1) & 0 & \cdots & 0 \\ 0 & k_{v_2}(\tilde{q}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{v_n}(\tilde{q}_n) \end{bmatrix}.$$

Assumption 3.1. *There exist positive constants k_{pl}, k_{pu}, k_{vl} and k_{vu}, where $k_{pu} > k_{pl} > 0$ and $k_{vu} > k_{vl} > 0$, such that $k_{pu} \geq k_{p_i}(\tilde{q}_i) \geq k_{pl}$ and $k_{vu} \geq k_{v_i}(\tilde{q}_i) \geq k_{vl}$, for all $\tilde{q}_i \in \mathbb{R}$ and $i = 1, \cdots, n$, where $k_{p_i}(\tilde{q}_i)$ and $k_{v_i}(\tilde{q}_i)$ are the i-th elements of the matrices $K_p(\tilde{\boldsymbol{q}})$ and $K_v(\tilde{\boldsymbol{q}})$ in Definition 3.1.*

Property 3.1. *(This property has been reported in [2], inspired in [37]). Constants k_{pl}, k_{pu}, k_{vl} and k_{vu} are the upper and lower bounds of certain continuously integrable functions, as follows:*

$$\frac{1}{2}k_{pl}\|\tilde{\boldsymbol{q}}\|^2 \leq \int_0^{\tilde{\boldsymbol{q}}} \boldsymbol{\sigma}^T K_p(\boldsymbol{\sigma}) d\boldsymbol{\sigma} \leq \frac{1}{2}k_{pu}\|\tilde{\boldsymbol{q}}\|^2,$$

$$\frac{1}{2}k_{vl}\|\tilde{\boldsymbol{q}}\|^2 \leq \int_0^{\tilde{\boldsymbol{q}}} \boldsymbol{\sigma}^T K_v(\boldsymbol{\sigma}) d\boldsymbol{\sigma} \leq \frac{1}{2}k_{vu}\|\tilde{\boldsymbol{q}}\|^2.$$

Property 3.2. *The minimum and maximum eigenvalues of* $K_p(\tilde{\boldsymbol{q}})$ *and of* $K_v(\tilde{\boldsymbol{q}})$ *are bounded as follows:*

$$k_{pl} \le \lambda_{\min}\{K_p(\tilde{\boldsymbol{q}})\} \le \lambda_{\max}\{K_p(\tilde{\boldsymbol{q}})\} \le k_{pu}, \quad \forall\, \tilde{\boldsymbol{q}} \in \mathbb{R}^n,$$
$$k_{vl} \le \lambda_{\min}\{K_v(\tilde{\boldsymbol{q}})\} \le \lambda_{\max}\{K_v(\tilde{\boldsymbol{q}})\} \le k_{vu}, \quad \forall\, \tilde{\boldsymbol{q}} \in \mathbb{R}^n.$$

Proof: Since $K_p(\tilde{\boldsymbol{q}})$ is a diagonal matrix, then $\lambda_{\min}\{K_p(\tilde{\boldsymbol{q}})\} = \min_i\{k_{p_i}(\tilde{q}_i)\}$. To complete the proof, notice that according to Assumption 1, $k_{pl} \le k_{p_i}(\tilde{q}_i)$ for all $\tilde{q}_i \in \mathbb{R}$ and $i = 1, \ldots, n$, and also notice that $\min_i\{k_{p_i}(\tilde{q}_i)\} \in \{k_{p_i}(\tilde{q}_i)\}$ for all $\tilde{q}_i \in \mathbb{R}$ and $i = 1, \ldots, n$. The same procedure can be followed for k_{pu}, k_{vl} and k_{vu}.

3.2. **Closed-loop equation.** By considering the output $k(t)$ of the i-th SOF tuner (see (1)) for proportional and derivative gains as the entry $k_{pi}(\tilde{q}_i)$ of matrix $K_p(\tilde{\boldsymbol{q}})$ and $k_{vi}(\tilde{q}_i)$ of matrix $K_v(\tilde{\boldsymbol{q}})$, for $i = 1, \ldots, n$, respectively, the proposed control law is

$$\boldsymbol{\tau} = K_p(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{q}} + K_v(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}}. \tag{3}$$

Taking the joint position error vector $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q}$ and the joint velocity error vector $\dot{\tilde{\boldsymbol{q}}} = \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}$ as state variables, from (2) and (3) we obtain the closed-loop equation

$$\frac{d}{dt}\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{\boldsymbol{q}}} \\ M(\boldsymbol{q})^{-1}\left[-K_p(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{q}} - K_v(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} + \boldsymbol{d}(t,\boldsymbol{x})\right] \end{bmatrix}, \tag{4}$$

where $\boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{q}} & \dot{\tilde{\boldsymbol{q}}} \end{bmatrix}^T$ is the state vector and

$$\boldsymbol{d}(t,\boldsymbol{x}) = M(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}), \tag{5}$$

can be considered a perturbation of the system. When $\boldsymbol{d}(t,\boldsymbol{x}) = 0 \in \mathbb{R}^n$, the origin of the state space is the only equilibrium point. However, since tracking control is being considered, $\dot{\boldsymbol{q}}_d$ is not zero and presumably $\ddot{\boldsymbol{q}}_d$ is not zero, then $\boldsymbol{d}(t,\boldsymbol{x})$ is not zero. Hence, the origin cannot be considered as an equilibrium point of the system. Therefore, in the current case, instead of studying the problem of stability of the origin, the problem of the uniform boundedness of the solutions of the closed-loop system should be studied. In the rest of the current section, an analysis is carried out in order to find conditions to assure that the solutions of the closed-loop system are bounded.

3.3. **Lyapunov function candidate.** The following Lyapunov function candidate is proposed:

$$V(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \int_0^{\tilde{\boldsymbol{q}}} \boldsymbol{\sigma}^T K_p(\boldsymbol{\sigma})d\boldsymbol{\sigma} + \alpha\tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \alpha\int_0^{\tilde{\boldsymbol{q}}} \boldsymbol{\sigma}^T K_v(\boldsymbol{\sigma})d\boldsymbol{\sigma},$$

where α is a positive constant. It can be proven that $V(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})$ can be bounded as follows:

$$\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T P_1 \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix} \le V(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) \le \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T P_2 \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}, \tag{6}$$

where

$$P_1 = \frac{1}{2}\begin{bmatrix} k_{pl} + \alpha k_{vl} & -\alpha\lambda_{\max}\{M\} \\ -\alpha\lambda_{\max}\{M\} & \lambda_{\min}\{M\} \end{bmatrix} \tag{7}$$

and

$$P_2 = \frac{1}{2}\begin{bmatrix} k_{pu} + \alpha k_{vu} & \alpha\lambda_{\max}\{M\} \\ \alpha\lambda_{\max}\{M\} & \lambda_{\max}\{M\} \end{bmatrix}, \tag{8}$$

where Properties 3.1 and 3.2 have been used. (6) can also be written as:

$$k_1\|\boldsymbol{x}\|^2 \le V(t,\boldsymbol{x}) \le k_2\|\boldsymbol{x}\|^2,$$

where k_1 and k_2 are the positive constants

$$k_1 = \lambda_{\min}\{P_1\}, \quad k_2 = \lambda_{\max}\{P_2\}. \tag{9}$$

To assure that (6) is positive definite, it is enough to prove that matrix P_1 is positive definite. It can be proven that P_1 is positive definite if we choose α such that $0 < \alpha < \alpha_1$, where

$$\alpha_1 = \frac{k_{vl}\lambda_{\min}\{M\}}{2\lambda_{\max}^2\{M\}} + \left[\frac{k_{vl}^2\lambda_{\min}^2\{M\}}{4\lambda_{\max}^4\{M\}} + \frac{k_{pl}\lambda_{\min}\{M\}}{\lambda_{\max}^2\{M\}}\right]^{\frac{1}{2}}. \tag{10}$$

3.4. **Time derivative of the Lyapunov function candidate.** The time derivative of (6) is:

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) = {} & \dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\ddot{\tilde{\boldsymbol{q}}} + \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T \dot{M}(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \alpha\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \alpha\tilde{\boldsymbol{q}}^T \dot{M}(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} \\ & + \alpha\tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\ddot{\tilde{\boldsymbol{q}}} + \tilde{\boldsymbol{q}}^T K_p(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} + \alpha\tilde{\boldsymbol{q}}^T K_v(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}},\end{aligned}$$

where the Leibnitz rule for derivation of integrals has been used. The derivative along the trajectories of the closed-loop system (4) is

$$\dot{V} = -\dot{\tilde{\boldsymbol{q}}}^T K_v(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} + \alpha\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} - \alpha\tilde{\boldsymbol{q}}^T K_p(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{q}} + \alpha\dot{\tilde{\boldsymbol{q}}}^T C(\boldsymbol{q}, \dot{\boldsymbol{q}})\tilde{\boldsymbol{q}} + \left[\dot{\tilde{\boldsymbol{q}}} + \alpha\tilde{\boldsymbol{q}}\right]^T \boldsymbol{d}(t, \boldsymbol{x}), \tag{11}$$

where the property of skew-symmetry of $\frac{1}{2}\dot{M}(\boldsymbol{q}) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ has been used (see [38]). The perturbation $\boldsymbol{d}(t, \boldsymbol{x})$ can be bounded as

$$\|\boldsymbol{d}(t, \boldsymbol{x})\| \leq k_M\|\ddot{\boldsymbol{q}}_d\|_M + k_g + k_c\|\dot{\boldsymbol{q}}_d\|_M^2 + k_c\|\dot{\boldsymbol{q}}_d\|_M\|\dot{\tilde{\boldsymbol{q}}}\|,$$

where $\|\dot{\boldsymbol{q}}_{\boldsymbol{d}}\|_M$ is the largest value of the desired joint velocities vector, $\|\ddot{\boldsymbol{q}}_d\|_M$ is the largest value of the desired joint accelerations vector, and upper bounds over terms $M(\boldsymbol{x})\boldsymbol{y}$, $C(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{z}$ and $\boldsymbol{g}(\boldsymbol{q})$ [1, 36, 39] have been used. By denoting d_1 and d_2 as

$$\begin{aligned} d_1 &= k_M\|\ddot{\boldsymbol{q}}_d\|_M + k_g + k_c\|\dot{\boldsymbol{q}}_d\|_M^2, \\ d_2 &= k_c\|\dot{\boldsymbol{q}}_d\|_M, \end{aligned} \tag{12}$$

the last term of (11) can be bounded as

$$\left[\dot{\tilde{\boldsymbol{q}}} + \alpha\tilde{\boldsymbol{q}}\right]^T \boldsymbol{d}(t, \boldsymbol{x}) \leq d_1\left[\|\dot{\tilde{\boldsymbol{q}}}\| + \alpha\|\tilde{\boldsymbol{q}}\|\right] + d_2\left[\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\|\right]. \tag{13}$$

After upper bounding the first four terms of (11) (by using Property 3.2 and the upper bound of the term $C(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{z}$), and substituting (13), $\dot{V}$ can be bounded as follows:

$$\begin{aligned}\dot{V} \leq {} & -k_{vl}\|\dot{\tilde{\boldsymbol{q}}}\|^2 - \alpha k_{pl}\|\tilde{\boldsymbol{q}}\|^2 + \alpha\lambda_{\max}\{M\}\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha k_c\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\|^2 \\ & + \alpha d_2\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\| + d_1\left[\|\dot{\tilde{\boldsymbol{q}}}\| + \alpha\|\tilde{\boldsymbol{q}}\|\right] + d_2\left[\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\|\right].\end{aligned} \tag{14}$$

For analysis purposes, it is useful to divide (14) in two parts,

$$\dot{V} \leq W_1 + W_2,$$

with

$$\begin{aligned} W_1 = {} & -k_{vl}\|\dot{\tilde{\boldsymbol{q}}}\|^2 - \alpha k_{pl}\|\tilde{\boldsymbol{q}}\|^2 + \alpha\lambda_{\max}\{M\}\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha k_c\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha d_2\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\| \\ & + d_2\left[\|\dot{\tilde{\boldsymbol{q}}}\|^2 + \alpha\|\tilde{\boldsymbol{q}}\|\|\dot{\tilde{\boldsymbol{q}}}\|\right] \end{aligned} \tag{15}$$

and

$$W_2 = d_1\left[\|\dot{\tilde{\boldsymbol{q}}}\| + \alpha\|\tilde{\boldsymbol{q}}\|\right]. \tag{16}$$

3.5. **Bounds over the solutions.** In the process of finding a negative definite function that upper bounds $\dot{V}$, we have to prove that W_1 is negative definite, then, we have to find conditions to assure that the sum $W_1 + W_2$ is negative definite. In order to reduce the degree of W_1 from third to second, let us consider a ball $B_r \subset \mathcal{D} \subset \mathbb{R}^n$ centered in the origin, with radius $r > 0$, such that

$$B_r = \left\{ \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}} \in \mathbb{R}^n : \left\| \begin{matrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \end{matrix} \right\| < r \right\},$$

inside of which W_1 is negative definite. Equation (15) can now be written as

$$W_1 = - \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T Q \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}, \tag{17}$$

where

$$Q = \begin{bmatrix} \alpha k_{pl} & -\alpha d_2 \\ -\alpha d_2 & k_{vl} - d_2 - \alpha \left[\lambda_{\max}\{M\} + k_c r\right] \end{bmatrix}. \tag{18}$$

It can be proven that the matrix Q in (18) is positive definite if α is chosen such that $0 < \alpha < \alpha_2$, where

$$\alpha_2 = \frac{[k_{vl} - d_2]\, k_{pl}}{d_2^2 + [\lambda_{\max}\{M\} + k_c r]\, k_{pl}}. \tag{19}$$

Equation (17) can now be written as

$$W_1 \leq -k_3 \|\boldsymbol{x}\|^2, \tag{20}$$

where $k_3 = \lambda_{\min}\{Q\}$. Taking in account the fact that $\|\tilde{\boldsymbol{q}}\| \leq \|\boldsymbol{x}\|$ and $\|\dot{\tilde{\boldsymbol{q}}}\| \leq \|\boldsymbol{x}\|$, by defining

$$b = [\alpha + 1]\, d_1, \tag{21}$$

W_2 in (16) can be bounded as

$$W_2 \leq b \|\boldsymbol{x}\|. \tag{22}$$

By using (20) and (22), $\dot{V}$ can be bounded as

$$\dot{V} \leq -k_3 \|\boldsymbol{x}\|^2 + b\|\boldsymbol{x}\|. \tag{23}$$

In order to cancel the effect of the positive term in (23) (by using a procedure that has been used in [40] and in [41]), we include a positive constant $\epsilon < 1$, such that

$$\dot{V}(t, \boldsymbol{x}) \leq -k_3(1-\epsilon)\|\boldsymbol{x}\|^2 - k_3\epsilon\|\boldsymbol{x}\|^2 + b\|\boldsymbol{x}\|. \tag{24}$$

By taking the two last terms of (24), we have that

$$-k_3\epsilon\|\boldsymbol{x}\|^2 + b\|\boldsymbol{x}\| < 0,$$

when

$$k_3\epsilon\|\boldsymbol{x}\|^2 > b\|\boldsymbol{x}\|,$$
$$\|\boldsymbol{x}\| > \frac{b}{k_3\epsilon}.$$

Therefore, by denoting

$$\mu = \frac{b}{k_3\epsilon}, \tag{25}$$

we can write

$$\dot{V}(t, \boldsymbol{x}) \leq -k_3(1-\epsilon)\|\boldsymbol{x}\|^2, \quad \forall\ \|\boldsymbol{x}\| > \mu. \tag{26}$$

Notice that the time derivative of the Lyapunov candidate function (26) is a negative definite function inside the ball B_r with radius r. From the previous analysis, we can now establish the stability result, stated in the following proposition.

Proposition 3.1. *Let consider the dynamic model of the robot (2) together with the control law (3). The structure of the variable gains matrices* $K_p(\tilde{\boldsymbol{q}})$ *and* $K_v(\tilde{\boldsymbol{q}})$ *is given in Definition 3.1. There exists a positive constant* $\alpha > 0$ *that satisfies*

$$\alpha < \min\{\alpha_1, \alpha_2\}, \tag{27}$$

where α_1 *and* α_2 *were defined in (10) and (19). Also, the parameter*

$$\mu = \frac{b}{k_3 \epsilon},$$

where $b = [\alpha + 1]\, d_1$*,* ϵ *is a positive constant lower than 1, and* $k_3 = \lambda_{\min}\{Q\}$*, satisfies*

$$\mu < \sqrt{\frac{k_1}{k_2}} r, \tag{28}$$

where k_1 *and* k_2 *were defined in (9), and* r *is the radius of the ball* B_r *inside which the time derivative of the Lyapunov function (26) is negative definite. Then, the closed-loop system (4) is stabilizable in the sense that there exists* $T \geq 0$ *such that the solution* $\boldsymbol{x}(t)$*, with initial state* $\boldsymbol{x}_0 = \boldsymbol{x}(t_0) < \sqrt{\frac{k_1}{k_2}} r$*, satisfies*

$$\|\boldsymbol{x}(t)\| \leq \sqrt{\frac{k_2}{k_1}} \mu, \quad \forall\, t > t_0 + T, \tag{29}$$

where (29) represents the ultimate uniform bound. A Lyapunov function to demonstrate this is (6).

Proof: The proof results from the Theorem 4.18 in [40]. In our case, the functions and parameters referred in the theorem are:

$$\begin{aligned}
\delta_1(\|\boldsymbol{x}\|) &= k_1 \|\boldsymbol{x}\|^2, \\
\delta_2(\|\boldsymbol{x}\|) &= k_2 \|\boldsymbol{x}\|^2, \\
W_3(\boldsymbol{x}) &= k_3(1 - \epsilon)\|\boldsymbol{x}\|^2, \\
\mu &= \frac{b}{k_3 \epsilon}.
\end{aligned}$$

Remark 3.1. *The ultimate uniform bound of* $\|\boldsymbol{x}(t)\|$ *is inside the ball* B_r *with radius* r*, that is,* $r > \sqrt{\frac{k_2}{k_1}} \mu$*. This means that the solution* $\boldsymbol{x}(t)$ *of the system will stay inside the ball* B_r*, centered in the origin, after a finite time* $t_0 + T$*.*

Remark 3.2. *Notice that since the parameter* μ *depends on the maximum desired velocities* $\|\dot{\boldsymbol{q}}_d\|_M$ *and accelerations* $\|\ddot{\boldsymbol{q}}_d\|_M$ *(see (12), (21) and (25)), if* $\|\dot{\boldsymbol{q}}_d\|_M$ *and* $\|\ddot{\boldsymbol{q}}_d\|_M$ *are large, the uniform ultimate bound* $\sqrt{\frac{k_2}{k_1}} \mu$ *over the solutions will be large, and inversely, if these are small, the uniform ultimate bound will be small. Given a certain desired trajectory, if we want to diminish the bound, we have to increase the values of* k_{vl} *and* k_{pl} *(see (7) and (9)). On the other hand, increasing* k_{vu} *and* k_{pu} *can result in a larger uniform ultimate bound (see (8) and (9)).*

The requirement of boundedness of the PD variable gains assumed in this analysis (see Assumption 3.1) is consistent with the practice of using gain values within a certain range in order to avoid eventual instabilities of the system such as large oscillations or high tracking errors. This requirement can be fulfilled through an appropriate design of the tuning mechanism. Notice that the result of this analysis can be generalized to others variable gains PD tracking control schemes, regardless of the mechanism used to adjust the gains, provided that these gains are lower and upper bounded. On the other hand, for practical applications, the bounds for the variable PD gains cannot be arbitrarily selected due to the physical limitations of the actuators and the sensors. This issue will be addressed in the following section.

4. **Tuning and Experiments.** The ultimate boundedness of the solutions is assured if (27) and (28) are satisfied. In this context, tuning stands for the appropriate selection of the minimum and maximum values of the variable PD gains that can be used to meet such criterion. Therefore, with the purpose of achieving this objective, a tuning procedure is proposed. The largest value of the desired velocities $\|\dot{\boldsymbol{q}}_{\boldsymbol{d}}\|_M$ and the largest value of the desired accelerations $\|\ddot{\boldsymbol{q}}_{\boldsymbol{d}}\|_M$ are supposed to be known for the designer, as well as some parameters from the dynamical model of the robot. These parameters are k_M, k_g, k_c, $\lambda_{\min}\{M\}$ and $\lambda_{\max}\{M\}$. It is important to remark that although these parameters (or an estimate of them) are required to assure that the solutions are uniformly ultimately bounded, they are not required to compute the control law. Moreover, in case of variation of these parameters, the uniform ultimate boundedness of the solutions can still be guaranteed by selecting large values of k_{vl} or k_{pl} and no too large values of k_{vu} and k_{pu} (see (25) and (28)).

4.1. **Suggested procedure.**

- Propose a value for the radius r.
- Propose bounds for the maximum and minimum eigenvalues of the variable gains matrices $K_p(\tilde{\boldsymbol{q}})$ and $K_v(\tilde{\boldsymbol{q}})$, that is, propose values for k_{pl}, k_{pu}, k_{vl} and k_{vu}.
- Compute α_1 and α_2 according to (10) and (19), so that (27) can be computed.
- For a set of chosen values of α according to the previous step, compute the constants b, k_1, k_2, k_3 and μ, so that the fulfillment of (28) can be verified. In case of no fulfillment of (28), the radius r can be increased, or else the maximum and minimum eigenvalues for the variable gains matrices can be changed.
- Finally, compute the ultimate uniform bound (29). Choose values of α and r that yield the best value for the ultimate bound.

4.2. **Tuning the gains for the CICESE robot of 2 d.o.f.** In the following, values for tuning the variable gains of the PD controller are proposed. The task of the controller is to obtain good trajectory tracking for a 2 d.o.f. vertical direct-drive robot, designed and built at CICESE Research Center, Ensenada, Mexico, which is a prototype robot for research purposes. This robot is at the Control Laboratory of the Instituto Tecnologico de La Laguna. The following dynamical parameters of this robot are required: $k_M = 2.533$ [kg m^2], $k_c = 0.336$ [kg m^2], $k_g = 40.334$ [kg m^2/s^2], $\lambda_{\min}\{M\} = 0.102$ [kg m^2] and $\lambda_{\max}\{M\} = 2.533$ [kg m^2] (see [36]).

The desired trajectories are the following (proposed in [36])

$$q_{d1} = a_1 + b_1\left[1 - e^{-\beta_1 t^3}\right] + c_1\left[1 - e^{-\beta_1 t^3}\right]\sin(\omega_1 t),$$
$$q_{d2} = a_2 + b_2\left[1 - e^{-\beta_2 t^3}\right] + c_2\left[1 - e^{-\beta_2 t^3}\right]\sin(\omega_2 t),$$

with coefficient values shown in Table 3. With these trajectories, the maximum values of

Table 3. Coefficients of the desired trajectory

	a_i	b_i	c_i	ω_i	β_i
$i = 1$	0.7854	0.2	0.17	1.0	0.5
$i = 2$	0.0872	0.2	0.2	1.0	0.5
units	[rad]	[rad]	[rad]	[rad/s]	

the desired velocities and accelerations are $\|\dot{\boldsymbol{q}}_d\|_M = 0.3985$ [rad/s] and $\|\ddot{\boldsymbol{q}}_d\|_M = 0.6012$ [rad/s^2].

Trivially we propose $\epsilon = 0.9999$. The procedure of choosing the bounds for the variable PD gains is outlined below.

- Selecting k_{pl}. In order to avoid the commanded torques from exceeding the largest allowed motor torques, which are $\tau_1^{\max} = 150$ [Nm] and $\tau_2^{\max} = 15$ [Nm], we compute the initial tracking position and velocity errors. With initial conditions $q_1 = 0.0$ [rad], $q_2 = 0.0$ [rad], $\dot{q}_1 = 0.0$ [rad/s] and $\dot{q}_2 = 0.0$ [rad/s], the initial position errors are $\tilde{q}_1 = 0.7854$ [rad], $\tilde{q}_2 = 0.0872$ [rad] and the initial velocity errors are $\dot{\tilde{q}}_1 = 0.0$ [rad/s], $\dot{\tilde{q}}_2 = 0.0$ [rad/s], which yield the following values for the maximum initial gains: $k_{p1} \leq 190.9$ [Nm/rad], $k_{p2} \leq 172.0$ [Nm/rad]. Thus we selected $k_{pl} = 150.0$ [Nm/rad].
- Selecting k_{pu}. This is the largest value that the proportional gains can take. Since large proportional gains produce small steady state error, it is desirable to set large values for this bound. However, since a large k_{pu} yields a large uniform ultimate bound for the solutions (see Remark 3.2), we chose a minimum value that yields small position errors, Thus we selected $k_{pu} = 1100.0$ [Nm/rad].
- Selecting k_{vl} and k_{vu}. Notice from (19) that k_{vl} should be larger than $d_2 = k_c\|\dot{\boldsymbol{q}}_d\|_M$ in order to assure the existence of a positive constant α_2 and hence the definite positiveness of Q. For the current desired trajectory, k_{vl} should be set greater than 0.1339 [Nm s/rad]. Large values of k_{vl} are desirable since the derivative action contributes to cope with a variable reference. On the other hand, in our experimental platform, joint velocities are estimated via the Euler algorithm, which in the practice produces a noisy output signal. This noise can be amplified when the velocity error is multiplied by the derivative gain, which may result in large distortions of the derivative control action and exceeding the maximum motor torques. In order to prevent that this condition happen, by trial and error, the values k_{vl} and k_{vu} were set to 10.0 [Nm s/rad] and 15.0 [Nm s/rad], respectively.

The selected values are shown in Table 4.

Table 4. Bounds for the gains

k_{pl}	k_{pu}	k_{vl}	k_{vu}
150.0 [Nm/rad]	1100.0 [Nm/rad]	10.0 [Nm s/rad]	15.0 [Nm s/rad]

By choosing $r = 5 \times 10^4$, from (10), (19) and (27) we get that α must be smaller than 5.8718×10^{-4}. If we select $\alpha = 5.871 \times 10^{-4}$, by using the minimum and maximum gains values just proposed, together with the dynamical robot parameters and the trajectory parameters, from (9) and (25) we obtain $\sqrt{\frac{k_2}{k_1}}\mu = 4.9975 \times 10^4$, thus the condition $r > \sqrt{\frac{k_2}{k_1}}\mu$ (28) is satisfied. The uniform ultimate bound over the solutions, according to (29), is $\sqrt{\frac{k_2}{k_1}}\mu = 4.9975 \times 10^4$.

4.3. **Experiments.** Experiments of the controller with gains according to the boundedness criterion were carried out.

- First, experiments of the SOF-PD were carried out. The bounds of the variable gains were the proposed in the tuning procedure. The initial values of the gains are shown in Table 5.

TABLE 5. Initial gains values

k_{p1}	k_{p2}	k_{v1}	k_{v2}
150.0 [Nm/rad]	150.0 [Nm/rad]	10.0 [Nm s/rad]	10.0 [Nm s/rad]

- Second, in order to compare results, experiments with a classic PD with constant gains were also done. The employed gain values are shown in Table 6. Notice that these gains also satisfy the boundedness criterion just obtained.

TABLE 6. PD classic gains

k_{p1}	k_{p2}	k_{v1}	k_{v2}
190.0 [Nm/rad]	170.0 [Nm/rad]	10.0 [Nm s/rad]	10.0 [Nm s/rad]

The experiments were carried out using WinMechLab, which is a real-time control test-oriented software for mechatronic systems [42]. The sampling period used is 2.5 [ms]. Notice that in the experiments of both controllers, we were careful in avoiding to exceed the maximum allowed torques of the servo-motors.

The tracking errors obtained with the SOF-PD and the classic PD are shown in Figures 4 and 5. The torques delivered by the motors with the SOF-PD controller are shown in Figures 6 and 7. The gains tuned by the SOF algorithm are shown in Figures 8 and 9.

4.4. **Remarks on the experimental results.** It can be seen in Figures 4 and 5 that the errors obtained with the SOF-PD are much smaller than those obtained with the classic PD. Torques delivered by the SOF-PD (see Figures 6 and 7) are kept under the maximum allowed torques of the motors. The tuned gains of the SOF-PD are kept within the established range.

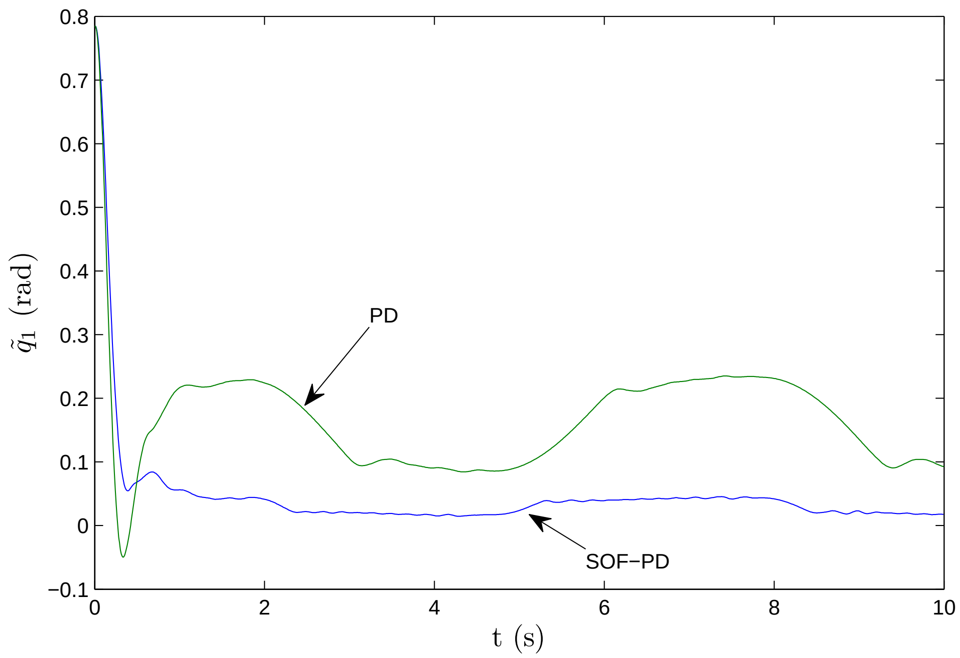

FIGURE 4. Tracking errors on q_1

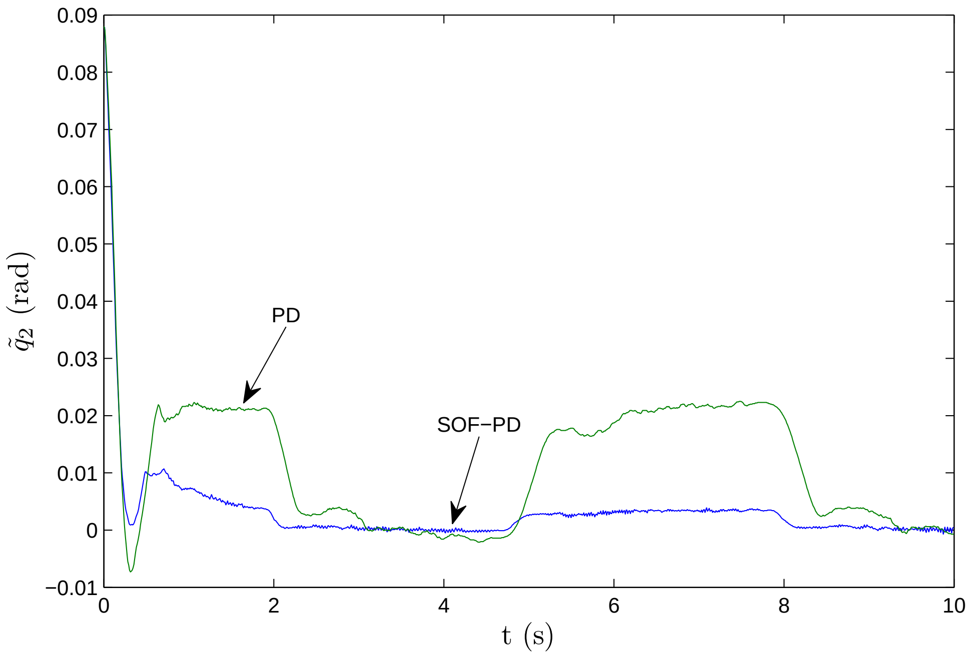

FIGURE 5. Tracking errors on q_2

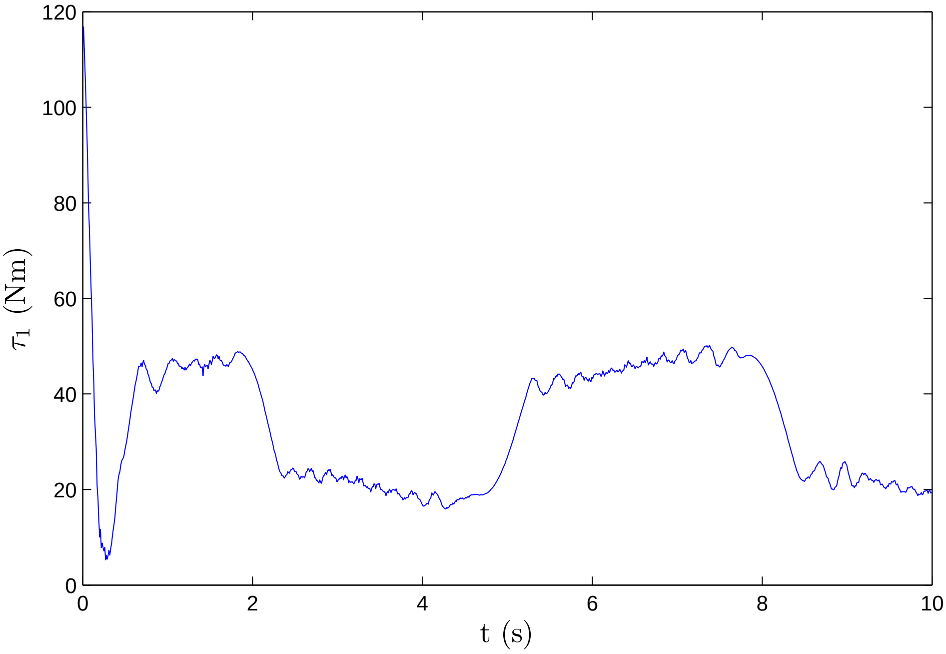

FIGURE 6. Torque τ_1 delivered by the SOF-PD

A significant quantitative comparison of the performance of the controllers can be done by using the $\mathcal{L}_2$ norm criterion, which has been used in [43]. The $\mathcal{L}_2$ norm of the tracking error can be obtained with the following formula:

$$\mathcal{L}_2\left[\tilde{\boldsymbol{q}}\right] = \sqrt{\frac{1}{T - t_0}\int_{t_0}^{T} \tilde{\boldsymbol{q}}^T \tilde{\boldsymbol{q}}\, dt}$$

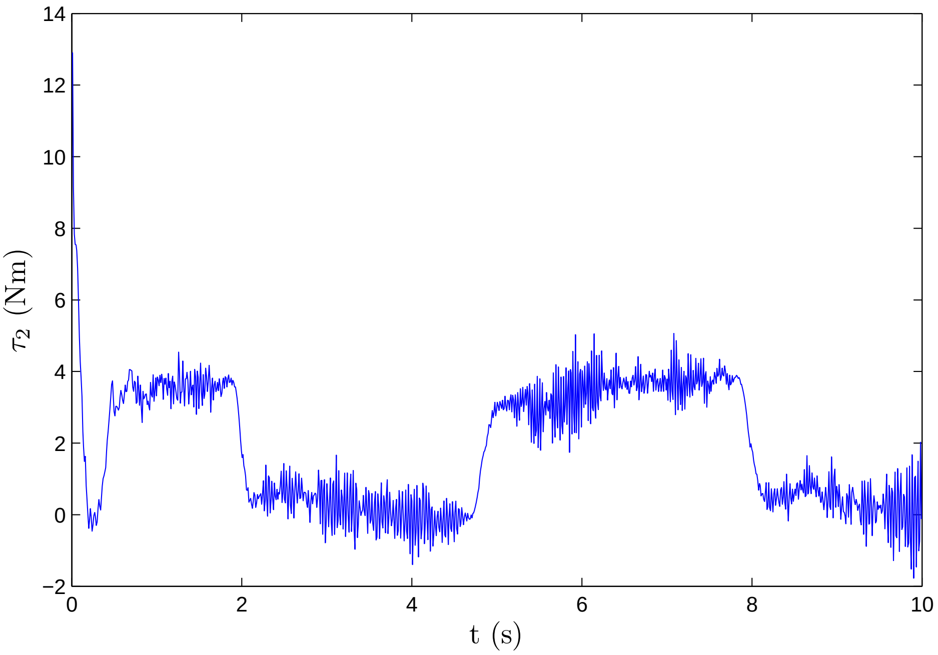

FIGURE 7. Torque τ_2 delivered by the SOF-PD

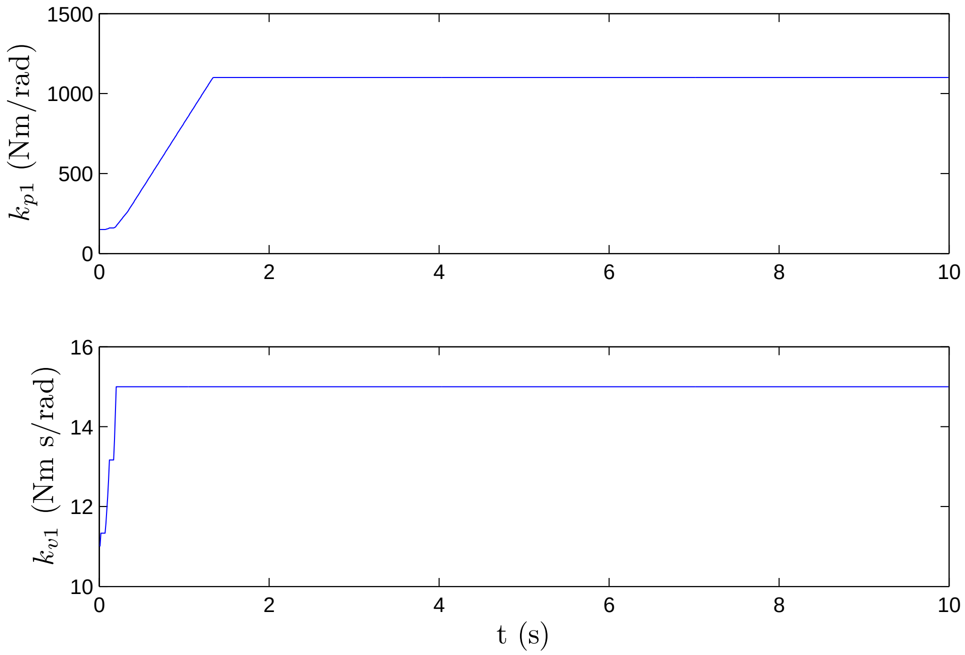

FIGURE 8. SOF-PD Gains for joint 1

where t_0 and T are the initial and final time of the period of time being considered. The result of computing this norm on the resulting tracking errors is shown in Figure 10.

4.5. **Further experiments.** Although good results were obtained from the experiments with the gains chosen to meet the tuning conditions given in the procedure, in order to show that the results of the proposed controller can be improved, additional experiments

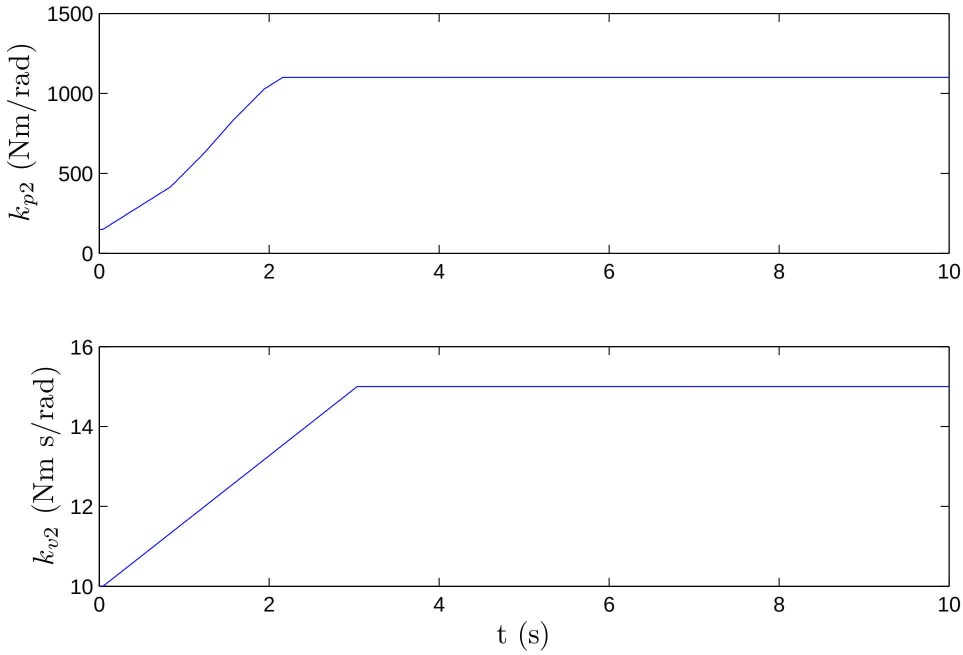

FIGURE 9. SOF-PD Gains for joint 2

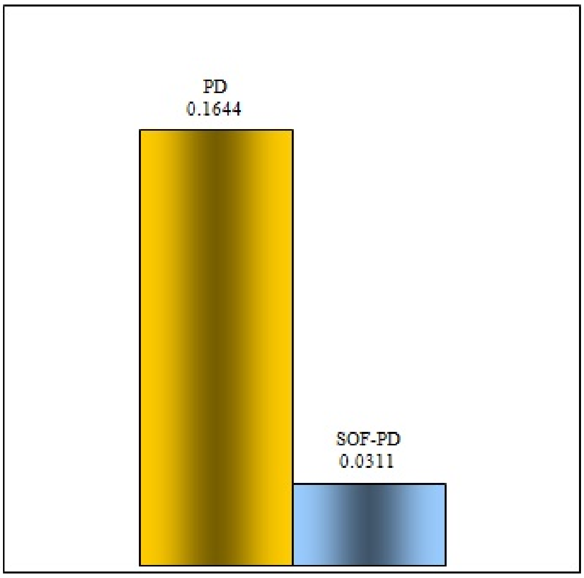

FIGURE 10. $\mathcal{L}_2$ norm of the tracking errors

with the SOF-PD controller were carried out, in which the gains were not subject to the boundedness criterion. The proposed bounds for the variable gains are shown in Table 7. The initial values were the same used in the previous experiments with the SOF-

TABLE 7. New bounds for the gains

k_{pl}	k_{pu}	k_{vl}	k_{vu}
150.0 [Nm/rad]	3500.0 [Nm/rad]	10.0 [Nm s/rad]	15.0 [Nm s/rad]

PD controller. The gains employed in the classic PD were the same used in the former

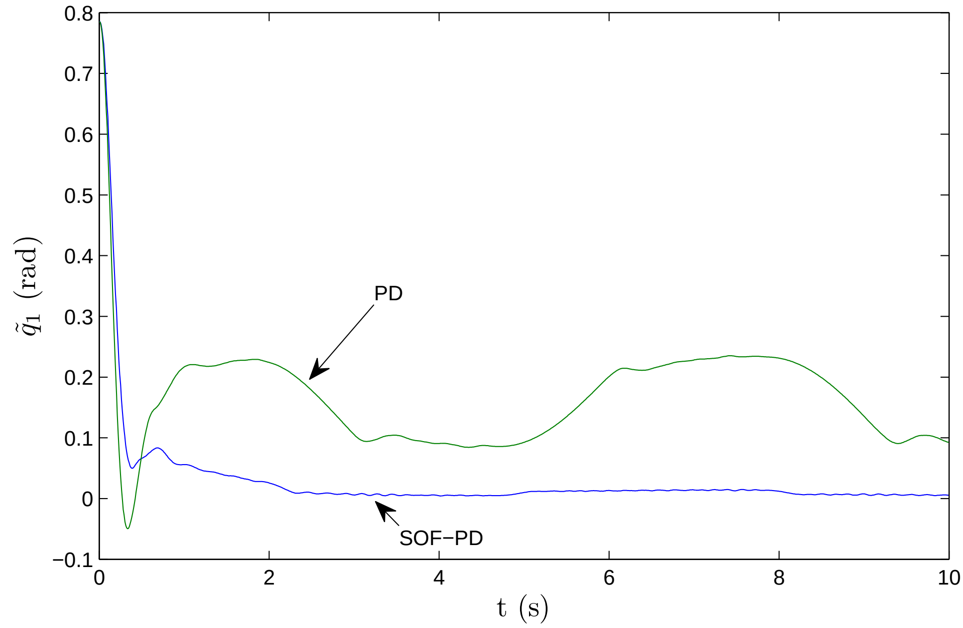

FIGURE 11. Tracking errors on q_1 with the new tuning of the SOF-PD

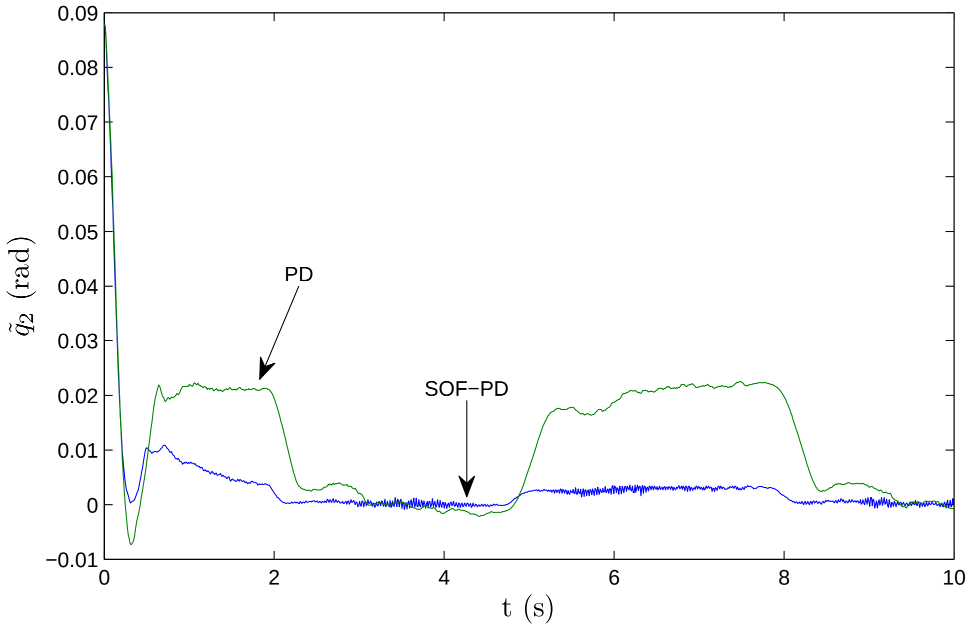

FIGURE 12. Tracking errors on q_2 with the new tuning of the SOF-PD

experiments. These values could not be increased for the desired trajectory without exceeding the maximum torques of the motors.

The tracking errors obtained with the SOF-PD and the classic PD are shown in Figures 11 and 12. The torques delivered by the motors with the SOF-PD controller are shown in Figures 13 and 14. A comparison of the $\mathcal{L}_2$ norm of the tracking errors obtained with the latter tuning of the SOF-PD controller and the tracking errors obtained with the PD controller is shown in Figure 15. It can be seen that, in the latter experiments,

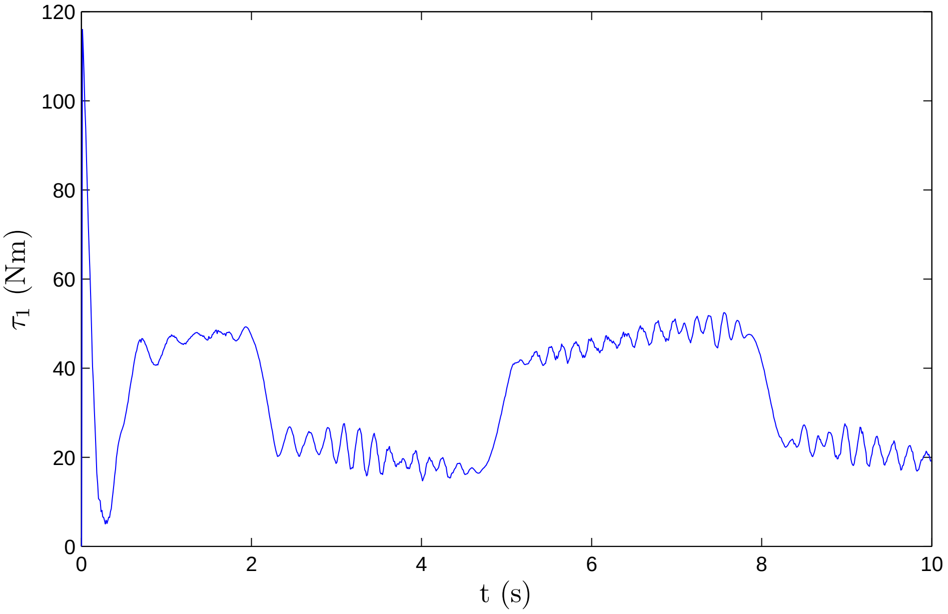

FIGURE 13. Torque τ_1 delivered with the new tuning of the SOF-PD

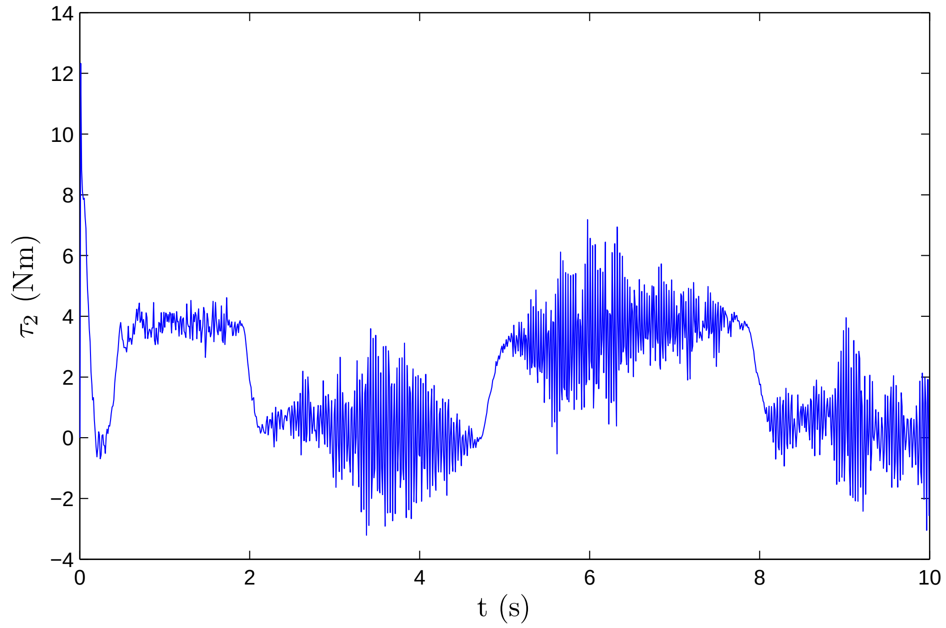

FIGURE 14. Torque τ_2 delivered with the new tuning of the SOF-PD

smaller tracking errors were obtained, outperforming the errors obtained with the SOF-PD controller in the previous experiments (see Figures 10 and 15), this is due to the fact that the gains could be adjusted without being subject to the tuning constraints. Nevertheless, it is convenient to remind that the tuning conditions given on the suggested tuning procedure (see Subsection 4.1) are sufficient but no necessary to guarantee uniform ultimate boundedness of the solutions of the closed-loop system.

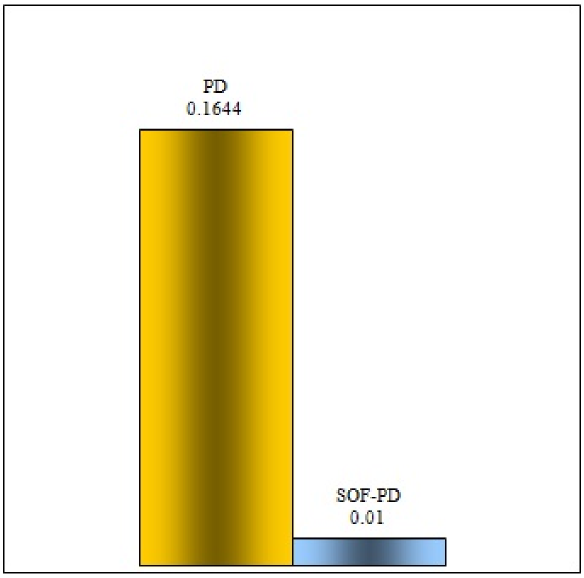

FIGURE 15. $\mathcal{L}_2$ norm of the tracking errors with the new tuning of the SOF-PD

5. **Conclusions.** In this paper we have presented a Self-Organizing Fuzzy PD tracking controller for robot manipulators with a gain-scheduling structure, in which a SOF system performs the tuning of the gains of the PD controller in the feedback loop, depending on the position errors. This SOF system has the ability to adjust its own inference rules, according to two performance index tables designed for independent tuning of P and D gains. This controller takes advantage of the the simplicity of the PD structure, and its performance is enhanced by the employment of the SOF system for gain tuning. Based on the Lyapunov theory, the uniform ultimate boundedness of the solutions of the closed-loop system of this controller is demonstrated, provided that, for a bounded trajectory, the variable gains bounds meet the developed boundedness criterion. This analysis, to the best knowledge of the authors, is carried out for the first time for this class of PD gain-scheduling controllers.

Moreover, in order to make easier the selection of the bounds of the variable gains that satisfy the boundedness of the solutions criterion, a tuning procedure is proposed.

Finally, the better performance of the proposed controller in comparison with a classic PD controller has been verified through real-time experiments on a vertical direct drive 2 d.o.f robot arm, using gain bounds that satisfy the tuning conditions from the boundedness criterion. Furthermore, additional experiments were carried-out, using gain bounds that exceed the boundedness criterion, in order to show that the controller performance can still be improved. The computation of the $\mathcal{L}_2$ norm of the resulting position errors confirms the superiority of the proposed controller.

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